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FIXED POINT RESULTS VIA TRIANGLE MULTIPLICATIVE METRIC SPACES

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Abstract This article's essential aim is to prove the existence and uniqueness of common fixed points for occasionally weakly biased mappings of type $(\mathbb{A})$ in triangular multiplicative metric spaces (shortly TMMSs). To make our results valid and significant, we produce two suitable examples and an application to integral equation. Our findings improve several works in metric as well as TMMSs.

1 Introduction

In 1922, the famous Polish mathematician Banach proved the existence and uniqueness of a fixed point for a contractive mapping in a complete metric space. This well-known theorem was improved and generalised by several authors. Until now, many researchers competed to improve and re-improve the famous contractive mapping theorem sometimes by increasing the number of mappings and sometimes by using other alternative spaces. For this end, in 2008, Bashirov et al. [3] presented an alternative space called multiplicative metric space which lies in changing the first and third axioms of the ordinary metric space. Recently in 2024, Berzig [4] introduced a new concept of TMMSs, and established a fixed point theorem in such spaces. According to him, it has been noticed that if $(\mathbf{X}, d)$ is a multiplicative metric space, then $(\mathbf{X}, d)$ is a classical metric space where $d(x, y) = \ln d(x, y)$ and to avoid this situation, he proposed a new space under the name "triangular multiplicative metric space (shortly TMMS)". In such spaces the second condition is $d(x, y) = 0$ which is different from the second axiom in multiplicative metric spaces which is $d(x, y) = 1$, according to Berzig, this condition prevents reducing TMMSs to multiplicative metric spaces. Note that, several authors investigated existence and uniqueness of fixed and common fixed points in different spaces, see for instance [2], [6], [7], [8], [10], [11], [12], [13], [14], [15] and [16].

Recently, in [5], we suggested the so-called "occasionally weakly biased mappings of type $(\mathbb{A})$", and compared it by the famous notions of weakly compatible mappings introduced by Jungck and Roades in [9] and occasionally weakly compatible mappings introduced by Al-Thagafi and Shahzad in [1], and we found that our concept is more general than the two mentioned ones.

In this article, we will utilise our concept to search shared fixed elements in complete TMMSs, and with suitable examples, we will show the applicability of our results.

2 Needed Preliminaries

Definition 2.1. Let $\mathbf{X} \neq \emptyset$ and $d : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$. Suppose $\forall\ u_1, u_2, u_3 \in \mathbf{X}$, we have

b_1: $d(u_1, u_2) = 0 \Leftrightarrow u_1 = u_2$,

b_2: $d(u_1, u_2) = d(u_2, u_1)$,

b_3: $d(u_1, u_2) \leq d(u_1, u_3) + d(u_3, u_2)$,

then, the pair $(\mathbf{X}, d)$ is called **metric space**.

Definition 2.2. ([3]) Let $\mathbf{X}$ be a non-empty set. Define a function d from $\mathbf{X} \times \mathbf{X}$ into $\mathbb{R}_+$ such that

(i) $\forall\ u_1, u_2 \in \mathbf{X}$, $d(u_1, u_2) \geq 1$,

(ii) $d(u_1, u_2) = 1$ if and only if $u_1 = u_2$,

(iii) $\forall\ u_1, u_2 \in \mathbf{X}$, $d(u_1, u_2) = d(u_2, u_1)$,

(iv) $\forall\ u_1, u_2, u_3 \in \mathbf{X}$, $d(u_1, u_3) \leq d(u_1, u_2).d(u_2, u_3)$.

The pair $(\mathbf{X}, d)$ is called a **multiplicative metric space**.

Definition 2.3. ([4]) Let $\mathbf{X}$ be a non-empty set, $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}_+$ be a function and $b \geq 1$ be a real constant. Then, d is called triangular multiplicative metric if $\forall\ u_1, u_2, u_3 \in \mathbf{X}$, $\forall\ u_3 \in \mathbf{X} \setminus \{u_1, u_2\}$, we have

(i) $d(u_1, u_2) \geq 0$,

(ii) $d(u_1, u_2) = 0$ if and only if $u_1 = u_2$,

(iii) $d(u_1, u_2) = d(u_2, u_1)$,

(iv) $d(u_1, u_2) \leq b.d(u_1, u_3).d(u_3, u_2)$.

The triplet $(\mathbf{X}, d, b)$ is a **TMMS**.

Example 2.4. Set $\mathbf{X} = \mathbb{R}$ and $d : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ such that

$$d(u_1, u_2) = \begin{cases} a^{|u_1 - u_2|} \text{ when } u_1 \neq u_2 \\ 0 \text{ when } u_1 = u_2, \end{cases}$$

where $a > 1$. Then, $(\mathbf{X}, d)$ is a TMMS.

Proof. We have:

(i) $d(u_1, u_2) = 0 \Leftrightarrow u_1 = u_2$,

(ii) $d(u_1, u_2) = d(u_2, u_1)$ for all $u_1, u_2 \in \mathbf{X}$,

(iii) now, for all $u_1, u_2, u_3 \in \mathbf{X}$ with $u_3 \in \mathbf{X} \backslash \{u_1, u_2\}$, for $a > 1$ and $b \geq 1$, we have

$$|u_1 - u_2| = |(u_1 - u_3) + (u_3 - u_2)| \leq |u_1 - u_3| + |u_3 - u_2|$$

which implies that

$$\begin{aligned} d(u_1, u_2) = a^{|u_1 - u_2|} &= a^{|(u_1 - u_3) + (u_3 - u_2)|} \\ &\leq a^{|u_1 - u_3| + |u_3 - u_2|} \\ &= a^{|u_1 - u_3|}.a^{|u_3 - u_2|} \\ &\leq b.a^{|u_1 - u_3|}.a^{|u_3 - u_2|} \\ &= b.d(u_1, u_3).d(u_3, u_2). \end{aligned}$$

□

Definition 2.5. ([5]) Let $(\mathbf{X}, d, b)$ be a TMMS and Let $\mathbf{U}, \mathbf{V} : (\mathbf{X}, d, b) \to (\mathbf{X}, d, b)$ be functions. $\mathbf{U}$ and $\mathbf{V}$ are **occasionally weakly U-biased (respectively V-biased) of type** $(\mathbb{A})$, iff, there exists $\varpi \in \mathbf{X}$ such that $\mathbf{C}\varpi = \mathbf{D}\varpi$ and

$$d(\mathbf{U}\mathbf{U}\varpi, \mathbf{V}\varpi) \leq d(\mathbf{V}\mathbf{U}\varpi, \mathbf{U}\varpi),$$
$$d(\mathbf{V}\mathbf{V}\varpi, \mathbf{U}\varpi) \leq d(\mathbf{U}\mathbf{V}\varpi, \mathbf{V}\varpi),$$

respectively.

3 Berzig's Results

In [4], Maher proposed and demonstrated the next result:

Theorem 3.1. *Assume that* $(\mathbf{X}, d)$ *is a complete TMMS and taking a mapping* $\mathbf{M}$ *from* $\mathbf{X}$ *into itself. Suppose there is* $0 \leq \sigma < 1$ *with*

$$d(\mathbf{M}x, \mathbf{M}y) \leq \sigma \max\{d(x, \mathbf{M}x), d(y, \mathbf{M}y)\}$$

$\forall\ x, y$ *in* $\mathbf{X}$. *Then* $\mathbf{M}$ *admits* x_* *as a sole fixed point and the sequence* $\{M^n x_0\}$ *converges to* x_* $\forall\ x_0$.

