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TRANSVERSALITY CONDITION FOR SOME SINGULAR SURFACES

Salim Usmanov and Ulugbek Navruzov

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Corresponding Author: Salim Usmanov

Abstract. This paper is devoted to the investigation of a class of parameterized singular surfaces in three-dimensional Euclidean space. Under appropriate assumptions, we establish that the transversality condition is satisfied at every regular point of these surfaces.

1 Introduction

The boundedness of maximal operators associated with hypersurfaces in Euclidean spaces is closely related to the geometric properties of the corresponding hypersurfaces. The geometric characteristics of hypersurfaces play a fundamental role in the analysis of these maximal operators (see, for example, [1, 2, 3, 4, 5, 6, 7]).

In particular, the curvature properties of hypersurfaces play an essential role in the analysis of the boundedness of maximal operators associated with hypersurfaces. In [4], Sogge established that the maximal operator is bounded on the Lebesgue space $L^p(\mathbb{R}^{n+1})$ for $p > 2$, under the condition that the hypersurface has at least one nonvanishing principal curvature.

The geometric properties of surfaces have been investigated in many papers. In the papers [8] and [9], some important results for ruled surfaces in $\mathbb{R}^3$ related to the curvatures of these surfaces were obtained. The articles [7] and [10] were devoted to the investigation of certain singular surfaces in three-dimensional Euclidean space defined by the parametric equations. It was shown that at each regular point of these surfaces, at least one of the principal curvatures is nonvanishing. Also, using this property of the singular hypersurfaces in $\mathbb{R}^{n+1}$, the boundedness of maximal operators associated with these hypersurfaces was established [7].

In the paper [11], it has been studied properties of the curvatures of rational surfaces. The authors showed that the Gaussian and the mean curvatures of these surfaces can be computed using a quaternion function.

Furthermore, the boundedness of maximal operators associated with certain hypersurfaces depends on a transversality condition. The papers [2] and [3] are devoted to the investigation of a class of smooth hypersurfaces satisfying this condition.

The transversality condition for singular surfaces is also important in the investigation of the boundedness problem for maximal operators associated with such surfaces. The boundedness problem of maximal operators associated with singular surfaces has been studied in several works, see, for example, [6, 7, 10, 12, 13, 14].

Next, we define a fractional power series by the following definition [15].

Definition 1.1. Let $V \subseteq \mathbb{R}^n_+$ be an open connected set such that $0 \in \bar{V}$, f is called a fractional power series in the set V if there is an open set $W \subseteq \mathbb{R}^n$, containing $\bar{V}$, a natural number N and a real analytic function g in $\Phi_N^{-1}(W)$ such that the identity $f = g \circ \Phi_{1/N}$ holds in the set V, where $\Phi_N : \mathbb{R}^n \to \mathbb{R}^n$ is a map, given by the formula $\Phi_N(x) = (x_1^N, x_2^N, ..., x_n^N)$.

Consider singular surfaces in $\mathbb{R}^3$ defined by the following parametric equations

$$x_1(u_1,u_2)=r_1+u_1^{a_1}u_2^{a_2}g_1(u_1,u_2),\ x_2(u_1,u_2)=r_2+u_1^{b_1}u_2^{b_2}g_2(u_1,u_2),$$

$$x_3(u_1,u_2)=r_3+u_1^{c_1}u_2^{c_2}g_3(u_1,u_2), \tag{1.1}$$

where r_1,r_2,r_3 are real numbers and the exponents a_1,a_2,b_1,b_2,c_1,c_2 are non-negative integers, $u_1\geq 0, u_2\geq 0$, $g_k(u_1,u_2)$ are fractional power series, k=1,2,3.

We introduce the following denotations:

$$B_1=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},\ B_2=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},\ B_3=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Remark 1.2. If at least one of the numbers B_1, B_2, and B_3 is nonzero, then points of the surface (1.1) that lie in a sufficiently small neighborhood of the point (r_1,r_2,r_3), outside the coordinate planes of the coordinate system that has its origin at the point (r_1,r_2,r_3), are regular (non-singular). Points of the surface (1.1) that lie on the coordinate planes of this coordinate system may be singular (see Remark 1, [6]).

In this paper, we will investigate the following condition of transversality for the surfaces (1.1): for any point an affine tangent plane $x+T_xS$ to S at the point x does not pass through the origin in $\mathbb{R}^{n+1}$ [2]. We also verify the transversality condition for certain surfaces derived from the surfaces (1.1).

2 Preliminaries

Following the papers [3], [15], and [16], we introduce the notion of the Newton polyhedron.

Let $\mathbb{N}_0\subset\mathbb{R}_+\subset\mathbb{R}$ be the sets of all non-negative integers, all non-negative real numbers, all real numbers, respectively. Suppose that $K\subset\mathbb{N}_0^n$ is a set. Newton polyhedron of the set K is defined as a convex hull in $\mathbb{R}_+^n$ of the set $\cup_{k\in K}(k+\mathbb{R}_+^n)$. The union of all compact faces of the Newton polyhedron of the set K is called the Newton diagram.

Let ψ be a smooth function defined at the origin. Consider the Maclaurin series of this function

$$\psi_x\approx\sum_{k\in\mathbb{N}_0^n}c_kx^k,\ c_k\in\mathbb{R},\ x^k:=x_1^{k_1}\cdot\ldots\cdot x_n^{k_n}.$$

The support of this series is defined by $supp(\psi_x)=\{k\in\mathbb{N}_0^n\setminus\{0\}:c_k\neq 0\}$. The Newton polyhedron of the series ψ is defined as the Newton polyhedron of the set $supp(\psi_x)$. And the Newton diagram of a smooth function ψ is defined as the Newton diagram of the set $supp(\psi_x)$.

Next, we present some essential facts from geometry, which will be used in the proof of the main result.

Let G_1 be a region in $\mathbb{R}^2$, and let the parametric equations of a certain regular surface in $\mathbb{R}^3$ be given by

$$x_1=x_1(u_1,u_2),\ x_2=x_2(u_1,u_2),\ x_3=x_3(u_1,u_2), \tag{2.1}$$

where $x_1(u_1,u_2), x_2(u_1,u_2), x_3(u_1,u_2)$ are differentiable functions in G_1.

