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Solitons on Kenmotsu Manifold with Semi-Zamkovoy Connection

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Abstract In this paper, we investigate the geometric properties of Kenmotsu manifolds endowed with the semi-Zamkovoy connection. We study almost Ricci solitons in this setting, and we determine the corresponding scalar curvature explicitly, analyze Ricci solitons, and obtain necessary and sufficient conditions for their existence. In addition, conditions for the existence of η-Ricci solitons are derived, and it is shown that the soliton parameters satisfy a specific linear relation. On the other hand, we examine Yamabe solitons with torse-forming potential vector fields and determine the corresponding existence and classification criteria. Finally, we presented an example of three-dimensional Kenmotsu manifolds with respect to semi-Zamkovoy connection $\hat{\nabla}$ to demonstrate our results.

1 Introduction

The systematic investigation of almost contact metric manifolds, particularly Kenmotsu manifolds, was first undertaken by K. Kenmotsu in 1972 [1]. It is well known that Kenmotsu manifolds naturally emerge from a warped product structure, which played a crucial role in their initial development. Motivated by these foundational contributions, several researchers [2, 3, 4, 5, 6, 24] have subsequently broadened the theory by introducing generalized and nearly Kenmotsu manifolds, along with the study of various affine connections, including semi-symmetric and Schouten-van Kampen connections. In 2021, A. Mandal [7] investigated LP-Sasakian manifolds endowed with the Zamkovoy connection and obtained the Riemannian curvature tensor with respect to the Zamkovoy connection $\mathcal{R}^*$, Ricci tensor S^*, and scalar curvature r^*. The study further examined various conharmonic curvature conditions, including ξ, ϕ, and quasi-conharmonically flat manifolds. Additionally, the notion of generalized ϕ-recurrent manifolds was discussed, and manifolds satisfying the curvature condition $K^*(\xi, U) \cdot \mathcal{R}^* = 0$ were also considered.

Recent years have witnessed growing interest in the investigation of Ricci and η-Ricci solitons associated with connections other than the Levi-Civita connection. In this context, Blaga [29] introduced the concept of almost η-Ricci solitons on $(LCS)_n$-manifolds. Thereafter, many authors investigated η-Ricci solitons on η-Einstein para-Kenmotsu manifolds obtained several geometric properties related to recurrent torse-forming vector fields [31] and geometric solitons on Sasakian and Kenmotsu manifolds equipped with various induced connections, including Tanaka–Webster, Schouten-van Kampen, Zamkovoy, and generalized symmetric metric connections. More recently, Mert et al. [30] obtained several results concerning solitons on Sasakian manifolds admitting generalized connections. On the other hand, Siddiqi and Bahadir [25, 27] studied Ricci and η-Ricci solitons on Kenmotsu manifolds endowed with generalized symmetric

metric connections under the curvature conditions $R \cdot S = 0$ and $S \cdot R = 0$. Furthermore, The geometry of Yamabe-type solitons, Ricci solitons and η-Ricci solitons on various contact metric manifolds has been investigated by many authors in recent years [32, 33, 34, 35, 36].

Inspired by the above investigations, the present work deals with Ricci-type solitons on Kenmotsu manifolds with a semi-Zamkovoy connection. As far as we are aware, this problem has not been considered earlier in the literature.

In 1982, R. Hamilton [8] introduced the concept of Ricci flow as a method for obtaining canonical metrics on smooth manifolds. Since then, Ricci flow has emerged as a fundamental tool in Riemannian geometry, particularly in the analysis of manifolds with positive curvature. The Ricci flow is an evolution equation governing the deformation of Riemannian metrics over time and is defined by

$$\frac{\partial}{\partial t}\mathrm{g}_{ij}(t) = -2\mathcal{S}_{ij}, \quad \mathrm{g}(0) = \mathrm{g}_0.$$

A Ricci soliton $(\mathrm{g}, \mathcal{Y}, \lambda)$ on a Riemannian manifold (M, g) is a natural generalization of an Einstein metric [9]

$$\mathcal{L}_{\mathcal{Y}}\mathrm{g} + 2\mathcal{S} + 2\lambda\mathrm{g} = 0. \tag{1.1}$$

Here, $\mathcal{S}$ denotes the Ricci tensor, $\mathcal{L}_{\mathcal{Y}}$ is the Lie derivative along the vector field $\mathcal{Y}$ on M, and λ is a real constant. The Ricci soliton is classified as shrinking, steady, or expanding depending on whether λ is negative, zero, or positive, respectively. Cho and Kimura [10] further generalized the concept of Ricci solitons by introducing the notion of an η-Ricci soliton, which incorporates the 1-form η corresponding to the structure vector field. A Riemannian manifold (M, g) is said to admit an almost η-Ricci soliton [26] if there exists a smooth vector field $\mathcal{Y}$ such that

$$\mathcal{L}_{\mathcal{Y}}\mathrm{g} + 2\mathcal{S} + 2\lambda\mathrm{g} + 2\kappa\eta \otimes \eta = 0. \tag{1.2}$$

Here, λ and κ are smooth functions on M. On the other hand, if both λ and κ are constant, then the metric g is called an η-Ricci soliton. Several geometers have studied Ricci solitons. For a comprehensive discussion on Ricci solitons, we refer the reader to the works of Hui et al. [11], Derdzinski [12], and Chow and Knopf [13]. Hamilton [8] introduced the notion of the Yamabe flow on Riemannian manifolds, given by

$$\frac{\partial}{\partial t}\mathrm{g}_{ij}(t) = -r(t), \quad \mathrm{g}(0) = \mathrm{g}_0. \tag{1.3}$$

Yamabe solitons can be viewed as self-similar outcomes of the Yamabe flow.
Let (M, g) be a Riemannian manifold. A triple $(\mathrm{g}, \mathcal{T}, \mu)$ is called an almost Yamabe soliton on M if [14, 28]

$$\frac{1}{2}\mathcal{L}_{\mathcal{T}}\mathrm{g} = (r - \mu)\mathrm{g}. \tag{1.4}$$

Here, r denotes the scalar curvature of the manifold (M, g), and μ is a smooth function. If μ is constant, the almost Yamabe soliton reduces to a Yamabe soliton. An almost Yamabe soliton is classified as expanding, steady, or shrinking depending on whether μ is positive, zero, or negative, respectively.
In 1944, Yano [18] introduced the concept of the form of a torse-forming vector field, that is

$$\nabla_{\mathcal{X}}\mathcal{T} = f\mathcal{X} + \omega(\mathcal{X})\mathcal{T}, \tag{1.5}$$

for all $\mathcal{X}, \mathcal{Y} \in \chi(M)$, where $\mathcal{T}$ on M is torse-forming and ω is an 1-form.
Moreover, when 1-form ω in (1.5) is zero, therefore $\mathcal{T}$ is concircular [19], for $f = 1$ and $\omega = 0$ therefore $\mathcal{T}$ is concurrent [20], now for $f = 0$ therefore $\mathcal{T}$ is recurrent, when $f = \omega = 0$, therefore $\mathcal{T}$ is parallel vector field. furthermore for $\omega(\mathcal{T}) = 0$, therefore $\mathcal{T}$ is torqued vector field[21]. In this context, the function f is referred to as the torqued function, while the 1-form ω is called the torqued form associated with $\mathcal{T}$.