From the above theorem, he derived the following result:

Corollary 3.2. *Taking a complete TMMS* $(\mathbf{X}, d)$ *and a given self-mapping of* $\mathbf{M}$. *Assume there is* $0 \leq \sigma < \frac{1}{2}$ *with*

$$d(\mathbf{M}x, \mathbf{M}y) \leq \sigma[d(x, \mathbf{M}x) + d(y, \mathbf{M}y)]$$

$\forall\ x, y$ *in* $\mathbf{X}$. *Then* $\mathbf{M}$ *admits* x_* *as a sole fixed point and the sequence* $\{M^n x_0\}$ *converges to* x_* *for every* x_0.

In the following lines, the above results will be improved and extended.

4 Results, Demonstrations and Examples

We start by giving our first result.

Theorem 4.1. *Define four mappings* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{S}$ *and* $\mathbf{T}$ *from a complete TMMS* $(\mathbf{X}, d, b)$ *into itself which satisfying the two conditions:*

(i) pairs $(\mathbf{A}, \mathbf{S})$ *and* $(\mathbf{B}, \mathbf{T})$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$,

(ii) $\forall\ x, y \in \mathbf{X}$,

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) \quad \leq \quad & \delta \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), \\ & d(\mathbf{A}x, \mathbf{T}y)\}^\gamma \end{aligned}$$

where $0 < \delta < \frac{1}{b^\gamma}$ *and* $0 < \gamma < \frac{1}{2}$.

Then, there is a sole element η *which verifies* $\mathbf{A}\eta = \mathbf{S}\eta = \eta = \mathbf{B}\eta = \mathbf{T}\eta$.

Proof. In the following, we will prove the existence and uniqueness of the common fixed point of the four mappings.

Existence: From the first condition, there are α, $\beta \in \mathbf{X}$ which verify $\mathbf{A}\alpha = \mathbf{S}\alpha$, $\mathbf{B}\beta = \mathbf{T}\beta$ and

$$\begin{aligned} d(\mathbf{A}\mathbf{A}\alpha, \mathbf{S}\alpha) &\leq d(\mathbf{S}\mathbf{A}\alpha, \mathbf{A}\alpha), \\ d(\mathbf{B}\mathbf{B}\beta, \mathbf{T}\beta) &\leq d(\mathbf{T}\mathbf{B}\beta, \mathbf{B}\beta). \end{aligned}$$

To prove the existence,we must go through three inevitable stages:

First stage: Assume that $\mathbf{A}\alpha \neq \mathbf{B}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad \leq \quad & \delta \max\{d(\mathbf{A}\alpha, \mathbf{B}\beta), d(\mathbf{A}\alpha, \mathbf{S}\alpha), d(\mathbf{B}\beta, \mathbf{T}\beta), d(\mathbf{S}\alpha, \mathbf{B}\beta), \\ & d(\mathbf{A}\alpha, \mathbf{T}\beta)\}^\gamma \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) &\leq \delta \max\{d(\mathbf{S}\alpha, \mathbf{T}\beta), 0\}^{\gamma} \\ &= \delta d^{\gamma}(\mathbf{S}\alpha, \mathbf{T}\beta) \\ &< d(\mathbf{S}\alpha, \mathbf{T}\beta) \end{aligned}$$

which is a contradiction, hence, $\mathbf{S}\alpha = \mathbf{T}\beta$.

Second stage: Suppose that $\mathbf{SS}\alpha \neq \mathbf{S}\alpha$, then, we have

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{T}\beta) \leq\ & \delta \max\{d(\mathbf{AS}\alpha, \mathbf{B}\beta), d(\mathbf{AS}\alpha, \mathbf{SS}\alpha), d(\mathbf{B}\beta, \mathbf{T}\beta), \\ & d(\mathbf{SS}\alpha, \mathbf{B}\beta), d(\mathbf{AS}\alpha, \mathbf{T}\beta)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \leq \delta \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{SS}\alpha), 0, d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\}^{\gamma}.$$

Using the fourth property of the space and the relationship between mappings $\mathbf{A}$ and $\mathbf{S}$, we obtain

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{S}\alpha) &\leq \delta \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), bd(\mathbf{AA}\alpha, \mathbf{S}\alpha)d(\mathbf{S}\alpha, \mathbf{SS}\alpha), 0, d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\}^{\gamma} \\ &\leq \delta \max\{d(\mathbf{SA}\alpha, \mathbf{A}\alpha), bd(\mathbf{SA}\alpha, \mathbf{A}\alpha)d(\mathbf{S}\alpha, \mathbf{SS}\alpha), 0, d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\}^{\gamma} \\ &= \delta \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), bd^{2}(\mathbf{SS}\alpha, \mathbf{S}\alpha), 0\}^{\gamma} \\ &= \delta.b^{\gamma}.d^{2\gamma}(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ &< d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \end{aligned}$$

which is a contradiction, hence, $\mathbf{SS}\alpha = \mathbf{S}\alpha$ and therefore $\mathbf{AS}\alpha = \mathbf{S}\alpha$.

Third stage: Now, assume that $\mathbf{TT}\beta \neq \mathbf{T}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{TT}\beta) \leq\ & \delta \max\{d(\mathbf{A}\alpha, \mathbf{BT}\beta), d(\mathbf{A}\alpha, \mathbf{S}\alpha), d(\mathbf{BT}\beta, \mathbf{TT}\beta), \\ & d(\mathbf{S}\alpha, \mathbf{BT}\beta), d(\mathbf{A}\alpha, \mathbf{TT}\beta)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$d(\mathbf{T}\beta, \mathbf{TT}\beta) \leq \delta \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), 0, d(\mathbf{BB}\beta, \mathbf{TT}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\}^{\gamma}.$$

Again, using the fourth property of the space and the relationship between mappings $\mathbf{B}$ and $\mathbf{T}$, we get

$$\begin{aligned} d(\mathbf{T}\beta, \mathbf{TT}\beta) &\leq \delta \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), 0, bd(\mathbf{BB}\beta, \mathbf{B}\beta)d(\mathbf{B}\beta, \mathbf{TT}\beta), \\ &\quad d(\mathbf{T}\beta, \mathbf{TT}\beta)\}^{\gamma} \\ &\leq \delta \max\{d(\mathbf{B}\beta, \mathbf{TB}\beta), 0, bd(\mathbf{TB}\beta, \mathbf{B}\beta)d(\mathbf{T}\beta, \mathbf{TT}\beta), \\ &\quad d(\mathbf{T}\beta, \mathbf{TT}\beta)\}^{\gamma} \\ &= \delta \max\{d(\mathbf{T}\beta, \mathbf{TT}\beta), 0, bd^{2}(\mathbf{TT}\beta, \mathbf{T}\beta)\}^{\gamma} \\ &= \delta.b^{\gamma}.d^{2.\gamma}(\mathbf{T}\beta, \mathbf{TT}\beta) \\ &< d(\mathbf{T}\beta, \mathbf{TT}\beta) \end{aligned}$$

which is a contradiction, thus, $\mathbf{T}\beta = \mathbf{TT}\beta$ and therefore $\mathbf{T}\beta = \mathbf{BT}\beta$. Consequently, $\eta = \mathbf{A}\alpha = \mathbf{S}\alpha = \mathbf{B}\beta = \mathbf{T}\beta$.