Suppose that the origin of the coordinate system in $\mathbb{R}^3$ lies on the tangent plane to the surface (2.1) at its point (x_1,x_2,x_3). Then, following [17] (p. 73), the equation of this tangent plane at the point (u_1,u_2) can be expressed in the form:

$$D_1(u_1,u_2)=\begin{vmatrix} x_1(u_1,u_2) & x_2(u_1,u_2) & x_3(u_1,u_2) \\ \frac{\partial x_1(u_1,u_2)}{\partial u_1} & \frac{\partial x_2(u_1,u_2)}{\partial u_1} & \frac{\partial x_3(u_1,u_2)}{\partial u_1} \\ \frac{\partial x_1(u_1,u_2)}{\partial u_2} & \frac{\partial x_2(u_1,u_2)}{\partial u_2} & \frac{\partial x_3(u_1,u_2)}{\partial u_2} \end{vmatrix}=0. \tag{2.2}$$

We denote the second-order minors of this determinant as follows:

$$M_{12}(u_1,u_2)=\frac{\partial x_1(u_1,u_2)}{\partial u_1}\cdot\frac{\partial x_2(u_1,u_2)}{\partial u_2}-\frac{\partial x_1(u_1,u_2)}{\partial u_2}\cdot\frac{\partial x_2(u_1,u_2)}{\partial u_1},$$

$$M_{13}(u_1,u_2)=\frac{\partial x_1(u_1,u_2)}{\partial u_1}\cdot\frac{\partial x_3(u_1,u_2)}{\partial u_2}-\frac{\partial x_1(u_1,u_2)}{\partial u_2}\cdot\frac{\partial x_3(u_1,u_2)}{\partial u_1},$$

$$M_{23}(u_1,u_2)=\frac{\partial x_2(u_1,u_2)}{\partial u_1}\cdot\frac{\partial x_3(u_1,u_2)}{\partial u_2}-\frac{\partial x_2(u_1,u_2)}{\partial u_2}\cdot\frac{\partial x_3(u_1,u_2)}{\partial u_1}.$$

We perform the change of variables

$$u_1=u_1(z_1,z_2),\ \ u_2=u_2(z_1,z_2) \tag{2.3}$$

in (2.1), where $u_1(z_1,z_2), u_2(z_1,z_2)$ are differentiable functions in some region $G_2\subset\mathbb{R}^2$.

Let $D_2(z_1,z_2)$ denote the value of $D_1(u_1,u_2)$ under the change of variables (2.3), and let J be the Jacobian matrix corresponding to the transformation (2.3). The subsequent lemma is then valid.

Lemma 2.1. *If $detJ\neq 0$, then under the change of variables (2.3), the following relation holds:*

$$D_2(z_1,z_2)=detJ\cdot D_1(u_1,u_2).$$

Proof. Indeed, by computing the second-order minors of $D_2(z_1,z_2)$, we have

$$N_{12}(z_1,z_2)=detJ\cdot M_{12}(u_1,u_2), N_{13}(z_1,z_2)=detJ\cdot M_{13}(u_1,u_2),$$

$$N_{23}(z_1,z_2)=detJ\cdot M_{23}(u_1,u_2).$$

From these relations, the desired equality follows. This completes the proof of Lemma 2.1.

3 On the transversality condition for the surfaces (1.1)

In this section, by applying the resolution of singularities [16], we show that, under suitable conditions on the surfaces (1.1), the transversality condition is satisfied.

A definition of a tangent cone to an affine variety at its any point is given in the paper [18] (chapter II, §1). Throughout this section we assume that the vector (r_1,r_2,r_3) is transversal to the tangent cone to the surface (1.1) at the origin.

The main result of this paper is contained in the following theorem.

Theorem 3.1. *Let $\{g_k(u_1,u_2)\}_{k=1}^3$ be fractional power series defined in a sufficiently small neighborhood U of the origin in $\mathbb{R}^2_+$. Suppose that $g_k(0,0)\neq 0, B_k\neq 0$ for each k, and at least one of the numbers r_1,r_2, and r_3 is nonzero. Then the surface (1.1) satisfies the transversality condition at all of its regular points.*

Proof. Suppose that the origin of the coordinate system in $\mathbb{R}^3$ lies on the tangent plane to the surface (1.1) at the point (x_1,x_2,x_3). Then, by (2.2), the equation of the tangent plane to this surface at the point (u_1,u_2) can be represented in the following form:

$$\begin{aligned}&x_1(u_1,u_2)\Big(B_2g_2(u_1,u_2)g_3(u_1,u_2)+\omega_1(u_1,u_2)\Big)u_1^{b_1+c_1-1}u_2^{b_2+c_2-1}-\\&-x_2(u_1,u_2)\Big(B_3g_1(u_1,u_2)g_3(u_1,u_2)+\omega_2(u_1,u_2)\Big)u_1^{a_1+c_1-1}u_2^{a_2+c_2-1}+\\&+x_3(u_1,u_2)\Big(B_1g_1(u_1,u_2)g_2(u_1,u_2)+\omega_3(u_1,u_2)\Big)u_1^{a_1+b_1-1}u_2^{a_2+b_2-1}=0,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\omega_1(u_1,u_2)&=b_1u_1u_2^2g_2(u_1,u_2)\frac{\partial g_3(u_1,u_2)}{\partial u_2}+c_2u_1^2u_2g_3(u_1,u_2)\frac{\partial g_2(u_1,u_2)}{\partial u_1}+\\&+u_1^2u_2^2\frac{\partial g_2(u_1,u_2)}{\partial u_1}\frac{\partial g_3(u_1,u_2)}{\partial u_2}-b_2u_1^2u_2g_2(u_1,u_2)\frac{\partial g_3(u_1,u_2)}{\partial u_1}-\\&-c_1u_1u_2^2g_3(u_1,u_2)\frac{\partial g_2(u_1,u_2)}{\partial u_2}-u_1^2u_2^2\frac{\partial g_2(u_1,u_2)}{\partial u_2}\frac{\partial g_3(u_1,u_2)}{\partial u_1},\end{aligned}$$