More recently, in 2020, A. Mandal [15] studied locally M-projectively ϕ-symmetric Sasakian manifolds with respect to the Zamkovoy connection and analyzed the relationship between the M-projective curvature tensor and the Riemannian curvature tensor with respect to the Zamkovoy connection $\overline{R}$. The author also examined manifolds satisfying the curvature condition $\overline{M}(\xi, U) \circ \overline{R} = 0$. Furthermore, Sandeep K. Mishra [23] investigated quasi-para-Sasakian

manifolds admitting the Zamkovoy connection, where various geometric properties were analyzed, and it was shown that these manifolds satisfy the condition $\bar{\tilde{V}}(\xi, U) \cdot \bar{\mathcal{S}}$, highlighting the behavior of the structure under this connection.
In 2009, Simeon Zamkovoy [16] introduced the Zamkovoy connection as a canonical paracontact connection, providing a tool to study the geometry of paracontact manifolds. Zamkovoy's connection is expressed as

$$\hat{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + (\nabla_{\mathcal{X}}\eta)(\mathcal{Y})\xi - \eta(\mathcal{Y})\nabla_{\mathcal{X}}\xi + \eta(\mathcal{X})\phi\mathcal{Y}, \tag{1.6}$$

where ∇ is the Lavi-Civita connection, $\hat{\nabla}$ is Zamkovoy connection M. In the present study, we introduce the semi-Zamkovoy connection, described [17] as follows

$$\hat{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + (\nabla_{\mathcal{X}}\eta)(\mathcal{Y})\xi - \eta(\mathcal{Y})\nabla_{\mathcal{X}}\xi, \tag{1.7}$$

where, ∇ is the Lavi-Civita connection, $\hat{\nabla}$ is semi-Zamkovoy connection on Kenmotsu manifold M. As a consequence Kenmotsu manifold, with ∇ representing the Levi-Civita connection corresponding to the metric g then it is demonstrate that

$$(\nabla_{\mathcal{X}}\eta)(\mathcal{Y}) = \mathrm{g}(\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{X})\eta(\mathcal{Y}), \tag{1.8}$$

for all $\mathcal{X}, \mathcal{Y} \in \chi(M)$.
This paper is organized as follows: In this paper, we focus on Kenmotsu manifolds equipped with the semi-Zamkovoy connection. In Section 3, we examine the curvature properties of the semi-Zamkovoy connection and establish that the corresponding curvature tensor satisfies Bianchi's first identity. Section 4 is devoted to the analysis of various types of Ricci solitons, where several significant results are obtained. Further, in Section 5, we derive different conditions characterizing η-Ricci solitons on Kenmotsu manifolds with respect to the semi-Zamkovoy connection. In Section 6, we determine the existence and classification criteria for torse-forming vector fields associated with Yamabe solitons on M, corresponding to different types of vector fields. Finally, we presented an example of a three-dimensional Kenmotsu manifold associated with the semi-Zamkovoy connection to verify some of the results obtained.

2 Preliminaries

An n-dimensional differentiable manifold (M, g) endowed with a structure $(\phi, \xi, \eta, \mathrm{g})$, where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form, and g is a Riemannian metric, is said to be an almost paracontact manifold [6], provided the following conditions are satisfied

$$\phi^2\mathcal{X} = -\mathcal{X} + \eta(\mathcal{X})\xi, \tag{2.1}$$

$$\mathrm{g}(\phi\mathcal{X}, \phi\mathcal{Y}) = \mathrm{g}(\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{X})\eta(\mathcal{Y}), \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

$$\phi(\xi) = 0, \quad \eta(\phi\mathcal{X}) = 0, \tag{2.4}$$

$$\mathrm{g}(\mathcal{X}, \xi) = \eta(\mathcal{X}), \tag{2.5}$$

for all vector fields $\mathcal{X}, \mathcal{Y} \in \chi(M)$. If an almost contact metric manifold meets the conditions of a para-Sasakian structure [1]

$$(\nabla_{\mathcal{X}}\phi)\mathcal{Y} = \mathrm{g}(\phi\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{Y})\phi\mathcal{X}, \tag{2.6}$$

$$\nabla_{\mathcal{X}}\xi = \mathcal{X} - \eta(\mathcal{X})\xi. \tag{2.7}$$

Furthermore, the curvature tensor $\mathcal{R}$, the Ricci tensor $\mathcal{S}$, and the Ricci operator Q fulfill the relations described in [1]

$$\mathcal{R}(\mathcal{X}, \mathcal{Y})\xi = \eta(\mathcal{X})\mathcal{Y} - \eta(\mathcal{Y})\mathcal{X}, \tag{2.8}$$

and

$$\mathcal{S}(\mathcal{X}, \xi) = -(n-1)\eta(\mathcal{X}). \tag{2.9}$$

Furthermore, $\mathcal{S}(\mathcal{X},\mathcal{Y}) = \mathrm{g}(Q\mathcal{X},\mathcal{Y})$, we obtain

$$\mathcal{S}(\phi\mathcal{X},\phi\mathcal{Y}) = \mathrm{g}(Q\phi\mathcal{X},\phi\mathcal{Y}). \tag{2.10}$$

View of (2.2) and (2.10), conclude that

$$\mathcal{S}(\phi\mathcal{X},\phi\mathcal{Y}) = \mathcal{S}(\mathcal{X},\mathcal{Y}) + (n-1)\eta(\mathcal{X})\eta(\mathcal{Y}). \tag{2.11}$$

Also putting $\mathcal{Y} = \xi$ in (1.2) and using (1.4), we conclude that

$$\hat{\nabla}_{\mathcal{X}}\xi = 0, \tag{2.12}$$

for all $\mathcal{X},\mathcal{Y} \in \chi(M)$.

3 Analytical Property of Kenmotsu Manifolds Associated with the semi-Zamkovoy Connection

Let $\hat{\mathcal{R}}$ represents the Riemannian curvature tensor associated with semi-Zamkovoy connection is described as:

$$\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} = \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\mathcal{Z} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}}\mathcal{Z} - \nabla_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}. \tag{3.1}$$

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z} = \hat{\nabla}_{\mathcal{X}}\hat{\nabla}_{\mathcal{Y}}\mathcal{Z} - \hat{\nabla}_{\mathcal{Y}}\hat{\nabla}_{\mathcal{X}}\mathcal{Z} - \hat{\nabla}_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}. \tag{3.2}$$

For every $\mathcal{X},\mathcal{Y},\mathcal{Z} \in M$. Based on (2.1) to (2.8) in equation (1.2), we exhibit

$$\begin{aligned}\hat{\nabla}_{\mathcal{X}}\hat{\nabla}_{\mathcal{Y}}\mathcal{Z} = {} & \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\mathcal{Z} + \mathrm{g}(\mathcal{X},\nabla_{\mathcal{Y}}\mathcal{Z})\xi - \eta(\nabla_{\mathcal{Y}}\mathcal{Z})\mathcal{X} + \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z})\xi + \mathrm{g}(\mathcal{Y},\nabla_{\mathcal{X}}\mathcal{Z})\xi \\ & + \mathrm{g}(\mathcal{Y},\mathcal{Z})\nabla_{\mathcal{X}}\xi + \eta(\mathcal{X})\mathrm{g}(\mathcal{Y},\mathcal{Z})\xi - \mathrm{g}(\mathcal{Y},\mathcal{Z})\mathcal{X} - \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Z},\xi)\mathcal{Y} - \mathrm{g}(\mathcal{Z},\nabla_{\mathcal{X}}\xi)\mathcal{Y} \\ & - \eta(\mathcal{Z})\nabla_{\mathcal{X}}\mathcal{Y} - \eta(\mathcal{Z})\mathrm{g}(\mathcal{X},\mathcal{Y})\xi + \eta(\mathcal{Y})\eta(\mathcal{Z})\mathcal{X},\end{aligned} \tag{3.3}$$

and

$$\hat{\nabla}_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z} = \nabla_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z} + \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z})\xi - \mathrm{g}(\nabla_{\mathcal{Y}}\mathcal{X},\mathcal{Z})\xi - \eta(\mathcal{Z})\nabla_{\mathcal{X}}\mathcal{Y} + \eta(\mathcal{Z})\nabla_{\mathcal{Y}}\mathcal{X}. \tag{3.4}$$