Uniqueness: At the end, suppose there is for instance $\mu \neq \eta$ such that $\mu = \mathbf{A}\mu = \mathbf{S}\mu = \mathbf{B}\mu = \mathbf{T}\mu$, then, we have

$$\begin{aligned} d(\mathbf{S}\eta, \mathbf{T}\mu) \leq\ & \delta \max\{d(\mathbf{A}\eta, \mathbf{B}\mu), d(\mathbf{A}\eta, \mathbf{S}\eta), d(\mathbf{B}\mu, \mathbf{T}\mu), d(\mathbf{S}\eta, \mathbf{B}\mu), \\ & d(\mathbf{A}\eta, \mathbf{T}\mu)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\eta,\mu) &\leq \delta \max\{d(\eta,\mu),0\}^{\gamma} \\ &= \delta d^{\gamma}(\eta,\mu) \\ &< d(\eta,\mu) \end{aligned}$$

this contradiction requires the equality between η and μ. □

The next example illustrates the above result.

Example 4.2. Taking $\mathbf{X} = \{0,3,5\}$. Let $d : \mathbf{X} \times \mathbf{X} \to [0,+\infty)$ such that

$$\begin{aligned} d(0,0) &= d(3,3) = d(5,5) = 0, \\ d(0,3) &= d(3,0) = 1, \\ d(0,5) &= d(5,0) = 4, \\ d(3,5) &= d(5,3) = 2. \end{aligned}$$

We have

$$\begin{aligned} 1 = d(0,3) &\leq 2[d(0,5)d(5,3)] = 16, \\ 4 = d(0,5) &\leq 2[d(0,3)d(3,5)] = 4, \\ 2 = d(3,5) &\leq 2[d(3,0)d(0,5)] = 8, \end{aligned}$$

thus, $(\mathbf{X}, d, b = 2)$ is a TMMS, and as $\mathbf{X}$ is a finite and discrete set, therefore, this space is complete.

Now, define four mappings as follows:

$$\mathbf{A}x = \begin{cases} 0 \text{ if } x = 0 \\ 5 \text{ if } x \in \{3,5\}, \end{cases} \quad \mathbf{B}x = \begin{cases} 0 \text{ if } x = 0 \\ x \text{ if } x \in \{3,5\}, \end{cases} \quad \mathbf{S}x = \begin{cases} 0 \text{ if } x = 0 \\ 3 \text{ if } x \in \{3,5\}, \end{cases}$$

$$\mathbf{T}x = 0 \text{ if } x \in \{0,3,5\}.$$

First of all, it is very clear to prove that the first hypothesis of the above theorem is satisfied.

Let the constructed table

♠	$x = y = 0$	$x, y \in \{3,5\}$	$x = 0$ and $y \in \{3,5\}$	$y = 0$, $x \in \{3,5\}$
$d(\mathbf{S}x, \mathbf{T}y)$	0	$1/1$	$0/0$	1
$d(\mathbf{A}x, \mathbf{B}y)$	0	$2/0$	$1/4$	4
$d(\mathbf{A}x, \mathbf{S}x)$	0	$2/2$	$0/0$	2
$d(\mathbf{B}y, \mathbf{T}y)$	0	$1/4$	$1/4$	0
$d(\mathbf{S}x, \mathbf{B}y)$	0	$0/2$	$1/4$	1
$d(\mathbf{A}x, \mathbf{T}y)$	0	$4/4$	$0/0$	4

Discussion: Choose $0 < \delta = \frac{3}{4} < \frac{1}{b^{\gamma}} = \frac{1}{2^{\frac{1}{3}}}$ and $0 < \gamma = \frac{1}{3} < \frac{1}{2}$, we get

First step: for $x = 0 = y$, $\mathbf{A}x = \mathbf{S}x = 0 = \mathbf{B}y = \mathbf{T}y$ and

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \delta \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}^{\gamma} \\ &= 0. \end{aligned}$$

Second step: If x and y belong to $\{3,5\}$, $\mathbf{A}x = 5$, $\mathbf{B}y = y$, $\mathbf{S}x = 3$, $\mathbf{T}y = 0$. For this step, we have two cases:

(i) **First case:** $By = 3$, then

$$\begin{aligned} 1 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{3}{4}\max\{2,2,1,0,4\}^{\frac{1}{3}} \\ &= \frac{3}{4}.4^{\frac{1}{3}} \\ &= \delta\max\{d(\mathbf{A}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{S}x), d(\mathbf{B}y,\mathbf{T}y), d(\mathbf{S}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{T}y)\}^{\gamma}. \end{aligned}$$

(ii) **Second case:** $By = 5$, then,

$$\begin{aligned} 1 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{3}{4}\max\{0,2,4,2,4\}^{\frac{1}{3}} \\ &= \frac{3}{4}.4^{\frac{1}{3}} \\ &= \delta\max\{d(\mathbf{A}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{S}x), d(\mathbf{B}y,\mathbf{T}y), d(\mathbf{S}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{T}y)\}^{\gamma}. \end{aligned}$$

Third step: If $x = 0$, y belongs to $\{3, 5\}$, $\mathbf{A}x = 0 = \mathbf{S}x$, $\mathbf{B}y = y$, $\mathbf{T}y = 0$. For this step, we have two cases:

(i) **First case:** $By = 1$,

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{3}{4}\max\{1,0,1,1,0\}^{\frac{1}{3}} \\ &= \frac{3}{4} \\ &= \delta\max\{d(\mathbf{A}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{S}x), d(\mathbf{B}y,\mathbf{T}y), d(\mathbf{S}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{T}y)\}^{\gamma}. \end{aligned}$$

(ii) **Second case:** $By = 2$,

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{3}{4}\max\{4,0,4,4,0\}^{\frac{1}{3}} \\ &= \frac{3}{4}.4^{\frac{1}{3}} \\ &= \delta\max\{d(\mathbf{A}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{S}x), d(\mathbf{B}y,\mathbf{T}y), d(\mathbf{S}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{T}y)\}^{\gamma}. \end{aligned}$$

Fourth step: If $y = 0$, x belongs to $\{3, 5\}$, $\mathbf{A}x = 5$, $\mathbf{S}x = 3$, $\mathbf{B}y = 0 = \mathbf{T}y$, and

$$\begin{aligned} 1 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{3}{4}\max\{4,2,0,1,4\}^{\frac{1}{3}} \\ &= \frac{3}{4}.4^{\frac{1}{3}} \\ &= \delta\max\{d(\mathbf{A}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{S}x), d(\mathbf{B}y,\mathbf{T}y), d(\mathbf{S}x,\mathbf{B}y), d(\mathbf{A}x,\mathbf{T}y)\}^{\gamma}, \end{aligned}$$

so, the second condition of the theorem is satisfied, therefore, the four mappings admit "zero" as the unique common fixed point.

Now, we give our second result.

Theorem 4.3. *Define four mappings* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{S}$ *and* $\mathbf{T}$ *from a complete TMMS* $(\mathbf{X}, d, b)$ *into itself such that*

(i) $(\mathbf{A}, \mathbf{S})$ *and* $(\mathbf{B}, \mathbf{T})$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$,

(ii) $\forall\, x,\, y \in \mathbf{X}$,

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) \quad \leq \quad & \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) \\ & + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma} \end{aligned}$$

where $0 < \varsigma < \frac{1}{2}$ *and* $0 < \sigma < \frac{1}{(3+b)^{\varsigma}}$.