$$\omega_2(u_1,u_2)=a_2u_1^2u_2g_1(u_1,u_2)\frac{\partial g_3(u_1,u_2)}{\partial u_1}+c_1u_1u_2^2g_3(u_1,u_2)\frac{\partial g_1(u_1,u_2)}{\partial u_2}+$$
$$+u_1^2u_2^2\frac{\partial g_1(u_1,u_2)}{\partial u_2}\frac{\partial g_3(u_1,u_2)}{\partial u_1}-a_1u_1u_2^2g_1(u_1,u_2)\frac{\partial g_3(u_1,u_2)}{\partial u_2}-$$
$$-c_2u_1^2u_2g_3(u_1,u_2)\frac{\partial g_1(u_1,u_2)}{\partial u_1}-u_1^2u_2^2\frac{\partial g_1(u_1,u_2)}{\partial u_1}\frac{\partial g_3(u_1,u_2)}{\partial u_2},$$
$$\omega_3(u_1,u_2)=a_1u_1u_2^2g_1(u_1,u_2)\frac{\partial g_2(u_1,u_2)}{\partial u_2}+b_2u_1^2u_2g_2(u_1,u_2)\frac{\partial g_1(u_1,u_2)}{\partial u_1}+$$
$$+u_1^2u_2^2\frac{\partial g_1(u_1,u_2)}{\partial u_1}\frac{\partial g_2(u_1,u_2)}{\partial u_2}-a_2u_1^2u_2g_1(u_1,u_2)\frac{\partial g_2(u_1,u_2)}{\partial u_1}-$$
$$-b_1u_1u_2^2g_2(u_1,u_2)\frac{\partial g_1(u_1,u_2)}{\partial u_2}-u_1^2u_2^2\frac{\partial g_1(u_1,u_2)}{\partial u_2}\frac{\partial g_2(u_1,u_2)}{\partial u_1},$$

$\omega_i(0,0)=0, i=1,2,3.$

Next, equation (3.1) can be expressed in the following form:

$$h_1(u_1,u_2)u_1^{m_1}u_2^{m_2}+h_2(u_1,u_2)u_1^{n_1}u_2^{n_2}+h_3(u_1,u_2)u_1^{l_1}u_2^{l_2}=0, \tag{3.2}$$

where $m_1=a_1+b_1-1$, $m_2=a_2+b_2-1$, $n_1=a_1+c_1-1$, $n_2=a_2+c_2-1$, $l_1=b_1+c_1-1$, $l_2=b_2+c_2-1$, $\tilde{g}(u_1,u_2)=g_1(u_1,u_2)g_2(u_1,u_2)g_3(u_1,u_2)$,

$$\begin{aligned}
h_1(u_1,u_2)&=r_3B_1g_1(u_1,u_2)g_2(u_1,u_2)+r_3\omega_3(u_1,u_2)\\
&\quad+\big(B_1\tilde{g}(u_1,u_2)+g_3(u_1,u_2)\omega_3(u_1,u_2)\big)u_1^{c_1}u_2^{c_2},\\
h_2(u_1,u_2)&=-r_2B_3g_1(u_1,u_2)g_3(u_1,u_2)-r_2\omega_2(u_1,u_2)\\
&\quad-\big(B_3\tilde{g}(u_1,u_2)-g_2(u_1,u_2)\omega_2(u_1,u_2)\big)u_1^{b_1}u_2^{b_2},\\
h_3(u_1,u_2)&=r_1B_2g_2(u_1,u_2)g_3(u_1,u_2)+r_1\omega_1(u_1,u_2)\\
&\quad+\big(B_2\tilde{g}(u_1,u_2)+g_1(u_1,u_2)\omega_1(u_1,u_2)\big)u_1^{a_1}u_2^{a_2}.
\end{aligned}$$

Suppose that $U=\{(u_1,u_2)\in\mathbb{R}^2: 0<u_1,u_2<\varepsilon\}$, where $\varepsilon>0$ is a sufficiently small. We will show that the left-hand side of equation (3.2) is nonzero in U. To this end, we consider several cases. First, suppose that $r_i\neq 0$ for each i. Then it follows that $h_i(0,0)\neq 0, i=1,2,3$.

Case 1. If either $min\{m_1,n_1,l_1\}=m_1$, $min\{m_2,n_2,l_2\}=m_2$, or $min\{m_1,n_1,l_1\}=n_1$, $min\{m_2,n_2,l_2\}=n_2$, or $min\{m_1,n_1,l_1\}=l_1$, $min\{m_2,n_2,l_2\}=l_2$, then the left side of the equation (3.2) is not equal to zero in U. In particular, if either $m_1=m_2=0$, or $n_1=n_2=0$, or $l_1=l_2=0$, then the left-hand side of (3.2) does not vanish in U.

Case 2. Let $min\{m_1,n_1,l_1\}=m_1$, $min\{m_2,n_2,l_2\}=n_2$. Then the left-hand side of the equation (3.2) can be written as:

$$h_1(u_1,u_2)u_1^{m_1}u_2^{m_2}+h_2(u_1,u_2)u_1^{n_1}u_2^{n_2}+h_3(u_1,u_2)u_1^{l_1}u_2^{l_2}=u_1^{m_1}u_2^{n_2}\phi(u_1,u_2),$$

where

$$\phi(u_1,u_2):=h_1(u_1,u_2)u_2^{m_2-n_2}+h_2(u_1,u_2)u_1^{n_1-m_1}+h_3(u_1,u_2)u_1^{l_1-m_1}u_2^{l_2-n_2} \tag{3.3}$$

is fractional power series, $m_2-n_2>0, n_1-m_1>0$.

In this case, to prove the theorem, we apply the resolution of singularities to $\phi(u_1,u_2)$ (see [16], p. 236-245). We consider the following two cases.

Case 2.1. Suppose that the Newton diagram of the function $\phi(u_1,u_2)$ consists of the segments γ_1, and γ_2, connecting the points (l_1-m_1,l_2-n_2), $(0,m_2-n_2)$ and (l_1-m_1,l_2-n_2), $(n_1-m_1,0)$. Then the point (l_1-m_1,l_2-n_2) lies below the line connecting the points $(n_1-m_1,0)$ and $(0,m_2-n_2)$. Obviously, the following inequality holds:

$$\frac{l_1-m_1}{n_1-m_1}+\frac{l_2-n_2}{m_2-n_2}<1. \tag{3.4}$$

Following the paper [16], we divide U into regions $V_1, X_1^j, X_2^j, W_1^j, W_2^j, V_2, V_3$, where

$$V_1=\big\{(u_1,u_2)\in U:\delta_1u_1^{s_2}<u_2<M_1u_1^{s_1}\big\},$$

corresponding to the vertex $(l_1 - m_1, l_2 - n_2)$, which is the intersection of two edges γ_1 and γ_2. Here $M_1 > 0, \delta_1 > 0$, $s_1 = \frac{l_1 - m_1}{m_2 - l_2}$, $s_2 = \frac{n_1 - l_1}{l_2 - n_2}$ and $s_1 < s_2$. The numbers $-\frac{1}{s_1}, -\frac{1}{s_2}$ are the slopes of the edges γ_1 and γ_2, respectively. Assume that the numbers $l_1 - m_1$, $m_2 - l_2$, $n_1 - l_1$, $l_2 - n_2$ are positive.