By applying equations (3.3) and (3.4) into equation (3.2), then after using (2.7), we derive the Riemannian curvature tensor $\hat{\mathcal{R}}$ of M corresponding to the semi-Zamkovoy connection $\hat{\nabla}$ as follows

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z} = \mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} + \mathrm{g}(\mathcal{Y},\mathcal{Z})\mathcal{X} - \mathrm{g}(\mathcal{X},\mathcal{Z})\mathcal{Y}, \tag{3.5}$$

where

$$\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} = \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\mathcal{Z} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}}\mathcal{Z} - \nabla_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}.$$

Equation (3.5) presents the relationship among the Riemannian curvature tensor defined with respect to the semi-Zamkovoy connection and the Levi-Civita connection. Also, we have

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y})\xi = \hat{\mathcal{R}}(\xi,\mathcal{X})\xi = 0. \tag{3.6}$$

On contracting both sides of (3.5) with the vector field V, one finds

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) = \mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) + \mathrm{g}(\mathcal{Y},\mathcal{Z})\mathrm{g}(\mathcal{X},V) - \mathrm{g}(\mathcal{X},\mathcal{Z})\mathrm{g}(\mathcal{Y},V), \tag{3.7}$$

where

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) = \mathrm{g}(\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z},V),$$

and

$$\mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) = \mathrm{g}(\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z},V).$$

By setting $\mathcal{X}$ and V as e_i in equation (3.7) and summing over $i = 1$ to n, we obtain the following result

$$\hat{\mathcal{S}}(\mathcal{Y},\mathcal{Z}) = \mathcal{S}(\mathcal{Y},\mathcal{Z}) + (n-1)\mathrm{g}(\mathcal{Y},\mathcal{Z}), \tag{3.8}$$

where $\hat{\mathcal{S}}$ and $\mathcal{S}$ represent the Ricci tensor with regard to the corresponding connections $\hat{\nabla}$ and ∇ respectively.
Once again $\mathcal{Y} = \mathcal{Z} = e_i$ in (3.8)

$$\hat{r} = r + n(n-1), \tag{3.9}$$

where $\hat{r}$ and r represent the Scalar curvature with respect to the corresponding connections $\hat{\nabla}$ and ∇ respectively.

$$\hat{Q}\mathcal{Y} = Q\mathcal{Y} + (n-1)\mathcal{Y}, \tag{3.10}$$

where $\mathcal{S}(\mathcal{Y}, \mathcal{Z}) = \mathsf{g}(Q\mathcal{Y}, \mathcal{Z})$. On the other hand, $\hat{Q}$ and Q denote the Ricci operators corresponding to the connections $\hat{\nabla}$ and ∇, respectively.
The following two relations are also obtained.

Proposition 3.1. *Let M be an n-dimensional Kenmostu manifold endowed with a semi-Zamkovoy connection, then we have*

$$\hat{\mathcal{S}}(\mathcal{Y}, \xi) = \mathcal{S}(\mathcal{Y}, \xi) + (n-1)\mathsf{g}(\mathcal{Y}, \xi), \tag{3.11}$$

and

$$\hat{\mathcal{S}}(\xi, \mathcal{Y}) = \mathcal{S}(\xi, \mathcal{Y}) + (n-1)\mathsf{g}(\xi, \mathcal{Y}), \tag{3.12}$$

then, we get

$$\hat{\mathcal{S}}(\mathcal{Y}, \xi) = \hat{\mathcal{S}}(\xi, \mathcal{Y}) = \hat{Q}\xi = 0. \tag{3.13}$$

Theorem 3.2. *For a Kenmotsu manifold M, the curvature tensor corresponding to the semi-Zamkovoy connection fulfills Bianchi's first identity*

$$\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} + \hat{\mathcal{R}}(\mathcal{Y}, \mathcal{Z})\mathcal{X} + \hat{\mathcal{R}}(\mathcal{Z}, \mathcal{X})\mathcal{Y} = 0. \tag{3.14}$$

Proof. By taking cyclic permutations of the vector fields $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ in equation (3.5), two additional equations are derived

$$\hat{\mathcal{R}}(\mathcal{Y}, \mathcal{Z})\mathcal{X} = \mathcal{R}(\mathcal{Y}, \mathcal{Z})\mathcal{X} + \mathsf{g}(\mathcal{Z}, \mathcal{X})\mathcal{Y} - \mathsf{g}(\mathcal{Y}, \mathcal{X})\mathcal{Z}, \tag{3.15}$$

and

$$\hat{\mathcal{R}}(\mathcal{Z}, \mathcal{X})\mathcal{Y} = \mathcal{R}(\mathcal{Z}, \mathcal{X})\mathcal{Y} + \mathsf{g}(\mathcal{X}, \mathcal{Y})\mathcal{Z} - \mathsf{g}(\mathcal{Z}, \mathcal{Y})\mathcal{X}. \tag{3.16}$$

By summing (3.5), (3.15), and (3.16) applying the identity $\mathcal{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} + \mathcal{R}(\mathcal{Y}, \mathcal{Z})\mathcal{X} + \mathcal{R}(\mathcal{Z}, \mathcal{X})\mathcal{Y} = 0$, the desired relation is obtained

$$\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} + \hat{\mathcal{R}}(\mathcal{Y}, \mathcal{Z})\mathcal{X} + \hat{\mathcal{R}}(\mathcal{Z}, \mathcal{X})\mathcal{Y} = 0. \tag{3.17}$$

Hence, the curvature tensor corresponding to Kenmotsu manifold on a semi-Zamkovoy connection fulfills the first Bianchi identity. □

Theorem 3.3. *Let a Kenmotsu manifold endowed with a semi-Zamkovoy connection, then its curvature tensor with a semi-Zamkovoy connection $\hat{\mathcal{R}}$ satisfies*

(i) it is skew-symmetric in the first two arguments, i.e., $\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, V) = -\hat{\mathcal{R}}(\mathcal{Y}, \mathcal{X}, \mathcal{Z}, V)$,

(ii) it is skew-symmetric in the last two arguments i.e., $\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, V) = -\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y}, V, \mathcal{Z})$,

(iii) it is symmetric under the interchange of the first and last pairs of arguments i.e., $\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, V) = \hat{\mathcal{R}}(\mathcal{Z}, V, \mathcal{X}, \mathcal{Y})$.