Then, there is a sole element η *which satisfies* $\mathbf{A}\eta = \mathbf{S}\eta = \eta = \mathbf{B}\eta = \mathbf{T}\eta$.

Proof. In the following, we will prove the existence and uniqueness of the common fixed point of the four mappings.

Existence: From the first condition, there are $\alpha,\, \beta \in \mathbf{X}$ with $\mathbf{A}\alpha = \mathbf{S}\alpha$, $\mathbf{B}\beta = \mathbf{T}\beta$ and

$$\begin{aligned} d(\mathbf{AA}\alpha, \mathbf{S}\alpha) &\leq d(\mathbf{SA}\alpha, \mathbf{A}\alpha), \\ d(\mathbf{BB}\beta, \mathbf{T}\beta) &\leq d(\mathbf{TB}\beta, \mathbf{B}\beta). \end{aligned}$$

To prove the existence,we must go through three inevitable stages:

First stage: Assume that $\mathbf{A}\alpha \neq \mathbf{B}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad \leq \quad & \sigma[d(\mathbf{A}\alpha, \mathbf{B}\beta) + d(\mathbf{A}\alpha, \mathbf{S}\alpha) + d(\mathbf{B}\beta, \mathbf{T}\beta) + d(\mathbf{S}\alpha, \mathbf{B}\beta) \\ & + d(\mathbf{A}\alpha, \mathbf{T}\beta)]^{\varsigma} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad &\leq \quad \sigma.3^{\varsigma}.d^{\varsigma}(\mathbf{S}\alpha, \mathbf{T}\beta) \\ &< \quad d(\mathbf{S}\alpha, \mathbf{T}\beta) \end{aligned}$$

which is a contradiction, hence, $\mathbf{S}\alpha = \mathbf{T}\beta$.

Second stage: Suppose that $\mathbf{SS}\alpha \neq \mathbf{S}\alpha$, then, we have

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{T}\beta) \quad \leq \quad & \sigma[d(\mathbf{AS}\alpha, \mathbf{B}\beta) + d(\mathbf{AS}\alpha, \mathbf{SS}\alpha) + d(\mathbf{B}\beta, \mathbf{T}\beta) \\ & + d(\mathbf{SS}\alpha, \mathbf{B}\beta) + d(\mathbf{AS}\alpha, \mathbf{T}\beta)]^{\varsigma} \end{aligned}$$

which is equivalent to

$$d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \leq \sigma[d(\mathbf{AA}\alpha, \mathbf{S}\alpha) + d(\mathbf{AA}\alpha, \mathbf{SS}\alpha) + d(\mathbf{SS}\alpha, \mathbf{S}\alpha) + d(\mathbf{AA}\alpha, \mathbf{S}\alpha)]^{\varsigma}.$$

The use of the fourth axiom of the space gives

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \quad \leq \quad & \sigma[d(\mathbf{AA}\alpha, \mathbf{S}\alpha) + bd(\mathbf{AA}\alpha, \mathbf{S}\alpha)d(\mathbf{S}\alpha, \mathbf{SS}\alpha) + d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ & + d(\mathbf{AA}\alpha, \mathbf{S}\alpha)]^{\varsigma}. \end{aligned}$$

Now, the use of the relationship between $\mathbf{A}$ and $\mathbf{S}$ gives

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \quad \leq \quad & \sigma[d(\mathbf{SA}\alpha, \mathbf{A}\alpha) + bd(\mathbf{SA}\alpha, \mathbf{A}\alpha)d(\mathbf{S}\alpha, \mathbf{SS}\alpha) + d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ & + d(\mathbf{SA}\alpha, \mathbf{A}\alpha)]^{\varsigma} \\ = \quad & \sigma[d(\mathbf{SS}\alpha, \mathbf{S}\alpha) + bd(\mathbf{SS}\alpha, \mathbf{S}\alpha)d(\mathbf{S}\alpha, \mathbf{SS}\alpha) + d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ & + d(\mathbf{SS}\alpha, \mathbf{S}\alpha)]^{\varsigma} \\ = \quad & \sigma[3d(\mathbf{SS}\alpha, \mathbf{S}\alpha) + bd^2(\mathbf{SS}\alpha, \mathbf{S}\alpha)]^{\varsigma} \\ \leq \quad & \sigma[3d^2(\mathbf{SS}\alpha, \mathbf{S}\alpha) + bd^2(\mathbf{SS}\alpha, \mathbf{S}\alpha)]^{\varsigma} \\ = \quad & \sigma.[3 + b]^{\varsigma}.d^{2.\varsigma}(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ < \quad & d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \end{aligned}$$

which is a contradiction, hence, $\mathbf{SS}\alpha = \mathbf{S}\alpha$ and therefore $\mathbf{AS}\alpha = \mathbf{S}\alpha$.

Third stage: Now, assume that $\mathbf{TT}\beta \neq \mathbf{T}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{TT}\beta) &\leq \sigma[d(\mathbf{A}\alpha, \mathbf{BT}\beta) + d(\mathbf{A}\alpha, \mathbf{S}\alpha) + d(\mathbf{BT}\beta, \mathbf{TT}\beta) \\ &\quad + d(\mathbf{S}\alpha, \mathbf{BT}\beta) + d(\mathbf{A}\alpha, \mathbf{TT}\beta)]^{\varsigma} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{T}\beta, \mathbf{TT}\beta) &\leq \sigma[d(\mathbf{T}\beta, \mathbf{BB}\beta) + d(\mathbf{BB}\beta, \mathbf{TT}\beta) + d(\mathbf{T}\beta, \mathbf{BB}\beta) \\ &\quad + d(\mathbf{T}\beta, \mathbf{TT}\beta)]^{\varsigma}. \end{aligned}$$

Using the definition of the space, we get

$$\begin{aligned} d(\mathbf{T}\beta, \mathbf{TT}\beta) &\leq \sigma[d(\mathbf{T}\beta, \mathbf{BB}\beta) + bd(\mathbf{BB}\beta, \mathbf{B}\beta)d(\mathbf{B}\beta, \mathbf{TT}\beta) \\ &\quad + d(\mathbf{T}\beta, \mathbf{BB}\beta) + d(\mathbf{T}\beta, \mathbf{TT}\beta)]^{\varsigma}. \end{aligned}$$

Now, the relationship between $\mathbf{B}$ and $\mathbf{T}$ gives

$$\begin{aligned} d(\mathbf{T}\beta, \mathbf{TT}\beta) &\leq \sigma[d(\mathbf{B}\beta, \mathbf{TB}\beta) + bd(\mathbf{TB}\beta, \mathbf{B}\beta)d(\mathbf{T}\beta, \mathbf{TT}\beta) \\ &\quad + d(\mathbf{B}\beta, \mathbf{TB}\beta) + d(\mathbf{T}\beta, \mathbf{TT}\beta)]^{\varsigma} \\ &= \sigma[3d(\mathbf{T}\beta, \mathbf{TT}\beta) + bd^2(\mathbf{TT}\beta, \mathbf{T}\beta)]^{\varsigma} \\ &\leq \sigma[3d^2(\mathbf{T}\beta, \mathbf{TT}\beta) + bd^2(\mathbf{TT}\beta, \mathbf{T}\beta)]^{\varsigma} \\ &= \sigma.[3+b]^{\varsigma}.d^{2.\varsigma}(\mathbf{TT}\beta, \mathbf{T}\beta)] \\ &< d(\mathbf{T}\beta, \mathbf{TT}\beta) \end{aligned}$$

which is a contradiction, thus, $\mathbf{T}\beta = \mathbf{TT}\beta$ and therefore $\mathbf{T}\beta = \mathbf{BT}\beta$. Consequently, $\eta = \mathbf{A}\alpha = \mathbf{S}\alpha = \mathbf{B}\beta = \mathbf{T}\beta$.