Now, we define regions X_1^j, X_2^j, W_1^j, and W_2^j. For small ε_0 we denote by D_1^c the curve $D_1^c = \{(u_1, cu_1^{s_1}) : 0 < u_1 < \varepsilon_0\}$. Then $\phi(u_1, u_2)$ on the curve D_1^c satisfies the following relation:

$$\phi(u_1, u_2) = \phi(u_1, cu_1^{s_1}) = \tau_1(c) u_1^{s_1(m_2 - n_2)} + o\big(u_1^{s_1(m_2 - n_2)}\big),$$

where $\tau_1(c) = d_1 c^{m_2 - n_2} + d_2 c^{l_2 - n_2}$ and d_1, d_2 are some nonzero real numbers.

By (3.4), we have $n_1 - m_1 > s_1(m_2 - n_2)$. If c is not a root of the polynomial $\tau_1(c)$, then on the curve D_1^c for small u_1, we can compare $\phi(u_1, u_2)$ with $u_1^{s_1(m_2 - n_2)}$. Otherwise, let $c_1^0, ..., c_1^l$ be the roots of $\tau_1(c)$. Suppose that for any j c_1^j lies inside the segment $[r_j, r_j']$, such that $\tau_1(c)$ no longer has roots in $[r_j, r_j']$, except c_1^j. The areas X_1^j are defined as follows:

$$X_1^j = \big\{(u_1, u_2) \in U : r_j u_1^{s_1} \le u_2 \le r_j' u_1^{s_1}\big\}.$$

We denote by D_2^c the curve given by $D_2^c = \{(u_1, cu_1^{s_2}) : 0 < u_1 < \varepsilon_0\}$. Similarly, one can define a polynomial $\tau_2(c)$ and the associated regions X_2^j.

Then, the set $U \backslash \big(V_1 \cup X_1^j \cup X_2^j\big)$ can be expressed as the union of the regions W_1^j and W_2^j. The regions W_1^j are defined as follows:

$$W_1^j = \big\{(u_1, u_2) \in U : s_j u_1^{s_1} < u_2 < s_j' u_1^{s_1}\big\},$$

where $s_j > 0, s_j' > 0$. The regions W_2^j are defined analogously to W_1^j.

If $\tau_1(c)$ and $\tau_2(c)$ have no roots, then we do not split $U \backslash V_1$ and define W_1^j to be W_1 and W_2^j to be W_2, where

$$W_1 = \big\{(u_1, u_2) \in U : M_1 u_1^{s_1} < u_2 < M_1' u_1^{s_1}\big\},$$
$$W_2 = \big\{(u_1, u_2) \in U : \delta_1' u_1^{s_2} < u_2 < \delta_1 u_1^{s_2}\big\},$$

the numbers $M_1, M_1', \delta_1, \delta_1'$ are positive.

Finally, the regions V_2 and V_3 are introduced as follows:

$$V_2 = \big\{(u_1, u_2) \in U : u_2 < \delta_1' u_1^{s_2}\big\},\ V_3 = \big\{(u_1, u_2) \in U : u_2 > M_1' u_1^{s_1}\big\},$$

corresponding to the vertices $(n_1 - m_1, 0)$ and $(0, m_2 - n_2)$, respectively.

Next, we represent the function $\phi(u_1, u_2)$ defined in (3.3) in the following form:

$$\phi(u_1, u_2) = \alpha(u_1, u_2) + \beta(u_1, u_2), \tag{3.5}$$

where

$$\alpha(u_1, u_2) = h_1(u_1, u_2) u_2^{m_2 - n_2} + 0,5 h_3(u_1, u_2) u_1^{l_1 - m_1} u_2^{l_2 - n_2},$$
$$\beta(u_1, u_2) = h_2(u_1, u_2) u_1^{n_1 - m_1} + 0,5 h_3(u_1, u_2) u_1^{l_1 - m_1} u_2^{l_2 - n_2}.$$

By performing the power transformation $u_1 = w_1$, $u_2 = w_1^{s_1} w_2$ in the region V_1, the function $\alpha(u_1, u_2)$ can be expressed in the following form:

$$\alpha(w_1, w_1^{s_1} w_2) = w_1^{l_1 - m_1 + s_1(l_2 - n_2)} w_2^{l_2 - n_2} \big(h_1(w_1, w_1^{s_1} w_2) w_2^{m_2 - l_2} + 0,5 h_3(w_1, w_1^{s_1} w_2)\big),$$

where $0 < w_1 < \varepsilon$, $\delta_1 w_1^{s_2 - s_1} < w_2 < M_1$, $l_1 - m_1 + s_1(l_2 - n_2) > 0$, $m_2 - l_2 > 0$. Suppose that M_1 is a sufficiently small. Therefore $\alpha(w_1, w_1^{s_1} w_2) \ne 0$.

If we replace the roles of the u_1 and u_2 axes, then we have

$$V_1' = \big\{(u_1, u_2) \in U : M_1^{-\frac{1}{s_1}} u_2^{\frac{1}{s_1}} < u_1 < \delta_1^{-\frac{1}{s_2}} u_2^{\frac{1}{s_2}}\big\}.$$

Performing the change of variables $u_1 = \nu_1 \nu_2^{\frac{1}{s_2}}$, $u_2 = \nu_2$ in V_1', the function $\beta(u_1, u_2)$ can be of the form:

$$\beta(\nu_1 \nu_2^{\frac{1}{s_2}}, \nu_2) = \nu_1^{l_1 - m_1} \nu_2^{\frac{1}{s_2}(n_1 - m_1)} \Big(h_2(\nu_1 \nu_2^{\frac{1}{s_2}}, \nu_2) \nu_1^{n_1 - l_1} + 0,5 h_3(\nu_1 \nu_2^{\frac{1}{s_2}}, \nu_2)\Big),$$

where $M_1^{-\frac{1}{s_1}}\nu_2^{\frac{1}{s_1}-\frac{1}{s_2}} < \nu_1 < \delta_1^{-\frac{1}{s_2}}$, $0 < \nu_2 < \varepsilon$, $n_1 - l_1 > 0$. Assume that the number δ_1 is sufficiently large. Hence $\beta\big(\nu_1\nu_2^{\frac{1}{s_2}}, \nu_2\big) \neq 0$.