Proof.
1. Upon exchanging $\mathcal{X}$ and $\mathcal{Y}$ in (3.7)

$$\hat{\mathcal{R}}(\mathcal{Y}, \mathcal{X}, \mathcal{Z}, V) = \mathcal{R}(\mathcal{Y}, \mathcal{X}, \mathcal{Z}, V) + \mathsf{g}(\mathcal{X}, \mathcal{Z})\mathsf{g}(\mathcal{Y}, V) - \mathsf{g}(\mathcal{Y}, \mathcal{Z})\mathsf{g}(\mathcal{X}, V). \tag{3.18}$$

By summing (3.7) and (3.18) and using $\mathcal{R}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, V) + \mathcal{R}(\mathcal{Y}, \mathcal{X}, \mathcal{Z}, V) = 0$, then

$$\hat{\mathcal{R}}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, V) = -\hat{\mathcal{R}}(\mathcal{Y}, \mathcal{X}, \mathcal{Z}, V). \tag{3.19}$$

2. Upon exchanging $\mathcal{Z}$ and V in (3.7)

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y},V,\mathcal{Z}) = \mathcal{R}(\mathcal{X},\mathcal{Y},V,\mathcal{Z}) + \mathrm{g}(\mathcal{Y},V)\mathrm{g}(\mathcal{X},\mathcal{Z}) - \mathrm{g}(\mathcal{X},V)\mathrm{g}(\mathcal{Y},\mathcal{Z}). \tag{3.20}$$

By summing (3.7) and (3.20) and using $\mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) + \mathcal{R}(\mathcal{X},\mathcal{Y},V,\mathcal{Z}) = 0$, then

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) = -\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y},V,\mathcal{Z}). \tag{3.21}$$

3. Upon exchanging pair of arguments in (3.7)

$$\hat{\mathcal{R}}(\mathcal{Z},V,\mathcal{X},\mathcal{Y}) = \mathcal{R}(\mathcal{Z},V,\mathcal{X},\mathcal{Y}) + \mathrm{g}(V,\mathcal{X})\mathrm{g}(\mathcal{Z},\mathcal{Y}) - \mathrm{g}(\mathcal{Z},\mathcal{X})\mathrm{g}(V,\mathcal{Y}). \tag{3.22}$$

By summing (3.7) and (3.22) and using $\mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) - \mathcal{R}(\mathcal{Z},V,\mathcal{X},\mathcal{Y}) = 0$, then

$$\hat{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{Z},V) = \hat{\mathcal{R}}(\mathcal{Z},V,\mathcal{X},\mathcal{Y}). \tag{3.23}$$

□

4 Ricci Solitons on Kenmotsu Manifolds with respect to semi-Zamkovoy Connection

In the following section, we investigate various types of Ricci solitons on an n-dimensional Kenmotsu manifold M associated with the semi-Zamkovoy connection.
Let an n-dimensional Kenmotsu manifold M admit the semi-Zamkovoy connection $\hat{\nabla}$ is metric, i.e., $\hat{\nabla}\mathrm{g} = 0$, and its torsion tensor $\hat{T} \neq 0$ then we obtain

$$(\hat{\mathcal{L}}_{\mathcal{Y}}\mathrm{g})(\mathcal{X},\mathcal{Z}) = \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z}) + \mathrm{g}(\mathcal{X},\nabla_{\mathcal{Z}}\mathcal{Y}) = (\mathcal{L}_{\mathcal{Y}}\mathrm{g})(\mathcal{X},\mathcal{Z}). \tag{4.1}$$

Here, $\mathcal{L}$ and $\hat{\mathcal{L}}$ denote the Lie derivatives on the n-dimensional Kenmotsu manifold M with respect to the Levi-Civita connection and the semi-Zamkovoy connection, respectively.
We examine (g,ξ,λ) an almost Ricci soliton on an n-dimensional Kenmotsu manifold with respect to semi-Zamkovoy connection, then in view of (1.1), thus

$$(\hat{\mathcal{L}}_{\mathcal{Y}}\mathrm{g})(\mathcal{X},\mathcal{Z}) + 2\hat{\mathcal{S}}(\mathcal{X},\mathcal{Z}) + 2\lambda\mathrm{g}(\mathcal{X},\mathcal{Z}) = 0. \tag{4.2}$$

As established in (4.1) and (4.2), yield

$$\mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z}) + \mathrm{g}(\mathcal{X},\nabla_{\mathcal{Z}}\mathcal{Y}) + 2\hat{\mathcal{S}}(\mathcal{X},\mathcal{Z}) + 2\lambda\mathrm{g}(\mathcal{X},\mathcal{Z}) = 0. \tag{4.3}$$

Setting $\mathcal{Y} = \xi$ in (4.3) and using (2.7)

$$\hat{\mathcal{S}}(\mathcal{X},\mathcal{Z}) = \eta(\mathcal{X})\eta(\mathcal{Z}) - (1+\lambda)\mathrm{g}(\mathcal{X},\mathcal{Z}). \tag{4.4}$$

Now view of (3.8), we obtain

$$\mathcal{S}(\mathcal{X},\mathcal{Z}) = \eta(\mathcal{X})\eta(\mathcal{Z}) - (\lambda+n)\mathrm{g}(\mathcal{X},\mathcal{Z}). \tag{4.5}$$

Let M be an n-dimensional Kenmotsu manifold which is η-Einstein manifold with respect to the semi-Zamkovoy connection then for $\mathcal{Z} = \xi$ in (4.5), it follows that

$$\mathcal{S}(\mathcal{X},\xi) = (1-\lambda-n)\eta(\mathcal{X}). \tag{4.6}$$

Now accordingly (2.9), we get

$$\lambda\eta(\mathcal{X}) = 0. \tag{4.7}$$

As $\eta(\mathcal{X}) \neq 0$ so $\lambda = 0$. As a result, the following theorem can be expressed.

Theorem 4.1. *An almost Ricci soliton is always steady with respect to the semi-Zamkovoy connection on an n-dimensional Kenmotsu manifold.*

Contracting with respect to $\mathcal{X}$ and $\mathcal{Z}$ in (4.4)

$$\hat{r} = 1 - n(1+\lambda). \tag{4.8}$$

As a result, the following theorem can be expressed.

Theorem 4.2. *The scalar curvature of an n-dimensional Kenmotsu manifold M with an almost Ricci soliton with respect to the semi-Zamkovoy connection is $\hat{r} = 1 - n(1+\lambda)$.*

Now by (2.9) Einstein manifold with respect to the Levi-Civita connection, and while the Ricci tensor is of the particular condition, we have

$$\mathcal{S}(\mathcal{X},\mathcal{Z}) = -(n-1)\mathrm{g}(\mathcal{X},\mathcal{Z}), \tag{4.9}$$

and we have

$$\hat{\mathcal{S}}(\mathcal{X},\mathcal{Z}) = \mathcal{S}(\mathcal{X},\mathcal{Z}) + (n-1)\mathrm{g}(\mathcal{X},\mathcal{Z}). \tag{4.10}$$

In view of (4.9), we get

$$\hat{\mathcal{S}}(\mathcal{X},\mathcal{Z}) = 0. \tag{4.11}$$

Combining (4.3) and (4.11),

$$\mathrm{g}(\hat{\nabla}_{\mathcal{X}}\xi,\mathcal{Z}) + \mathrm{g}(\mathcal{X},\hat{\nabla}_{\mathcal{Z}}\xi) + 2\lambda\mathrm{g}(\mathcal{X},\mathcal{Z}) = 0. \tag{4.12}$$

Setting $\mathcal{X} = \mathcal{Z} = \xi$ in (4.12) and by (2.1) and (2.2), we get the following relation $\mathrm{g}(\hat{\nabla}_{\xi}\xi,\xi) = -\lambda$ However, we have $\mathrm{g}(\hat{\nabla}_{X}\xi,\xi) = 0$. Thus, we acquire (i) now as a result, (4.12) yields

$$\mathrm{g}(\hat{\nabla}_{\mathcal{X}}\xi,\mathcal{Z}) + \mathrm{g}(\mathcal{X},\hat{\nabla}_{\mathcal{Z}}\xi) = 0. \tag{4.13}$$

Place $\mathcal{X} = \xi$ in (4.13) and by (2.1)–(2.7), then we obtain $\mathrm{g}(\hat{\nabla}_{\xi}\xi,\mathcal{Z}) = 0$ with every vector field $\mathcal{Z}$ on M thus, we get $\nabla_{\xi}\xi = 0$ that is ξ is a geodesic vector field so we acquire (ii).
As a result, the following theorem can be expressed.