Uniqueness: At the end, suppose there is for instance $\mu \neq \eta$ such that $\mathbf{A}\mu = \mathbf{S}\mu = \mu = \mathbf{B}\mu = \mathbf{T}\mu$, then, we get

$$\begin{aligned} d(\mathbf{S}\eta, \mathbf{T}\mu) &\leq \sigma[d(\mathbf{A}\eta, \mathbf{B}\mu) + d(\mathbf{A}\eta, \mathbf{S}\eta) + d(\mathbf{B}\mu, \mathbf{T}\mu) + d(\mathbf{S}\eta, \mathbf{B}\mu) \\ &\quad + d(\mathbf{A}\eta, \mathbf{T}\mu)]^{\varsigma} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\eta, \mu) &\leq \sigma.[3.d(\eta, \mu)]^{\varsigma} \\ &= \sigma.3^{\varsigma}.d^{\varsigma}(\eta, \mu) \\ &< d(\eta, \mu) \end{aligned}$$

this contradiction requires the equality between η and μ. □

To illustrate the above theorem, we present the following example.

Example 4.4. Let $\mathbf{X} = \{0, 1, 2\}$. Define $d : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ by

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(0,1) &= d(1,0) = 1, \\ d(0,2) &= d(2,0) = 2, \\ d(1,2) &= d(2,1) = \frac{3}{2}. \end{aligned}$$

We have

$$\begin{aligned} 1 = d(0,1) &\leq 2[d(0,2)d(2,1)] = 6, \\ 2 = d(0,2) &\leq 2[d(0,1)d(1,2)] = 3, \\ \frac{3}{2} = d(1,2) &\leq 2[d(1,0)d(0,2)] = 4, \end{aligned}$$

then, $(\mathbf{X}, d, b = 2)$ is a TMMS, and as $\mathbf{X}$ is a finite and discrete set, therefore, this space is complete.

Now, define four mappings as follows:

$$\mathbf{A}x = \begin{cases} 0 \text{ if } x = 0 \\ 2 \text{ if } x \in \{1,2\}, \end{cases} \quad \mathbf{B}x = \begin{cases} 0 \text{ if } x = 0 \\ x \text{ if } x \in \{1,2\}, \end{cases}$$

$$\mathbf{S}x = \mathbf{T}x = 0 \text{ if } x \in \{0,1,2\}.$$

First of all, it is very clear to prove that the first hypothesis of the above theorem is satisfied.

Let the constructed table

♠	$x = y = 0$	$x, y \in \{1,2\}$	$x = 0$ and $y \in \{1,2\}$	$y = 0$, $x \in \{1,2\}$
$d(\mathbf{S}x, \mathbf{T}y)$	0	$0/0$	$0/0$	0
$d(\mathbf{A}x, \mathbf{B}y)$	0	$\frac{3}{2}/0$	$1/2$	2
$d(\mathbf{A}x, \mathbf{S}x)$	0	$2/2$	$0/0$	2
$d(\mathbf{B}y, \mathbf{T}y)$	0	$1/2$	$1/2$	0
$d(\mathbf{S}x, \mathbf{B}y)$	0	$1/2$	$1/2$	0
$d(\mathbf{A}x, \mathbf{T}y)$	0	$2/2$	$0/0$	2

Discussion: Choose $0 < \varsigma = \frac{1}{3} < \frac{1}{2}$ and $0 < \sigma = \frac{1}{2} < \frac{1}{(3+b)^{\varsigma}} = \left(\frac{1}{5}\right)^{\frac{1}{3}}$, we get

First step: for $x = 0 = y$, $\mathbf{A}x = \mathbf{S}x = 0 = \mathbf{B}y = \mathbf{T}y$ and

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma} \\ &= 0. \end{aligned}$$

Second step: If x and y belong to $\{1,2\}$, $\mathbf{A}x = 2$, $\mathbf{B}y = y$, $\mathbf{S}x = 0$, $\mathbf{T}y = 0$. For this step, we have two cases:

(i) **First case:** $By = 1$, then

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{2}\left[\frac{3}{2} + 2 + 1 + 1 + 2\right]^{\frac{1}{3}} \\ &= \frac{1}{2}.\left(\frac{15}{2}\right)^{\frac{1}{3}} \\ &= \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma}. \end{aligned}$$

(ii) **Second case:** $By = 2$, then

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{2}\left[0 + 2 + 2 + 2 + 2\right]^{\frac{1}{3}} \\ &= \frac{1}{2}.8^{\frac{1}{3}} \\ &= \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma}. \end{aligned}$$

Third step: If $x = 0$, y belongs to $\{1,2\}$, $\mathbf{A}x = 0 = \mathbf{S}x$, $\mathbf{B}y = y$, $\mathbf{T}y = 0$. For this step, we have two cases:

(i) **First case:** $By = 1$, then,

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{2}\left[1+0+1+1+0\right]^{\frac{1}{3}} \\ &= \frac{1}{2}.3^{\frac{1}{3}} \\ &= \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma}. \end{aligned}$$

(ii) **Second case:** $By = 2$, then,

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{2}\left[2+0+2+2+0\right]^{\frac{1}{3}} \\ &= \frac{1}{2}.6^{\frac{1}{3}} \\ &= \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma}. \end{aligned}$$

Fourth step: If $y = 0$, x belongs to $\{1, 2\}$, $\mathbf{A}x = 2$, $\mathbf{S}x = 0$, $\mathbf{B}y = 0 = \mathbf{T}y$, and

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{2}\left[2+2+0+0+2\right]^{\frac{1}{3}} \\ &= \frac{1}{2}.6^{\frac{1}{3}} \\ &= \sigma[d(\mathbf{A}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{S}x) + d(\mathbf{B}y, \mathbf{T}y) + d(\mathbf{S}x, \mathbf{B}y) + d(\mathbf{A}x, \mathbf{T}y)]^{\varsigma}, \end{aligned}$$

so, the second condition of the theorem is satisfied, therefore, the four mappings admit "zero" as the unique common fixed point.

In the sequel, we propose our third and last result.

Theorem 4.5. *Define four mappings* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{S}$ *and* $\mathbf{T}$ *from a complete TMMS* $(\mathbf{X}, d, b)$ *into itself such that*

(i) couples $(\mathbf{A}, \mathbf{S})$ *and* $(\mathbf{B}, \mathbf{T})$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$*,*

(ii) $\forall$ x *and* y *in* $\mathbf{X}$*,*

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) \leq \max\{&d(\mathbf{A}x, \mathbf{B}y), \theta_1 d(\mathbf{A}x, \mathbf{S}x), \theta_2 d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), \\ &d(\mathbf{A}x, \mathbf{T}y)\}^{\gamma} \end{aligned}$$

where $0 < \theta_1, \theta_2 < \frac{1}{b}$ *and* $0 < \gamma < \frac{1}{2}$.