Thus, taking into account Lemma 2.1 and the equality (3.5), we have $\phi(u_1, u_2) \neq 0$ in the region V_1.

Next, following [16], we make the power transformation $u_1 = v_1$, $u_2 = v_1^{s_1}v_2$ in the domain W_1^j and the function $\phi(u_1, u_2)$ defined in (3.3) is expressed in the following form:

$$\phi(v_1, v_1^{s_1}v_2) = \tilde{h}(v_1, v_2)v_1^{s_1(m_2-n_2)} + h_2(v_1, v_1^{s_1}v_2)v_1^{n_1-m_1},$$

where $0 < v_1 < \varepsilon$, $s_j \leq v_2 \leq s_j'$, $\tilde{h}(v_1, v_2) = h_1(v_1, v_1^{s_1}v_2)v_2^{m_2-n_2} + h_3(v_1, v_1^{s_1}v_2)v_2^{l_2-n_2}$.

From the inequality (3.4) it follows that $n_1 - m_1 > s_1(m_2 - n_2)$. By the definition of W_1^j, $\tilde{h}(v_1, v_2)$ does not have any zeros on $[s_j, s_j']$. Hence, $\tilde{h}(v_1, v_2) \neq 0$ on $[s_j, s_j']$ and $\phi(v_1, v_1^{s_1}v_2) \neq 0$.

By an analogous argument and in view of Lemma 2.1, one concludes that $\phi(u_1, u_2) \neq 0$ in the domains W_2^j, W_1, and W_2.

Taking into account (3.3) and making the change of variables $u_1 = \sigma_1$, $u_2 = \sigma_1^{s_2}\sigma_2$ in the domain V_2, we have

$$\phi(\sigma_1, \sigma_1^{s_2}\sigma_2) = \sigma_1^{n_1-m_1}\Big(h_1(\sigma_1, \sigma_1^{s_2}\sigma_2)\sigma_1^{s_2(m_2-n_2)+m_1-n_1}\sigma_2^{m_2-n_2} +$$

$$+h_2(\sigma_1, \sigma_1^{s_2}\sigma_2) + h_3(\sigma_1, \sigma_1^{s_2}\sigma_2)\sigma_2^{l_2-n_2}\Big),$$

where $0 < \sigma_1 < \varepsilon$, $0 < \sigma_2 < \delta_1'$. From the inequality (3.4) it follows that $s_2(m_2-n_2)+m_1-n_1 > 0$. Hence, from the last equality, it follows that $\phi(\sigma_1, \sigma_1^{s_2}\sigma_2) \neq 0$ in V_2.

Transforming the variables $u_1 = z_1 z_2^{\frac{1}{s_1}}$, $u_2 = z_2$ in the domain V_3, we have

$$\phi\Big(z_1 z_2^{\frac{1}{s_1}}, z_2\Big) = z_2^{m_2-n_2}\Big(h_1\big(z_1 z_2^{\frac{1}{s_1}}, z_2\big) + h_2\big(z_1 z_2^{\frac{1}{s_1}}, z_2\big)z_1^{n_1-m_1}z_2^{\frac{n_1-m_1}{s_1}+n_2-m_2} +$$

$$+h_3\big(z_1 z_2^{\frac{1}{s_1}}, z_2\big)z_1^{l_1-m_1}\Big),$$

where $\frac{1}{s_1}(n_1 - m_1) + n_2 - m_2 > 0$, $0 < z_1 < (M_1')^{-\frac{1}{s_1}}$, $0 < z_2 < \varepsilon$. Hence $\phi\big(z_1 z_2^{\frac{1}{s_1}}, z_2\big) \neq 0$ in V_3.

Following the paper [16] (p. 242-244) and in view of Lemma 2.1, one can verify that $\phi(u_1, u_2) \neq 0$ in the domains X_1^j and X_2^j.

Note that if any of the numbers $l_1 - m_1, m_2 - l_2, n_1 - l_1, l_2 - n_2, n_1 - m_1, m_2 - n_2$ vanish, it can still be shown that $\phi(u_1, u_2) \neq 0$ throughout U.

Case 2.2. Suppose that the Newton diagram of the function $\phi(u_1, u_2)$ in (3.3) is a segment connecting the points $(n_1 - m_1, 0)$ and $(0, m_2 - n_2)$. Following the argument in Case 2.1, one can conclude that the function $\phi(u_1, u_2)$ is nonvanishing throughout the domain U.

Analogously, the subsequent cases are analyzed.

If either $min\{m_1, n_1, l_1\} = m_1$, $min\{m_2, n_2, l_2\} = l_2$, or $min\{m_1, n_1, l_1\} = n_1$, $\min\{m_2, n_2, l_2\} = m_2$, or $min\{m_1, n_1, l_1\} = l_1$, $min\{m_2, n_2, l_2\} = m_2$, or $min\{m_1, n_1, l_1\} = n_1$, $min\{m_2, n_2, l_2\} = l_2$, or $min\{m_1, n_1, l_1\} = l_1$, $min\{m_2, n_2, l_2\} = n_2$, then $\phi(u_1, u_2) \neq 0$ in U.

If two of the numbers r_i equal zero and one of them is nonzero, then it is easy to see that $\phi(u_1, u_2) \neq 0$. If one of the numbers r_i is equal to zero and two of them are nonzero, then using the resolution of singularities, we can show that $\phi(u_1, u_2) \neq 0$ in U. Theorem 3.1 has been proved. □

Remark 3.2. If we perform the change of variables $u_1 = v_1^{\frac{1}{n}}$, $u_2 = v_2^{\frac{1}{n}}$, $n \in \mathbb{N}$ in (1.1), then by Lemma 2.1, Theorem 3.1 is valid. Therefore, if the exponents a_i, b_i, c_i are non-negative rational numbers, then Theorem 3.1 remains true.

Considering equation (3.1), it is possible to verify that the following statements are true.