Theorem 4.3. *States that if (g,ξ,λ) is a Ricci soliton on a Kenmotsu manifold M with respect to the semi-Zamkovoy connection, then*

(i) λ=0

(ii) ξ *defines a geodesic vector field.*

As a consequence of (1.7) and (4.2), we obtain

$$\begin{aligned}(\hat{\mathcal{L}}_{Y}\mathrm{g})(\mathcal{X},\mathcal{Z}) &= \mathrm{g}(\hat{\nabla}_{\mathcal{X}}Y,\mathcal{Z}) + \mathrm{g}(\mathcal{X},\hat{\nabla}_{\mathcal{Z}}\mathcal{Y}) \\ &= \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Y} + \mathrm{g}(\mathcal{X},\mathcal{Y})\xi - \eta(\mathcal{Y})\mathcal{X},\mathcal{Z}) \\ &+ \mathrm{g}(\mathcal{X},\nabla_{\mathcal{Z}}\mathcal{Y} + \mathrm{g}(\mathcal{Z},\mathcal{Y})\xi - \eta(\mathcal{Y})\mathcal{Z}) \\ &= \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z}) + \mathrm{g}(\mathcal{X},\nabla_{\mathcal{Z}}\mathcal{Y}) + \eta(\mathcal{Z})\mathrm{g}(\mathcal{X},\mathcal{Y}) - \mathrm{g}(\mathcal{X},\mathcal{Z})\eta(\mathcal{Y}) \\ &+ \eta(\mathcal{X})\mathrm{g}(\mathcal{Y},\mathcal{Z}) - \mathrm{g}(\mathcal{X},\mathcal{Z})\eta(\mathcal{Y}).\end{aligned} \tag{4.14}$$

In view of (4.1) and (4.14), we get

$$(\hat{\mathcal{L}}_{\mathcal{Y}}\mathrm{g})(\mathcal{X},\mathcal{Z}) = (\mathcal{L}_{\mathcal{Y}}\mathrm{g})(\mathcal{X},\mathcal{Z}) + \eta(\mathcal{Z})\mathrm{g}(\mathcal{X},\mathcal{Y}) + \eta(\mathcal{X})\mathrm{g}(\mathcal{Y},\mathcal{Z}) - 2\mathrm{g}(\mathcal{X},\mathcal{Z})\eta(\mathcal{Y}). \tag{4.15}$$

By (4.2) and (4.15) and applying (3.8), yields

$$\begin{aligned}(\mathcal{L}_{\mathcal{Y}}\mathrm{g})(\mathcal{X},\mathcal{Z}) + 2\mathcal{S}(\mathcal{X},\mathcal{Z}) + 2\lambda\mathrm{g}(\mathcal{X},\mathcal{Z}) + \eta(\mathcal{Z})\mathrm{g}(\mathcal{X},\mathcal{Y}) \\ + \eta(\mathcal{X})\mathrm{g}(\mathcal{Y},\mathcal{Z}) + 2\{(n-1) - \eta(\mathcal{Y})\}\mathrm{g}(\mathcal{X},\mathcal{Z}) = 0.\end{aligned} \tag{4.16}$$

For a Kenmotsu manifold endowed with a Ricci soliton $(\mathrm{g},\mathcal{Y},\lambda)$ relative to the Levi-Civita connection, (1.1) necessarily holds. Hence, combining (1.1) and (4.16), we arrive at the following result

$$\eta(\mathcal{Z})\mathrm{g}(\mathcal{X},Y) + \eta(\mathcal{X})\mathrm{g}(\mathcal{Y},\mathcal{Z}) + 2\{n - 1 - \eta(\mathcal{Y})\}\mathrm{g}(\mathcal{X},\mathcal{Z}) = 0. \tag{4.17}$$

As a result, the following theorem can be expressed.

Theorem 4.4. *A Ricci soliton $(\mathrm{g},\mathcal{Y},\lambda)$ on a Kenmotsu manifold is associated with respect to a semi-Zamkovoy connection if and only if the condition below is satisfied.*

$$\eta(\mathcal{Z})\mathrm{g}(\mathcal{X},\mathcal{Y}) + \eta(\mathcal{X})\mathrm{g}(\mathcal{Y},\mathcal{Z}) + 2\{n - 1 - \eta(\mathcal{Y})\}\mathrm{g}(\mathcal{X},\mathcal{Z}) = 0.$$

It is satisfied for all arbitrary vector fields $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ defined on M.

Now from (4.2), which it follows that

$$g(\hat{\nabla}_{\mathcal{X}}\xi, \mathcal{Z}) + g(\mathcal{X}, \hat{\nabla}_{\mathcal{Z}}\xi) + 2\hat{\mathcal{S}}(\mathcal{X}, \mathcal{Z}) + 2\lambda g(\mathcal{X}, \mathcal{Z}) = 0. \tag{4.18}$$

Applying (2.12) to (4.18), it follows that

$$\hat{\mathcal{S}}(\mathcal{X}, \mathcal{Z}) + \lambda g(\mathcal{X}, \mathcal{Z}) = 0. \tag{4.19}$$

Moreover, applying (3.8) to (4.19), it follows that

$$\mathcal{S}(\mathcal{X}, \mathcal{Z}) + (n - 1 + \lambda)g(\mathcal{X}, \mathcal{Z}) = 0. \tag{4.20}$$

Contracting with $\mathcal{X}$ and $\mathcal{Z}$ in (4.20), Upon summing over $i, 1 \leq i \leq n$, it follows that

$$\lambda = -\frac{(r + (n - 1)n)}{n}. \tag{4.21}$$

As a result, in view (4.20) and (4.21), the following theorem can be expressed.

Theorem 4.5. *Let M be a Kenmotsu manifold admitting a Ricci soliton with respect to the semi-Zamkovoy connection. Then M is necessarily an Einstein manifold, and the Ricci soliton is shrinking, steady, or expanding according as if $r > -n(n-1)$ then $\lambda < 0$, if $r = -n(n-1)$ then $\lambda = 0$, and $r < -n(n-1)$ then $\lambda > 0$ respectively.*

5 η-Ricci Solitons on Kenmotsu Manifolds with respect to semi-Zamkovoy Connection

In the following section, we investigate various types of η-Ricci solitons on an n-dimensional Kenmotsu manifold M associated with the semi-Zamkovoy connection.
In view of (1.2), the η-Ricci soliton on an n-dimensional Kenmotsu manifold with respect to the semi-Zamkovoy connection, we have

$$\mathcal{L}_{\mathcal{Y}}g + 2\mathcal{S} + 2\lambda g + 2\kappa\eta \otimes \eta = 0. \tag{5.1}$$

In view of (4.1) and (5.1)

$$g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) + g(\mathcal{X}, \nabla_{\mathcal{Z}}\mathcal{Y}) + 2\hat{\mathcal{S}}(\mathcal{X}, \mathcal{Z}) + 2\lambda g(\mathcal{X}, \mathcal{Z}) + 2\kappa\eta(\mathcal{X})\eta(\mathcal{Z}) = 0. \tag{5.2}$$

Setting $\mathcal{Y} = \xi$ and from (2.7) in (5.2), we derive

$$\hat{\mathcal{S}}(\mathcal{X}, \mathcal{Z}) = -(1 + \lambda)g(\mathcal{X}, \mathcal{Z}) - (\kappa - 1)\eta(\mathcal{X})\eta(\mathcal{Z}). \tag{5.3}$$

As a result, the following theorem can be expressed.