Then, there is only one point η *which satisfies* $\mathbf{A}\eta = \mathbf{S}\eta = \eta = \mathbf{B}\eta = \mathbf{T}\eta$.

Proof. In the following, we will prove the existence and uniqueness of the common fixed point of the four mappings.

Existence: As above, there exist α and $\beta \in \mathbf{X}$ with $\mathbf{A}\alpha = \mathbf{S}\alpha$, $\mathbf{B}\beta = \mathbf{T}\beta$ and

$$\begin{aligned} d(\mathbf{A}\mathbf{A}\alpha, \mathbf{S}\alpha) \leq d(\mathbf{S}\mathbf{A}\alpha, \mathbf{A}\alpha), \\ d(\mathbf{B}\mathbf{B}\beta, \mathbf{T}\beta) \leq d(\mathbf{T}\mathbf{B}\beta, \mathbf{B}\beta). \end{aligned}$$

To prove the existence,we must go through three inevitable stages:

First stage: Assume that $\mathbf{A}\alpha \neq \mathbf{B}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad \leq \quad & \max\{d(\mathbf{A}\alpha, \mathbf{B}\beta), \theta_1 d(\mathbf{A}\alpha, \mathbf{S}\alpha), \theta_2 d(\mathbf{B}\beta, \mathbf{T}\beta), d(\mathbf{S}\alpha, \mathbf{B}\beta), \\ & d(\mathbf{A}\alpha, \mathbf{T}\beta)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad &\leq \quad \max\{d(\mathbf{S}\alpha, \mathbf{T}\beta), 0\}^{\gamma} \\ &= \quad d^{\gamma}(\mathbf{S}\alpha, \mathbf{T}\beta) \\ &< \quad d(\mathbf{S}\alpha, \mathbf{T}\beta) \end{aligned}$$

which is a contradiction, hence, $\mathbf{S}\alpha = \mathbf{T}\beta$.

Second stage: Suppose that $\mathbf{SS}\alpha \neq \mathbf{S}\alpha$, then, we have

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{T}\beta) \quad \leq \quad & \max\{d(\mathbf{AS}\alpha, \mathbf{B}\beta), \theta_1 d(\mathbf{AS}\alpha, \mathbf{SS}\alpha), \theta_2 d(\mathbf{B}\beta, \mathbf{T}\beta), \\ & d(\mathbf{SS}\alpha, \mathbf{B}\beta), d(\mathbf{AS}\alpha, \mathbf{T}\beta)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \leq \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), \theta_1 d(\mathbf{AA}\alpha, \mathbf{SS}\alpha), 0, d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\}^{\gamma}.$$

Using the fourth property of the space and the relationship between mappings $\mathbf{A}$ and $\mathbf{S}$, we obtain

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \quad &\leq \quad \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), \theta_1 b d(\mathbf{AA}\alpha, \mathbf{S}\alpha) d(\mathbf{S}\alpha, \mathbf{SS}\alpha), 0, d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\}^{\gamma} \\ &\leq \quad \max\{d(\mathbf{SA}\alpha, \mathbf{A}\alpha), \theta_1 b d(\mathbf{SA}\alpha, \mathbf{A}\alpha) d(\mathbf{S}\alpha, \mathbf{SS}\alpha), 0, d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\}^{\gamma} \\ &= \quad \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), \theta_1 b d^2(\mathbf{SS}\alpha, \mathbf{S}\alpha), 0\}^{\gamma} \\ &= \quad (\theta_1 . b)^{\gamma} . d^{2\gamma}(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ &< \quad d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \end{aligned}$$

which is a contradiction, hence, $\mathbf{SS}\alpha = \mathbf{S}\alpha$ and therefore $\mathbf{AS}\alpha = \mathbf{S}\alpha$.

Third stage: Now, assume that $\mathbf{TT}\beta \neq \mathbf{T}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{TT}\beta) \quad \leq \quad & \max\{d(\mathbf{A}\alpha, \mathbf{BT}\beta), \theta_1 d(\mathbf{A}\alpha, \mathbf{S}\alpha), \theta_2 d(\mathbf{BT}\beta, \mathbf{TT}\beta), \\ & d(\mathbf{S}\alpha, \mathbf{BT}\beta), d(\mathbf{A}\alpha, \mathbf{TT}\beta)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$d(\mathbf{T}\beta, \mathbf{TT}\beta) \leq \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), 0, \theta_2 d(\mathbf{BB}\beta, \mathbf{TT}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\}^{\gamma}.$$

Again, using the fourth property of the space and the relationship between mappings $\mathbf{B}$ and $\mathbf{T}$, we get

$$\begin{aligned} d(\mathbf{T}\beta, \mathbf{TT}\beta) \quad &\leq \quad \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), 0, \theta_2 b d(\mathbf{BB}\beta, \mathbf{B}\beta) d(\mathbf{B}\beta, \mathbf{TT}\beta), \\ & \qquad d(\mathbf{T}\beta, \mathbf{TT}\beta)\}^{\gamma} \\ &\leq \quad \max\{d(\mathbf{B}\beta, \mathbf{TB}\beta), 0, \theta_2 b d(\mathbf{TB}\beta, \mathbf{B}\beta) d(\mathbf{T}\beta, \mathbf{TT}\beta), \\ & \qquad d(\mathbf{T}\beta, \mathbf{TT}\beta)\}^{\gamma} \\ &= \quad \max\{d(\mathbf{T}\beta, \mathbf{TT}\beta), 0, \theta_2 b d^2(\mathbf{TT}\beta, \mathbf{T}\beta)\}^{\gamma} \\ &= \quad (\theta_2 . b)^{\gamma} . d^{2.\gamma}(\mathbf{T}\beta, \mathbf{TT}\beta) \\ &< \quad d(\mathbf{T}\beta, \mathbf{TT}\beta) \end{aligned}$$

which is a contradiction, thus, $\mathbf{T}\beta = \mathbf{TT}\beta$ and therefore $\mathbf{T}\beta = \mathbf{BT}\beta$. Consequently, $\eta = \mathbf{A}\alpha = \mathbf{S}\alpha = \mathbf{B}\beta = \mathbf{T}\beta$.

Uniqueness: Lastly, suppose there is $\mu \neq \eta$ such that $\mathbf{A}\mu = \mathbf{S}\mu = \mu = \mathbf{B}\mu = \mathbf{T}\mu$, then, we have

$$\begin{aligned} d(\mathbf{S}\eta, \mathbf{T}\mu) &\leq \max\{d(\mathbf{A}\eta, \mathbf{B}\mu), \theta_1 d(\mathbf{A}\eta, \mathbf{S}\eta), \theta_2 d(\mathbf{B}\mu, \mathbf{T}\mu), d(\mathbf{S}\eta, \mathbf{B}\mu), \\ &\quad d(\mathbf{A}\eta, \mathbf{T}\mu)\}^{\gamma} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\eta, \mu) &\leq \max\{d(\eta, \mu), 0\}^{\gamma} \\ &= d^{\gamma}(\eta, \mu) \\ &< d(\eta, \mu) \end{aligned}$$

this contradiction requires the equality between η and μ. □

The next example illustrates the above theorem.