Corollary 3.3. *Let $g_i(u_1,u_2)\}_{i=1}^3$ be fractional power series defined in a small neighborhood of the origin in $\mathbb{R}^2_+$ such that $g_i(0,0)\neq 0$. Suppose that one of the following conditions is satisfied:*

1. $r_1=0, r_2=0, r_3B_1\neq 0, (c_1,c_2)\neq(0,0)$;

2. $r_1=0, r_3=0, r_2B_3\neq 0, (b_1,b_2)\neq(0,0)$;

3. $r_2=0, r_3=0, r_1B_2\neq 0, (a_1,a_2)\neq(0,0)$.

Then the surface (1.1), in a sufficiently small neighborhood of the point (r_1,r_2,r_3), satisfies the transversality condition at each of its regular points.

Corollary 3.4. *Let $\{g_i(u_1,u_2)\}_{i=1}^3$ be fractional power series at the origin in $\mathbb{R}^2_+$ such that $g_i(0,0)\neq 0$. Assume that $B_i\neq 0$ and two of the numbers r_1,r_2, and r_3 are equal to zero and one of them is nonzero. Then, for the surface (1.1) in a sufficiently small neighborhood of the point (r_1,r_2,r_3), i.e., at regular points of this surface, the transversality condition is satisfied.*

Corollary 3.5. *If $\{g_i(u_1,u_2)\}_{i=1}^3$ are real analytic functions at the origin in $\mathbb{R}^2_+$ such that $g_i(0,0)\neq 0$, then all the above results will be true.*

4 Tansversality condition for some parameterized surfaces

In this section, we present additional results of this paper. Suppose that at least two of the numbers r_i are nonzero. Let $(e_{ij})_{i,j=1}^3$ denote an orthogonal rotation matrix as in [6]. Under this rotation of the coordinate system, the equations in (1.1) are transformed into the following new form:

$$\begin{aligned}
\tilde{x}_1(u_1,u_2) &= e_{11}u_1^{a_1}u_2^{a_2}g_1(u_1,u_2)+e_{12}u_1^{b_1}u_2^{b_2}g_2(u_1,u_2)+e_{13}u_1^{c_1}u_2^{c_2}g_3(u_1,u_2),\\
\tilde{x}_2(u_1,u_2) &= e_{21}u_1^{a_1}u_2^{a_2}g_1(u_1,u_2)+e_{22}u_1^{b_1}u_2^{b_2}g_2(u_1,u_2)+e_{23}u_1^{c_1}u_2^{c_2}g_3(u_1,u_2),\\
\tilde{x}_3(u_1,u_2) &= d+e_{31}u_1^{a_1}u_2^{a_2}g_1(u_1,u_2)+e_{32}u_1^{b_1}u_2^{b_2}g_2(u_1,u_2)+e_{33}u_1^{c_1}u_2^{c_2}g_3(u_1,u_2),
\end{aligned}\tag{4.1}$$

where $d=\sqrt{r_1^2+r_2^2+r_3^2}$.

Considering the vector $(0,0,d)$ is transversal to the tangent cone to the surface (4.1) at the origin, we now proceed to prove the following proposition.

Proposition 4.1. *Let $\{g_i(u_1,u_2)\}_{i=1}^3$ be fractional power series defined in a small neighborhood of the origin in $\mathbb{R}^2_+$ such that $g_i(0,0)\neq 0$. At least two of the numbers r_i are nonzero and $B_1+B_2-B_3\neq 0$. Then the surface (4.1) in a sufficiently small neighborhood of the point $(0,0,d)$ satisfies the transversality condition.*

Proof. Suppose that the origin of the coordinate system in $\mathbb{R}^3$ lies on the tangent plane of the surface (4.1) at its point $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)$. Then, by (2.2), we obtain the equation of the tangent plane of the form:

$$\tilde{x}_1(u_1,u_2)M_{23}(u_1,u_2)-\tilde{x}_2(u_1,u_2)M_{13}(u_1,u_2)+\tilde{x}_3(u_1,u_2)M_{12}(u_1,u_2)=0,\tag{4.2}$$

where

$$\begin{aligned}
M_{23}(u_1,u_2) &= \frac{\partial\tilde{x}_2(u_1,u_2)}{\partial u_1}\cdot\frac{\partial\tilde{x}_3(u_1,u_2)}{\partial u_2}-\frac{\partial\tilde{x}_2(u_1,u_2)}{\partial u_2}\cdot\frac{\partial\tilde{x}_3(u_1,u_2)}{\partial u_1},\\
M_{13}(u_1,u_2) &= \frac{\partial\tilde{x}_1(u_1,u_2)}{\partial u_1}\cdot\frac{\partial\tilde{x}_3(u_1,u_2)}{\partial u_2}-\frac{\partial\tilde{x}_3(u_1,u_2)}{\partial u_1}\cdot\frac{\partial\tilde{x}_1(u_1,u_2)}{\partial u_2},\\
M_{12}(u_1,u_2) &= \frac{\partial\tilde{x}_1(u_1,u_2)}{\partial u_1}\cdot\frac{\partial\tilde{x}_2(u_1,u_2)}{\partial u_2}-\frac{\partial\tilde{x}_1(u_1,u_2)}{\partial u_2}\cdot\frac{\partial\tilde{x}_2(u_1,u_2)}{\partial u_1}.
\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}
M_{23}(u_1,u_2) &= (e_{21}e_{32}-e_{22}e_{31})u_1^{a_1+b_1-1}u_2^{a_2+b_2-1}\Big(B_1g_1(u_1,u_2)g_2(u_1,u_2)+\varphi_1(u_1,u_2)\Big)+\\
&+(e_{21}e_{33}-e_{23}e_{31})u_1^{a_1+c_1-1}u_2^{a_2+c_2-1}\Big(B_3g_1(u_1,u_2)g_3(u_1,u_2)+\varphi_2(u_1,u_2)\Big)+
\end{aligned}$$

$$+(e_{22}e_{33}-e_{23}e_{32})u_1^{b_1+c_1-1}u_2^{b_2+c_2-1}\Big(B_2g_2(u_1,u_2)g_3(u_1,u_2)+\varphi_3(u_1,u_2)\Big),$$

$$M_{13}(u_1,u_2)=(e_{11}e_{32}-e_{12}e_{31})u_1^{a_1+b_1-1}u_2^{a_2+b_2-1}\Big(B_1g_1(u_1,u_2)g_2(u_1,u_2)+\varphi_1(u_1,u_2)\Big)+$$

$$+(e_{11}e_{33}-e_{13}e_{31})u_1^{a_1+c_1-1}u_2^{a_2+c_2-1}\Big(B_3g_1(u_1,u_2)g_3(u_1,u_2)+\varphi_2(u_1,u_2)\Big)+$$