Theorem 5.1. *Suppose M is an n-dimensional Kenmotsu manifold admitting an η-Ricci soliton. It follows that M is an η-Einstein manifold with respect to the semi-Zamkovoy connection.*

Now by (5.3) and (4.10), then

$$\mathcal{S}(\mathcal{X}, \mathcal{Z}) = -(\lambda + n)\eta(\mathcal{X}) - (\kappa - 1)\eta(\mathcal{X}). \tag{5.4}$$

Setting $\mathcal{Z} = \xi$ in (5.4), we obtain

$$\mathcal{S}(\mathcal{X}, \xi) = -(\lambda + n)\eta(\mathcal{X}) - (\kappa - 1)\eta(\mathcal{X}). \tag{5.5}$$

In view of (2.9) then after simplification, we get

$$(\lambda + \kappa)\eta(\mathcal{X}) = 0. \tag{5.6}$$

As $\eta \neq 0$ therefore, $(\lambda + \kappa) = 0$. As a result, the following theorem can be expressed.

Theorem 5.2. *Let M be an n-dimensional Kenmotsu manifold admitting an η-Ricci soliton with a semi-Zamkovoy connection, then the scalar value λ and κ are connected by the relation $(\lambda + \kappa) = 0$.*

Making use of $\lambda = -\kappa$ in the (5.3), we deduce that

$$\hat{\mathcal{S}}(\mathcal{X}, \mathcal{Z}) = (\kappa - 1)\mathrm{g}(\mathcal{X}, \mathcal{Z}) - (\kappa - 1)\eta(\mathcal{X})\eta(\mathcal{Z}), \tag{5.7}$$

which is of the form

$$\hat{\mathcal{S}}(\mathcal{X}, \mathcal{Z}) = \gamma_1 \mathrm{g}(\mathcal{X}, \mathcal{Z}) + \gamma_2 \eta(\mathcal{X})\eta(\mathcal{Z}), \tag{5.8}$$

where $\gamma_1 = (\kappa - 1),\ \gamma_2 = -(\kappa - 1)$ and soliton constants satisfy $\lambda = -\kappa$.

Corollary 5.3. *Let M be an n-dimensional Kenmotsu manifold admitting an η-Ricci soliton with respect to the semi-Zamkovoy connection. Then M is an η-Einstein manifold.*

Moreover, the η-Ricci soliton is shrinking, steady, or expanding according as $\kappa > 0,\ \kappa = 0,\ \kappa < 0$, respectively.

6 Yamabe Solitons on Kenmotsu Manifolds with respect to semi-Zamkovoy Connection

In the following section, we investigate various types of Yamabe solitons on an n-dimensional Kenmotsu manifold M with the semi-Zamkovoy connection.
In view of (1.4), we have

$$\frac{1}{2}(\hat{\mathcal{L}}_{\mathcal{T}}\mathrm{g})(\mathcal{X}, \mathcal{Z}) = (\hat{r} - \mu)\mathrm{g}(\mathcal{X}, \mathcal{Z}). \tag{6.1}$$

From (4.1), we get

$$\frac{1}{2}(\mathcal{L}_{\mathcal{T}}\mathrm{g})(\mathcal{X}, \mathcal{Z}) = (\hat{r} - \mu)\mathrm{g}(\mathcal{X}, \mathcal{Z}). \tag{6.2}$$

Applying (3.9) to (6.2), it follows that

$$\frac{1}{2}(\mathcal{L}_{\mathcal{T}}\mathrm{g})(\mathcal{X}, \mathcal{Z}) = (r + n(n-1) - \mu)\mathrm{g}(\mathcal{X}, \mathcal{Z}). \tag{6.3}$$

As a result, the following theorem can be expressed.

Theorem 6.1. *An almost Yamabe soliton on an n-dimensional Kenmotsu manifold associated with respect to the semi-Zamkovoy connection if and only if $n(n-1) = 0$.*

Now consider

$$\mathcal{L}_{\mathcal{T}}\mathrm{g}(\mathcal{X}, \mathcal{Z}) = \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{T}, \mathcal{Z}) + \mathrm{g}(\mathcal{X}, \nabla_{\mathcal{Z}}\mathcal{T}). \tag{6.4}$$

By using (1.5), yields

$$\mathcal{L}_{\mathcal{T}}\mathrm{g}(\mathcal{X}, \mathcal{Z}) = 2f\mathrm{g}(\mathcal{X}, \mathcal{Z}) + \omega(\mathcal{X})\mathrm{g}(\mathcal{T}, \mathcal{Z}) + \omega(\mathcal{Z})\mathrm{g}(\mathcal{T}, \mathcal{X}). \tag{6.5}$$

Now using (1.4) and (1.7), yields

$$\begin{aligned}(\hat{\mathcal{L}}_{\mathcal{T}}\mathrm{g})(\mathcal{X}, \mathcal{Z}) &= \mathrm{g}(\hat{\nabla}_{\mathcal{X}}\mathcal{T}, \mathcal{Z}) + \mathrm{g}(\mathcal{X}, \hat{\nabla}_{\mathcal{Z}}\mathcal{T}) \\ &= \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{T} + \mathrm{g}(\mathcal{X}, \mathcal{T})\xi - \eta(\mathcal{T})\mathcal{X}, \mathcal{Z}) \\ &+ \mathrm{g}(\mathcal{X}, \nabla_{\mathcal{Z}}\mathcal{T} + \mathrm{g}(\mathcal{Z}, \mathcal{T})\xi - \eta(\mathcal{T})\mathcal{Z}) \\ &= \mathrm{g}(\nabla_{\mathcal{X}}\mathcal{T}, \mathcal{Z}) + \mathrm{g}(\mathcal{X}, \nabla_{\mathcal{Z}}\mathcal{T}) + \eta(\mathcal{Z})\mathrm{g}(\mathcal{X}, \mathcal{T}) - \mathrm{g}(\mathcal{X}, \mathcal{Z})\eta(\mathcal{T}) \\ &+ \eta(\mathcal{X})\mathrm{g}(\mathcal{T}, \mathcal{Z}) - \mathrm{g}(\mathcal{X}, \mathcal{Z})\eta(\mathcal{T}).\end{aligned} \tag{6.6}$$

In view of (4.8) and (6.1) in (6.6), we get

$$\begin{aligned}\frac{1}{2}(\mathcal{L}_{\mathcal{T}}\mathrm{g})(\mathcal{X}, \mathcal{Z}) &= (r - \mu)\mathrm{g}(\mathcal{X}, \mathcal{Z}) + \{n(n-1) + \eta(\mathcal{T})\}\mathrm{g}(\mathcal{X}, \mathcal{Z}) \\ &- \frac{1}{2}\{\eta(\mathcal{Z})\mathrm{g}(\mathcal{X}, \mathcal{T}) + \eta(\mathcal{X})\mathrm{g}(\mathcal{Z}, \mathcal{T})\}.\end{aligned} \tag{6.7}$$

Setting (6.5) and (6.7), we obtain

$$\{r-\mu-f+n(n-1)+\eta(\mathcal{T})\}\mathsf{g}(\mathcal{X},\mathcal{Z})-\frac{1}{2}\{\omega(\mathcal{Z})+\eta(\mathcal{Z})\}\mathsf{g}(\mathcal{T},\mathcal{X}) - \frac{1}{2}\{\omega(\mathcal{X})+\eta(\mathcal{X})\}\mathsf{g}(\mathcal{Z},\mathcal{T})=0. \quad (6.8)$$

Upon taking the contraction of (6.8) over $\mathcal{X}$ and $\mathcal{Y}$, we arrive

$$n\{r-\mu-f+n(n-1)\}+n(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})=0. \quad (6.9)$$

As a consequence, we have

$$r-f+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}\lessgtr 0, \quad (6.10)$$

whenever $r-f+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}$ is constant. As a result, the following result can be expressed.