Example 4.6. Let $(\mathbf{X} = [0, +\infty), d(x, y) = e^{|x-y|})$ be a complete TMMS. Consider the next four mappings:

$$\mathbf{S}x = \begin{cases} \frac{x+19}{20} \text{ when } 0 \leq x < 1, \\ 1 \text{ when } 1 \leq x < +\infty, \end{cases} \quad \mathbf{T}x = \begin{cases} \frac{x+20}{21} \text{ when } 0 \leq x < 1, \\ 1 \text{ when } 1 \leq x < +\infty, \end{cases}$$

$$\mathbf{A}x = \begin{cases} 200 \text{ when } 0 \leq x < 1, \\ \frac{1}{x} \text{ when } 1 \leq x < +\infty, \end{cases} \quad \mathbf{B}x = \begin{cases} 2000 \text{ when } 0 \leq x < 1, \\ \frac{1}{x} \text{ when } 1 \leq x < +\infty. \end{cases}$$

First of all, it is very clear to prove that the first hypothesis of the above theorem is satisfied.

Let the constructed table

♠	$x, y \in [0, 1)$	$x, y \in [1, +\infty)$	$x \in [0, 1), y \in [1, +\infty)$	$y \in [0, 1), x \in [1, +\infty)$
$d(\mathbf{S}x, \mathbf{T}y)$	$e^{\left\lvert \frac{x+19}{20} - \frac{y+20}{21} \right\rvert}$	$e^0 = 1$	$e^{\left\lvert 1 - \frac{x+19}{20} \right\rvert}$	$e^{\left\lvert 1 - \frac{y+20}{21} \right\rvert}$
$d(\mathbf{A}x, \mathbf{B}y)$	e^{1800}	$e^{\left\lvert \frac{1}{x} - \frac{1}{y} \right\rvert}$	$e^{\left\lvert 200 - \frac{1}{y} \right\rvert}$	$e^{\left\lvert 2000 - \frac{1}{x} \right\rvert}$
$d(\mathbf{A}x, \mathbf{S}x)$	$e^{\left\lvert 200 - \frac{x+19}{20} \right\rvert}$	$e^{\left\lvert 1 - \frac{1}{x} \right\rvert}$	$e^{\left\lvert 200 - \frac{x+19}{20} \right\rvert}$	$e^{\left\lvert 1 - \frac{1}{x} \right\rvert}$
$d(\mathbf{B}y, \mathbf{T}y)$	$e^{\left\lvert 2000 - \frac{y+20}{21} \right\rvert}$	$e^{\left\lvert 1 - \frac{1}{y} \right\rvert}$	$e^{\left\lvert 1 - \frac{1}{y} \right\rvert}$	$e^{\left\lvert 2000 - \frac{y+20}{21} \right\rvert}$
$d(\mathbf{S}x, \mathbf{B}y)$	$e^{\left\lvert 2000 - \frac{x+19}{20} \right\rvert}$	$e^{\left\lvert 1 - \frac{1}{y} \right\rvert}$	$e^{\left\lvert \frac{x+19}{20} - \frac{1}{y} \right\rvert}$	e^{1999}
$d(\mathbf{A}x, \mathbf{T}y)$	$e^{\left\lvert 200 - \frac{y+20}{21} \right\rvert}$	$e^{\left\lvert 1 - \frac{1}{x} \right\rvert}$	e^{199}	$e^{\left\lvert \frac{1}{x} - \frac{y+20}{21} \right\rvert}$

Discussion: Choose $b = 2 \geq 1$, $0 < \theta_1 = \frac{1}{4}, \theta_2 = \frac{1}{3} < \frac{1}{b} = \frac{1}{2}$ and $0 < \gamma = \frac{1}{3} < \frac{1}{2}$, we get

(1) If $0 \leq x, y < 1$, we have $\mathbf{S}x = \frac{x+19}{20}$, $\mathbf{T}y = \frac{y+20}{21}$, $\mathbf{A}x = 200$, $\mathbf{B}y = 2000$ and

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) &= e^{\left\lvert \frac{x+19}{20} - \frac{y+20}{21} \right\rvert} \\ &\leq \max\left\{e^{1800}, \frac{1}{4}e^{\left\lvert 200 - \frac{x+19}{20} \right\rvert}, \frac{1}{3}e^{\left\lvert 2000 - \frac{y+20}{21} \right\rvert}, e^{\left\lvert 2000 - \frac{x+19}{20} \right\rvert}, e^{\left\lvert 200 - \frac{y+20}{21} \right\rvert}\right\}^{\frac{1}{3}} \\ &= \max\{d(\mathbf{A}x, \mathbf{B}y), \theta_1 d(\mathbf{A}x, \mathbf{S}x), \theta_2 d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}^{\gamma}. \end{aligned}$$

(2) When $1 \leq x, y < +\infty$, $\mathbf{S}x = 1$, $\mathbf{T}y = 1$, $\mathbf{A}x = \frac{1}{x}$, $\mathbf{B}y = \frac{1}{y}$, then,

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) &= 1 \\ &\leq \max\left\{e^{\left\lvert \frac{1}{x} - \frac{1}{y} \right\rvert}, \frac{1}{4}e^{\left\lvert 1 - \frac{1}{x} \right\rvert}, \frac{1}{3}e^{\left\lvert 1 - \frac{1}{y} \right\rvert}, e^{\left\lvert 1 - \frac{1}{y} \right\rvert}, e^{\left\lvert 1 - \frac{1}{x} \right\rvert}\right\}^{\frac{1}{3}} \\ &= \max\{d(\mathbf{A}x, \mathbf{B}y), \theta_1 d(\mathbf{A}x, \mathbf{S}x), \theta_2 d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}^{\gamma}. \end{aligned}$$

(3) Now, if $0 \leq x < 1$, $1 \leq y < +\infty$, we have $\mathbf{S}x = \frac{x+19}{20}$, $\mathbf{T}y = 1$, $\mathbf{A}x = 200$, $\mathbf{B}y = \frac{1}{y}$ and

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) &= e^{\left|1-\frac{x+19}{20}\right|} \\ &\leq \max\left\{e^{\left|200-\frac{1}{y}\right|}, \frac{1}{4}e^{\left|200-\frac{x+19}{20}\right|}, \frac{1}{3}e^{\left|1-\frac{1}{y}\right|}, e^{\left|\frac{x+19}{20}-\frac{1}{y}\right|}, e^{199}\right\}^{\frac{1}{3}} \\ &= \max\{d(\mathbf{A}x, \mathbf{B}y), \theta_1 d(\mathbf{A}x, \mathbf{S}x), \theta_2 d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}^{\gamma}. \end{aligned}$$

(4) At the end, when $1 \leq x < +\infty$ and $0 \leq y < 1$, $\mathbf{S}x = 1$, $\mathbf{T}y = \frac{y+20}{21}$, $\mathbf{A}x = \frac{1}{x}$, $\mathbf{B}y = 2000$ and

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) &= e^{\left|1-\frac{y+20}{21}\right|} \\ &\leq \max\left\{e^{\left|2000-\frac{1}{x}\right|}, \frac{1}{4}e^{\left|1-\frac{1}{x}\right|}, \frac{1}{3}e^{\left|2000-\frac{y+20}{21}\right|}, e^{1999}, e^{\left|\frac{1}{x}-\frac{y+20}{21}\right|}\right\}^{\frac{1}{3}} \\ &= \max\{d(\mathbf{A}x, \mathbf{B}y), \theta_1 d(\mathbf{A}x, \mathbf{S}x), \theta_2 d(\mathbf{B}y, \mathbf{T}y), d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}^{\gamma}, \end{aligned}$$

we conclude that the two conditions of our above theorem are satisfied and 1 is the unique element which verifies that $\mathbf{S}(1) = \mathbf{T}(1) = \mathbf{A}(1) = \mathbf{B}(1) = 1$.