$$+(e_{12}e_{33}-e_{13}e_{32})u_1^{b_1+c_1-1}u_2^{b_2+c_2-1}\Big(B_2g_2(u_1,u_2)g_3(u_1,u_2)+\varphi_3(u_1,u_2)\Big),$$

$$M_{12}(u_1,u_2)=(e_{11}e_{22}-e_{12}e_{21})u_1^{a_1+b_1-1}u_2^{a_2+b_2-1}\Big(B_1g_1(u_1,u_2)g_2(u_1,u_2)+\varphi_1(u_1,u_2)\Big)+$$

$$+(e_{11}e_{23}-e_{13}e_{21})u_1^{a_1+c_1-1}u_2^{a_2+c_2-1}\Big(B_3g_1(u_1,u_2)g_3(u_1,u_2)+\varphi_2(u_1,u_2)\Big)+$$

$$+(e_{12}e_{23}-e_{13}e_{22})u_1^{b_1+c_1-1}u_2^{b_2+c_2-1}\Big(B_2g_2(u_1,u_2)g_3(u_1,u_2)+\varphi_3(u_1,u_2)\Big),$$

where

$$\varphi_1(u_1,u_2)=a_1g_1(u_1,u_2)u_2\frac{\partial g_2(u_1,u_2)}{\partial u_2}+b_2g_2(u_1,u_2)u_1\frac{\partial g_1(u_1,u_2)}{\partial u_1}-$$

$$-b_1g_2(u_1,u_2)u_2\frac{\partial g_1(u_1,u_2)}{\partial u_2}-a_2g_1(u_1,u_2)u_1\frac{\partial g_2(u_1,u_2)}{\partial u_1}+$$

$$+u_1u_2\left(\frac{\partial g_1(u_1,u_2)}{\partial u_1}\frac{\partial g_2(u_1,u_2)}{\partial u_2}-\frac{\partial g_1(u_1,u_2)}{\partial u_2}\frac{\partial g_2(u_1,u_2)}{\partial u_1}\right),$$

$$\varphi_2(u_1,u_2)=a_1g_1(u_1,u_2)u_2\frac{\partial g_3(u_1,u_2)}{\partial u_2}+c_2g_3(u_1,u_2)u_1\frac{\partial g_1(u_1,u_2)}{\partial u_1}-$$

$$-c_1g_3(u_1,u_2)u_2\frac{\partial g_1(u_1,u_2)}{\partial u_2}-a_2g_1(u_1,u_2)u_1\frac{\partial g_3(u_1,u_2)}{\partial u_1}+$$

$$+u_1u_2\left(\frac{\partial g_1(u_1,u_2)}{\partial u_1}\frac{\partial g_3(u_1,u_2)}{\partial u_2}-\frac{\partial g_1(u_1,u_2)}{\partial u_2}\frac{\partial g_3(u_1,u_2)}{\partial u_1}\right),$$

$$\varphi_3(u_1,u_2)=b_1g_2(u_1,u_2)u_2\frac{\partial g_3(u_1,u_2)}{\partial u_2}+c_2g_3(u_1,u_2)u_1\frac{\partial g_2(u_1,u_2)}{\partial u_1}-$$

$$-c_1g_3(u_1,u_2)u_2\frac{\partial g_2(u_1,u_2)}{\partial u_2}-b_2g_2(u_1,u_2)u_1\frac{\partial g_3(u_1,u_2)}{\partial u_1}+$$

$$+u_1u_2\left(\frac{\partial g_2(u_1,u_2)}{\partial u_1}\frac{\partial g_3(u_1,u_2)}{\partial u_2}-\frac{\partial g_2(u_1,u_2)}{\partial u_2}\frac{\partial g_3(u_1,u_2)}{\partial u_1}\right).$$

Substituting the values of $\tilde{x}_1(u_1,u_2)$, $\tilde{x}_2(u_1,u_2)$, $\tilde{x}_3(u_1,u_2)$, $M_{23}(u_1,u_2)$, $M_{13}(u_1,u_2)$ and $M_{12}(u_1,u_2)$ into (4.2), we obtain

$$d\cdot M_{12}(u_1,u_2)+u_1^{a_1+b_1+c_1-1}u_2^{a_2+b_2+c_2-1}\tilde{g}(u_1,u_2)=0, \tag{4.3}$$

where $\tilde{g}(u_1,u_2)=(B_1+B_2-B_3)g_1(u_1,u_2)g_2(u_1,u_2)g_3(u_1,u_2)+h(u_1,u_2)$,

$$h(u_1,u_2)=g_1(u_1,u_2)\varphi_3(u_1,u_2)+g_2(u_1,u_2)\varphi_2(u_1,u_2)+g_3(u_1,u_2)\varphi_1(u_1,u_2).$$

It follows from equation (4.3) that, if $M_{12}(u_1,u_2)=0$ and $B_1+B_2-B_3\neq 0$, then the left-hand side of this equation is nonvanishing. If $M_{12}(u_1,u_2)\neq 0$, then the left-hand side of the equation (4.3) does not vanish in U. This completes the proof of Proposition 4.1. □

Under the assumption $B_1+B_2-B_3\neq 0$, the left-hand side of the equation (4.2) is nonvanishing. It then follows that the minors $M_{23}(u_1,u_2)$, $M_{13}(u_1,u_2)$, and $M_{12}(u_1,u_2)$ cannot all simultaneously vanish, that is, at least one of these minors must be nonzero.

Remark 4.2. Under the condition $B_1+B_2-B_3\neq 0$, points of the surface (4.1) that lie in a sufficiently small neighborhood of the point $(0,0,d)$, and off the coordinate planes are regular. In contrast, points of the surface (4.1) that lie on the coordinate planes may be singular.

Remark 4.3. If $\{g_i(u_1,u_2)\}_{i=1}^3$ are real analytic functions defined in a small neighborhood of the origin in $\mathbb{R}^2_+$, then the Proposition 4.1 remains true.