Lemma 6.2. *If $\mathcal{T}$ is torse forming, then the Yamabe soliton on M with respect to the semi-Zamkovoy $\hat{\nabla}$ is shrinking, steady, and expanding depending upon*

$$r-f+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}\lessgtr 0.$$

According the result of (6.9), therefore the following theorem can be expressed.

Theorem 6.3. *Consider $(\mathsf{g},\mathcal{T},\mu)$ be a Yamabe soliton on M with respect to the semi-Zamkovoy $\hat{\nabla}$ therefore, it follows that*

$\mathcal{T}$	Condition of existence	Conditions of shrinking, steady and expanding
torse-forming	$r-f+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}=$ constant	$r-f+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}\lessgtr 0$
concircular	$r-f+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}=$ constant	$r-f+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}\lessgtr 0$
concurrent	$r-1+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}=$ constant	$r-1+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}\lessgtr 0$
recurrent	$r+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}=$ constant	$r+n(n-1)+\frac{1}{n}\{(n-1)\eta(\mathcal{T})-\omega(\mathcal{T})\}\lessgtr 0$
parallel	$r+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}=$ constant	$r+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}\lessgtr 0$
torqued	$r-f+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}=$ constant	$r-f+(n-1)\{n+\frac{1}{n}\eta(\mathcal{T})\}\lessgtr 0$

Table 1. Existence and classification conditions corresponding to different types of vector fields.

7 Example of a three-dimensional Kenmotsu Manifolds with respect to semi-Zamkovoy Connection

Examine a Kenmotsu manifold M with three-dimensions that is $M=\{(u,v,w)\in\mathcal{R}^3\}$. Where u,v,w is the standard vector coordinates in $\mathcal{R}^3$ then the vector fields is chosen,

$$\mathsf{e}_1=\mathsf{e}^{-w}(1+u)\frac{\partial}{\partial u},\quad \mathsf{e}_2=\mathsf{e}^{-w}(1+v)\frac{\partial}{\partial v},\quad \mathsf{e}_3=\frac{\partial}{\partial w}.$$

The vectors are linearly independent for any point of M^n.
The Lorentzian metric can be described by

$$\begin{aligned}&\mathsf{g}(\mathsf{e}_1,\mathsf{e}_1)=1,\quad \mathsf{g}(\mathsf{e}_2,\mathsf{e}_2)=1,\quad \mathsf{g}(\mathsf{e}_3,\mathsf{e}_3)=1,\\ &\mathsf{g}(\mathsf{e}_1,\mathsf{e}_2)=0,\quad \mathsf{g}(\mathsf{e}_2,\mathsf{e}_3)=0,\quad \mathsf{g}(\mathsf{e}_1,\mathsf{e}_3)=0.\end{aligned} \quad (7.1)$$

Assume that η is a particular 1-form as $\eta(\mathcal{X}) = \mathsf{g}(\mathcal{X}, \xi) = \mathsf{g}(\mathcal{X}, \mathsf{e}_3) \quad \forall \mathcal{X}$.
The $(1,1)$-tensor field ϕ is described by

$$\phi(\mathsf{e}_1) = -\mathsf{e}_2, \quad \phi(\mathsf{e}_2) = -\mathsf{e}_1, \quad \phi(\mathsf{e}_3) = 0. \tag{7.2}$$

By virtue of ϕ and η is the 1-form satisfying

$$\eta(\mathsf{e}_3) = 1, \quad \phi^2\mathcal{X} = -\mathcal{X} + \eta(\mathcal{X})\xi, \quad \mathsf{g}(\phi\mathcal{X}, \phi\mathcal{Y}) = \mathsf{g}(\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{X})\eta(\mathcal{Y}).$$

From explicit computations, it follows that

$$[\mathsf{e}_1, \mathsf{e}_2] = 0, \quad [\mathsf{e}_1, \mathsf{e}_3] = \mathsf{e}_1, \quad [\mathsf{e}_2, \mathsf{e}_3] = \mathsf{e}_2. \tag{7.3}$$

With the help of Koszul's formula, we obtain

$$\begin{aligned} \nabla_{\mathsf{e}_1}\mathsf{e}_1 &= -\mathsf{e}_3, \quad \nabla_{\mathsf{e}_1}\mathsf{e}_2 = 0, \quad \nabla_{\mathsf{e}_1}\mathsf{e}_3 = \mathsf{e}_1, \\ \nabla_{\mathsf{e}_2}\mathsf{e}_1 &= 0, \quad \nabla_{\mathsf{e}_2}\mathsf{e}_2 = -\mathsf{e}_3, \quad \nabla_{\mathsf{e}_2}\mathsf{e}_3 = \mathsf{e}_2, \\ \nabla_{\mathsf{e}_3}\mathsf{e}_1 &= 0, \quad \nabla_{\mathsf{e}_3}\mathsf{e}_2 = 0, \quad \nabla_{\mathsf{e}_3}\mathsf{e}_3 = 0. \end{aligned} \tag{7.4}$$

Consequently, M is a Kenmotsu manifold then combining (1.7) and (7.4), we deduce

$$\begin{aligned} \hat{\nabla}_{\mathsf{e}_1}\mathsf{e}_1 &= 0, \quad \hat{\nabla}_{\mathsf{e}_1}\mathsf{e}_2 = 0, \quad \hat{\nabla}_{\mathsf{e}_1}\mathsf{e}_3 = 0, \\ \hat{\nabla}_{\mathsf{e}_2}\mathsf{e}_1 &= 0, \quad \hat{\nabla}_{\mathsf{e}_2}\mathsf{e}_2 = 0, \quad \hat{\nabla}_{\mathsf{e}_2}\mathsf{e}_3 = 0, \\ \hat{\nabla}_{\mathsf{e}_3}\mathsf{e}_1 &= 0, \quad \hat{\nabla}_{\mathsf{e}_3}\mathsf{e}_2 = 0, \quad \hat{\nabla}_{\mathsf{e}_3}\mathsf{e}_3 = 0. \end{aligned} \tag{7.5}$$

Then by (3.1), (7.3) and (7.4), we get

$$\begin{aligned} \mathcal{R}(\mathsf{e}_1, \mathsf{e}_2)\mathsf{e}_1 &= \mathsf{e}_2, \quad \mathcal{R}(\mathsf{e}_1, \mathsf{e}_2)\mathsf{e}_2 = -\mathsf{e}_1, \quad \mathcal{R}(\mathsf{e}_1, \mathsf{e}_2)\mathsf{e}_3 = 0, \\ \mathcal{R}(\mathsf{e}_1, \mathsf{e}_3)\mathsf{e}_1 &= \mathsf{e}_3, \quad \mathcal{R}(\mathsf{e}_1, \mathsf{e}_3)\mathsf{e}_2 = 0, \quad \mathcal{R}(\mathsf{e}_1, \mathsf{e}_3)\mathsf{e}_3 = -\mathsf{e}_1, \\ \mathcal{R}(\mathsf{e}_2, \mathsf{e}_3)\mathsf{e}_1 &= 0, \quad \mathcal{R}(\mathsf{e}_2, \mathsf{e}_3)\mathsf{e}_2 = \mathsf{e}_3, \quad \mathcal{R}(\mathsf{e}_2, \mathsf{e}_3)\mathsf{e}_3 = -\mathsf{e}_2. \end{aligned} \tag{7.6}$$