We finish this contribution by giving an application to an integral equation.

5 Application of our Third Result to Integral Equation

Taking the following equation:

$$\mathbf{E}(x) = \mathbf{P}(x) + \int_0^x \mathbf{Q}(x, y).\mathbf{R}(y, \mathbf{E}(y))dy \tag{5.1}$$

for $x \in [0, \tau]$, where $\tau > 0$, $\mathbf{P} : [0, \tau] \to \mathbb{R}$, $\mathbf{Q} : [0, \tau] \times [0, \tau] \to \mathbb{R}$, $\mathbf{R} : [0, \tau] \times \mathbb{R} \to \mathbb{R}$ are continuous mappings. Endow $\mathbf{X} = C([0, \tau], \mathbb{R})$ with the following triangular multiplicative metric:

$$d(\mathbf{E}, \mathbf{E}') = e^{\sup\limits_{0 \leq x \leq \tau} |\mathbf{E}(x) - \mathbf{E}'(x)|},$$

for all $\mathbf{E}$, $\mathbf{E}'$ belong to $\mathbf{X}$. Let us define self-operator $\mathbf{S}$ on $\mathbf{X}$ as follows:

$$\mathbf{S}\mathbf{E}(x) = \mathbf{P}(x) + \int_0^x \mathbf{Q}(x, y).\mathbf{R}(y, \mathbf{E}(y))dy.$$

Theorem 5.1. *Assume that the two conditions are true:*

(i) $\sup\limits_{0 \leq x \leq \tau} \int_0^x |\mathbf{Q}(x, y)|dy \leq 1$,

(ii) $|\mathbf{R}(y, \mathbf{E}(y)) - \mathbf{R}(y, \mathbf{E}'(y))| \leq \gamma |\mathbf{E} - \mathbf{E}|$, *for each* $y \in [0, \tau]$, $\mathbf{E}$, $\mathbf{E}' \in \mathbf{X}$, *where* $\gamma \in \left(0, \frac{1}{2}\right)$,

then, integral equation (5.1) *admits only one solution in* $\mathbf{X}$.

Proof. For all $\mathbf{E}$, $\mathbf{E}'$ belong to $\mathbf{X}$, we have

$$\begin{aligned}
d(\mathbf{SE},\mathbf{SE}') &= e^{\sup\limits_{0\le x\le\tau}|\mathbf{SE}(x)-\mathbf{SE}'(x)|}\\
&= e^{\sup\limits_{0\le x\le\tau}\left|\int_0^x \mathbf{Q}(x,y).[\mathbf{R}(y,\mathbf{E}(y))-\mathbf{R}(y,\mathbf{E}'(y))]dy\right|}\\
&\le e^{\sup\limits_{0\le x\le\tau}\int_0^x|\mathbf{Q}(x,y)|.|\mathbf{R}(y,\mathbf{E}(y))-\mathbf{R}(y,\mathbf{E}'(y))|dy}\\
&\le e^{\gamma.\int_0^x|\mathbf{Q}(x,y)|.\sup\limits_{0\le x\le\tau}|\mathbf{E}(y)-\mathbf{E}'(y)|dy}\\
&\le \left[e^{\sup\limits_{0\le x\le\tau}|\mathbf{E}(y)-\mathbf{E}'(y)|}\right]^{\gamma}\\
&= d^{\gamma}(\mathbf{E},\mathbf{E}')\\
&\le \max\{d(\mathbf{E},\mathbf{E}'),\theta_1 d(\mathbf{E},\mathbf{SE}),\theta_2 d(\mathbf{E}',\mathbf{SE}'),d(\mathbf{SE},\mathbf{E}'),\\
&\quad d(\mathbf{E},\mathbf{SE}')\}^{\gamma},
\end{aligned}$$

so, all the conditions of Theorem 4.5 (with $\mathbf{S}=\mathbf{T}$ and $\mathbf{A}=\mathbf{B}=\mathbf{I}$: the identity mapping on $\mathbf{X}$) are satisfied, consequently, integral equation (5.1) possesses only one solution in $\mathbf{X}$. □

6 Conclusion

In this contribution, we proved the existence and uniqueness of common fixed points for two pairs of occasionally weakly biased mappings of type $(\mathbb{A})$ in complete TMMSs. Our theorems improve some existing results for instance, Theorem 2.1 and Corollary 2.2 of [4].

References

[1] M.A. Al-Thagafi and N. Shahzad, *Generalized I-nonexpansive selfmaps and invariant approximations*, Acta Math. Sin. (Engl. Ser.), **24**(5), 867–876, (2008).

[2] J. Balamurugan and C. Rajesh, *Unique fixed point results for contraction maps in b-metric spaces*, Palest. J. Math., **14**(Special Issue II), 420–426, (2025).

[3] A.E. Bashirov, E.M. Kurpınar, and A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337**(1), 36–48, (2008).

[4] M. Berzig, *Fixed point theorems in new multiplicative metric spaces*, Lett. Nonlinear Anal. Appl., **2**(2), 91–96, (2024).

[5] H. Bouhadjera, *Fixed points for occasionally weakly biased mappings of type $(\mathcal{A})$*, Math. Morav., **26**(1), 113–122, (2022).

[6] R. Bouhafs, W. Shatanawi and A. Tallafha, *Some fixed point results for nonlinear contractive type mapping on b-metric spaces with altering distance functions*, Palest. J. Math., **11**(2), 582–591, (2022).

[7] S.S. Chauhan(Gonder), R. Sharma, M. Asim and M. Imdad, *A New approach of fixed point in parametric b-metric space*, Palest. J. Math., **13**(4), 441–408, (2024).

[8] K. Fallahi and A. Aghanians, *Boyd and Wong's fixed point theorem in metric spaces endowed with a graph*, Palest. J. Math., **6**(2), 496–501, (2017).

[9] G. Jungck and B.E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29**(3), 227–238, (1998).

[10] M.V.R. Kameswari, *Fixed point theorems on generalized $\alpha-\eta$ extended Z-contractions in extended b-metric spaces*, Palest. J. Math., **12**(1), 848–858, (2023).

[11] C. Rajesh, R.D. Mohanraj and S. Thalapathiraj, *Few results on fixed point theorems in multiplicative 2-metric space*, Palest. J. Math., **14**(Special Issue II), 275–279, (2025).

[12] A.N. Sakelo, K.K. Tola, M.A. Mamud and M.T. A/mecha, *Common fixed point results for generalized $L*$-contraction in b-rectangular metric spaces*, Palest. J. Math., **13**(4), 848–858, (2024).

[13] G.S. Saluja, *Some xed point results for generalized contractions in cone b-metric spaces*, Palest. J. Math., **6**(1), 76–83, (2017).

[14] K.P.R. Sastry, M.V.R. Kameswari and D.M.K. Kiran, *Common fixed point theorems for generalized TAC contraction condition in b-metric spaces*, Palest. J. Math., **9**(1), 354–370, (2020).

[15] R. Shah, *Some new fixed point results in b-metric-like spaces*, Palest. J. Math., **11**(1), 378–384, (2022).

[16] Koti N.V.V. Vara Prasad, *A fixed point theorem on weak contraction condition (B) in complete metric spaces with w- distance*, Palest. J. Math., **10**(2), 609–613, (2021).

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