Let $B_1 \neq 0$, $g_i(0,0) \neq 0$, $r_3 \neq 0$. Assume that at least one of the numbers r_1, and r_2 is nonzero. Then, after an appropriate change of variables and a rotation of the coordinate system, the surfaces defined by equations (1.1) can be reduced to the following form (see section 3, [6]):

$$\begin{gathered}\hat{x}_1(w_1,w_2) = e_{11}w_1 + e_{12}w_2 + e_{13}w_1^{\frac{-B_2}{B_1}} w_2^{\frac{B_3}{B_1}} g(w_1,w_2),\\ \hat{x}_2(w_1,w_2) = e_{21}w_1 + e_{22}w_2 + e_{23}w_1^{\frac{-B_2}{B_1}} w_2^{\frac{B_3}{B_1}} g(w_1,w_2),\\ \hat{x}_3(w_1,w_2) = c + e_{31}w_1 + e_{32}w_2 + e_{33}w_1^{\frac{-B_2}{B_1}} w_2^{\frac{B_3}{B_1}} g(w_1,w_2),\end{gathered} \tag{4.4}$$

where $(e_{ij})_{i,j=1}^3$ is an orthogonal matrix, $c = \sqrt{4^{a_1j_1+a_2j_2} + 4^{b_1j_1+b_2j_2} + 4^{c_1j_1+c_2j_2}}$,

$$\begin{gathered}(w_1,w_2) \in M = \{(w_1,w_2) \in \mathbb{R}^2 : 2^{-(a_1+a_2)} \leq w_1 \leq 2^{a_1+a_2}, 2^{-(b_1+b_2)} \leq w_2 \leq 2^{b_1+b_2}\},\\ g(w_1,w_2) = \big(\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2)\big)^{c_1}\big(\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\big)^{c_2}\times\\ g_3\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2), 2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),\\ \tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2) = 1 + 2^{-j_1}\tilde{h}_1(2^{-j_1},2^{-j_2},w_1,w_2) + 2^{-j_2}\tilde{h}_2(2^{-j_1},2^{-j_2},w_1,w_2),\\ \tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2) = 1 + 2^{-j_1}\tilde{\rho}_1(2^{-j_1},2^{-j_2},w_1,u_2) + 2^{-j_2}\tilde{\rho}_2(2^{-j_1},2^{-j_2},w_1,w_2),\end{gathered} \tag{4.5}$$

$\tilde{h}_1, \tilde{h}_2, \tilde{\rho}_1, \tilde{\rho}_2$ are smooth functions, j_1, j_2 are large positive integers.

Since $\tilde{g}_1(0,0,w_1,w_2) \neq 0, \tilde{g}_2(0,0,w_1,w_2) \neq 0$ and $g_3(0,0) \neq 0$, it follows from (4.5) that $g(w_1,w_2) \neq 0$.

Proposition 4.4. *Let $g(w_1,w_2)$ be a fractional power series defined on the set M, and assume that $B_1 \neq 0$ and $B_1 + B_2 - B_3 \neq 0$. Further, suppose that $r_3 \neq 0$ and that at least one of the numbers r_1, and r_2 is nonzero. Then the surface (4.4) satisfies the transversality condition.*

Proof. Let the origin of the coordinate system in $\mathbb{R}^3$ be located on the tangent plane to the surface (4.4) at a point $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$. Then, in accordance with (2.2), the equation of the tangent plane to the surface at the point (w_1,w_2) can be represented in the form:

$$c \cdot M_{12}(w_1,w_2) + w_1^{-\frac{B_2}{B_1}} w_2^{\frac{B_3}{B_1}} \left(C \cdot g(w_1,w_2) - w_1\frac{\partial g(w_1,w_2)}{\partial w_1} - w_2\frac{\partial g(w_1,w_2)}{\partial w_2}\right) = 0, \tag{4.6}$$

where

$$C = \frac{B_1+B_2-B_3}{B_1},\ M_{12}(w_1,w_2) = \frac{\partial \hat{x}_1(w_1,w_2)}{\partial w_1}\cdot\frac{\partial \hat{x}_2(w_1,w_2)}{\partial w_2} - \frac{\partial \hat{x}_1(w_1,w_2)}{\partial w_2}\cdot\frac{\partial \hat{x}_2(w_1,w_2)}{\partial w_1}.$$

Next, assume that $M_{12}(w_1,w_2) = 0$. Then, taking into account (4.5), the corresponding partial derivative can be computed as follows:

$$\begin{gathered}\frac{\partial g(w_1,w_2)}{\partial w_1} = 2^{-j_1}\tilde{g}_1^{c_1-1}\tilde{g}_2^{c_2-1}\left(c_1\tilde{g}_2g_3\frac{\partial \tilde{h}_1}{\partial w_1} + c_2\tilde{g}_1g_3\frac{\partial \tilde{\rho}_1}{\partial w_1} + \tilde{g}_1\tilde{g}_2\frac{\partial g_3}{\partial z_1}\frac{\partial z_1}{\partial w_1}\right) +\\ +2^{-j_2}\tilde{g}_1^{c_1-1}\tilde{g}_2^{c_2-1}\left(c_1\tilde{g}_2g_3\frac{\partial \tilde{h}_2}{\partial w_1} + c_2\tilde{g}_1g_3\frac{\partial \tilde{\rho}_2}{\partial w_1} + \tilde{g}_1\tilde{g}_2\frac{\partial g_3}{\partial z_2}\frac{\partial z_2}{\partial w_1}\right),\end{gathered}$$

where

$$z_1 = 2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ \ z_2 = 2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2).$$

Since j_1, j_2 are sufficiently large, it follows from the previous relation that the partial derivative $\frac{\partial g(w_1,w_2)}{\partial w_1}$ is correspondingly small. By a similar argument, the partial derivative $\frac{\partial g(w_1,w_2)}{\partial w_2}$ is also small. Thus, under the condition $B_1 + B_2 - B_3 \neq 0$, the left-hand side of equation (4.6) remains nonvanishing.

Suppose that $M_{12}(w_1,w_2) \neq 0$. In this case, since d is sufficiently large, the left-hand side of the equation (4.6) is nonvanishing throughout the set M. Consequently, the surface (4.4) satisfies the transversality condition. This completes the proof of Proposition 4.4. □

Remark 4.5. Note that under the assumptions of Proposition 4.4, the surface (4.1) is regular.

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Author information

Salim Usmanov, Department of Mathematics, Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.
Department of Exact Sciences, Kimyo International University in Tashkent, Tashkent, Uzbekistan.
E-mail: usmanov-salim@mail.ru

Ulugbek Navruzov, Department of Mathematics and Natural sciences in Primary Education, Chirchik State Pedagogical University, Uzbekistan.
E-mail: ulugbeknavrozov1@gmail.com

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