Moreover $\hat{\mathcal{R}}(\mathsf{e}_k, \mathsf{e}_i)\mathsf{e}_j$ for $i, j, k = 1, 2, 3$ of $\hat{\nabla}$, then from (3.2), (7.3) and (7.5).
We conclude that

$$\begin{aligned} \hat{\mathcal{R}}(\mathsf{e}_1, \mathsf{e}_2)\mathsf{e}_1 &= 0, \quad \hat{\mathcal{R}}(\mathsf{e}_1, \mathsf{e}_2)\mathsf{e}_2 = 0, \quad \hat{\mathcal{R}}(\mathsf{e}_1, \mathsf{e}_2)\mathsf{e}_3 = 0, \\ \hat{\mathcal{R}}(\mathsf{e}_1, \mathsf{e}_3)\mathsf{e}_1 &= 0, \quad \hat{\mathcal{R}}(\mathsf{e}_1, \mathsf{e}_3)\mathsf{e}_2 = 0, \quad \hat{\mathcal{R}}(\mathsf{e}_1, \mathsf{e}_3)\mathsf{e}_3 = 0, \\ \hat{\mathcal{R}}(\mathsf{e}_2, \mathsf{e}_3)\mathsf{e}_1 &= 0, \quad \hat{\mathcal{R}}(\mathsf{e}_2, \mathsf{e}_3)\mathsf{e}_2 = 0, \quad \hat{\mathcal{R}}(\mathsf{e}_2, \mathsf{e}_3)\mathsf{e}_3 = 0. \end{aligned} \tag{7.7}$$

Based on the obtained curvature tensors, yields

$$\mathcal{S}(\mathsf{e}_i, \mathsf{e}_j) = \sum_{k=1}^{3} \mathsf{g}(\mathcal{R}(\mathsf{e}_k, \mathsf{e}_i)\mathsf{e}_j, \mathsf{e}_k), \tag{7.8}$$

$$\text{and} \quad \hat{\mathcal{S}}(\mathsf{e}_i, \mathsf{e}_j) = \sum_{k=1}^{3} \mathsf{g}(\hat{\mathcal{R}}(\mathsf{e}_k, \mathsf{e}_i)\mathsf{e}_j, \mathsf{e}_k).$$

After expanding (7.8) then using (7.6) and (7.7), we obtain

$$\mathcal{S}(\mathsf{e}_1, \mathsf{e}_1) = -2, \quad \mathcal{S}(\mathsf{e}_2, \mathsf{e}_2) = -2, \quad \mathcal{S}(\mathsf{e}_3, \mathsf{e}_3) = -2, \tag{7.9}$$

and

$$\hat{\mathcal{S}}(\mathsf{e}_1, \mathsf{e}_1) = 0, \quad \hat{\mathcal{S}}(\mathsf{e}_2, \mathsf{e}_2) = 0, \quad \hat{\mathcal{S}}(\mathsf{e}_3, \mathsf{e}_3) = 0. \tag{7.10}$$

Based on the obtained scalar curvature, we conclude that.

$$r = \sum_{i=1}^{3} \mathcal{S}(\mathsf{e}_i, \mathsf{e}_i), \tag{7.11}$$

$$\text{and} \quad \hat{r} = \sum_{i=1}^{3} \hat{\mathcal{S}}(\mathsf{e}_i, \mathsf{e}_i).$$

From (7.11), then using (7.9) and (7.10), therefore, $r = -6$ and $\hat{r} = 0$. From (3.9), the result $\hat{r} = 0$ holds for three-dimensional manifolds ($n = 3$).
Using (7.9) and (7.10) in (4.10), therefore $\hat{S}(\mathcal{X}, \mathcal{Z}) = 0$ for $n = 3$, then (4.11) is satisfied.
From (4.4), we get $\hat{\mathcal{S}}(\mathsf{e}_3, \mathsf{e}_3) = -\lambda$; consequently, (7.10) implies that $\lambda = 0$, hence almost Ricci Soliton on an n-dimensional Kenmotsu manifold endowed with a $\hat{\nabla}$ is steady; therefore, the Theorem 4.1 is satisfied.
Using (7.4), we have $\nabla_{\mathsf{e}_3}\mathsf{e}_3 = 0$ and $\hat{\nabla}_{\mathsf{e}_3}\mathsf{e}_3 = 0$; therefore, ξ defines a geodesic vector field: Hence, the Theorem 4.3: (i) and (ii) are satisfied.
Furthermore, view of (5.3), we have $\hat{\mathcal{S}}(\mathsf{e}_3, \mathsf{e}_3) = -(\lambda + \kappa)$ using (7.10), which implies that $\lambda + \kappa = 0$.
Similarly, viewing (5.4) for $n = 3$, we obtain $\mathcal{S}(\mathsf{e}_3, \mathsf{e}_3) = -(\lambda+\kappa+2)$ using (7.9), which implies that $\lambda + \kappa = 0$: Hence, the Theorem 5.2 is satisfied.

8 Conclusions

In the present work, we have examined the geometric behavior of Kenmotsu manifolds endowed with the semi-Zamkovoy connection in the context of various soliton structures. First, it has been shown that the curvature tensor corresponding to the semi-Zamkovoy connection on a Kenmotsu manifold satisfies Bianchi's first identity. Further, the study of almost Ricci solitons reveals that such solitons are necessarily steady with respect to the semi-Zamkovoy connection. In this setting, the scalar curvature of an n-dimensional Kenmotsu manifold is explicitly determined as $\hat{r} = 1 - n(1 + \lambda)$, establishing a direct relation between curvature and the soliton constant also in the case of Ricci solitons, we proved that the soliton constant must vanish, i.e., $\lambda = 0$, and the characteristic vector field ξ turns out to be geodesic. It is further demonstrated that the Kenmotsu manifold admitting a Ricci soliton endowed with the semi-Zamkovoy connection becomes an Einstein manifold.
The nature of the soliton (shrinking, steady, or expanding) is completely determined by the scalar curvature, depending on whether $r > -n(n-1)$, $r = -n(n-1)$, or $r < -n(n-1)$, respectively. In addition, for η-Ricci solitons, the manifold is shown to be η-Einstein with respect to the semi-Zamkovoy connection, and the soliton parameters satisfy the relation $\lambda + \kappa = 0$, revealing a structural link between these constants. Furthermore, it is established that the existence of an almost Yamabe soliton on an n-dimensional Kenmotsu manifold with respect to the semi-Zamkovoy connection imposes the restrictive condition $n(n-1) = 0$, indicating strong geometric constraints. Finally, to support the present study and verify the obtained results, an example of a three-dimensional Kenmotsu manifold with respect to a semi-Zamkovoy connection has been constructed.
These results highlight the significant influence of the semi-Zamkovoy connection on the curvature properties and soliton structures of Kenmotsu manifolds, and provide a deeper understanding of the interplay between connection theory and geometric flows.

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