Palestine Journal of Mathematics

Vol.15(3), (2026) , 640–650 © Palestine Polytechnic University-PPU 2026

On the class of n-quasi-$[m, C]$-symmetric operators

Imene Djaber and Messaoud Guesba

Communicated by Fuad Kittaneh

MSC 2020 Classifications: Primary 47A05; Secondary 47A12; 47A30; 47B15.

Keywords and phrases: n-quasi-$[m, C]$-symmetric operator, perturbation, nilpotent operator.

The authors thank the editor and the referee for the helpful comments that improved our paper.

Corresponding Author: M. Guesba

Abstract. In this paper, we introduce and investigate a new class of bounded linear operators on Hilbert spaces, called n-quasi-$[m, C]$-symmetric operators, which generalizes of the $[m, C]$-symmetric operator, we define it as follows: an operator $T \in \mathcal{B}(\mathcal{H})$ is called n-quasi-$[m, C]$-symmetric operator if there exists some conjugation C such that

$$T^{*^n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{^{m-j}}CT^j\right)T^n = 0,$$

for some positive integers m and n.

Some basic structural properties of this class are established with the help of operator matrix representation. In particular, we obtain that a power of an n-quasi-$[m, C]$-symmetric operator is again an n-quasi-$[m, C]$-symmetric operator. Furthermore, we study the perturbation of an n-quasi-$[m, C]$-symmetric operator with a nilpotent operator. The product and tensor products of two n-quasi-$[m, C]$-symmetries are also investigated.

1 Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert with inner product $\langle ., . \rangle$, $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For every $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* the range, the null space and the adjoint of T, respectively. The closure of a set M will be denoted by $\overline{M}$.

In [12], Helton initiated the study of m-symmetric operator, for a positive integer m, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m-symmetric if form

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*^{m-j}}T^j = 0, \tag{1.1}$$

where $\binom{m}{j}$is the binomial coefficient. For more facts about this class of operators and some of its extensions, we refer the reader to [8, 11, 13].

An antilinear operator C on $\mathcal{H}$ is said to be conjugation if C satisfies $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. Recall that if C is a conjugation on $\mathcal{H}$, then $\|C\| = 1$, $(CTC)^k = CT^kC$ and $(CTC)^* = CT^*C$ for every positive integer k (see [9, 10] for more details). An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be complex symmetric if $CTC = T^*$. In light of complex symmetric operators, using the identity (1.1) and a conjugation C, Chō et al. [3] defined m-complex symmetric operators as follows: an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an m-complex symmetric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*^{m-j}}CT^jC = 0 \tag{1.2}$$

for some positive integer m. Several authors have studied the m-complex symmetric operator, we refer the reader to [2, 4, 5, 14] for further details.

In [6], Chō et al. introduced $[m, C]$-symmetric operators with conjugation C as follows, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an $[m, C]$-symmetric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}CT^j = 0$$

for some $m \in \mathbb{N}$.

Set

$$\delta_m (T, C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}CT^j.$$

Then, T is an $[m, C]$-symmetric operator if and only if $\delta_m (T, C) = 0$. Hence if T is complex symmetric and $[m, C]$-symmetric, then T is m-symmetric. It also holds that

$$CTC\delta_m (T, C) - \delta_m (T, C) T = \delta_{m+1} (T, C).$$

Moreover, if T is $[m, C]$-symmetric, then T is $[n, C]$-symmetric for all $n \geqslant m$.

In [16], Zuo and Mecheri, introduced the class of n-quasi-m-symmetric operators which generalizes the class of m-symmetric operators. For positive integers m and n, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an n-quasi-m-symmetric operator if

$$T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*^{m-j}} T^j \right) T^n = 0. \tag{1.3}$$

Further results on this class were established recently by Djaballah and Guesba (see [7]). Using (1.3) and a conjugation C, Al-Dohiman et al. [1] defined the class of n-quasi-m-complex symmetric operators as follows: an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an n-quasi-m-complex symmetric operator if there exists some conjugation C such that

$$T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*^{m-j}} CT^j C \right) T^n = 0 \tag{1.4}$$

for some positive integers m and n.

According to the definitions of complex symmetric, m-complex symmetric, $[m, C]$-symmetric, n-quasi-m-symmetric operator and n-quasi-m-complex symmetric, we define a new class of operators called n-quasi-$[m, C]$-symmetric operators with conjugation C as follows:

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an n-quasi-$[m, C]$-symmetric operator if there exists some conjugation C such that

$$T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}CT^j \right) T^n = 0$$

for some positive integers n and m.

In particular, for the case $m = 2$, an operator T is n-quasi-$[2, C]$-symmetric with conjugation C if

$$T^{*^n} \left(CT^2C - 2CTCT + T^2 \right) T^n = 0$$

and for $m = 3$, an operator T is n-quasi-$[3, C]$-symmetric with conjugation C if

$$T^{*^n} \left(CT^3C - 3CT^2CT + 3CTCT^2 - T^3 \right) T^n = 0.$$

Remark 1.2. (i) An 1-quasi-$[m, C]$-symmetric operator, i.e., $T^* (CTC - T) T = 0$, is called quasi-$[m, C]$-symmetric operator.

(ii) If T is an n-quasi-$[m,C]$-symmetric operator such that $\mathcal{R}(T^n)$ is dense, then T is an $[m,C]$-symmetric operator.

(iii) It is obvious that every $[m,C]$-symmetric operator is an n-quasi-$[m,C]$-symmetric operator and a quasi-$[m,C]$-symmetric operator is an n-quasi-$[m,C]$-symmetric operator.

(iv) Let $T \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Observe that

$$\begin{aligned} & T^{*^n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}CT^j\right)T^n \\ = \; & C\left(CT^*C\right)^n\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}C\left(CTC\right)^{m-j}C\left(CTC\right)^j\right)\left(CTC\right)^nC. \end{aligned}$$

It is clear that T is an n-quasi-$[m,C]$-symmetric operator if and only if CTC is an n-quasi-$[m,C]$-symmetric operator.

The following example shows that an n-quasi-$[m,C]$-symmetric operator is not necessary $[m,C]$-symmetric.

Example 1.3. Let C be a conjugation on $\mathbb{C}^3$ defined by $C(z_1,z_2,z_3)=(\overline{z_3},\overline{z_2},\overline{z_1})$ and consider the operator $T=\begin{pmatrix}0&0&0\\0&1&0\\1&0&0\end{pmatrix}$ on $\mathbb{C}^3$. A direct calculation shows that

$$T^*\left(CT^2C-2CTCT+T^2\right)T=0$$

and

$$CT^2C-2CTCT+T^2\neq 0.$$

Thus, T is a quasi-$[2,C]$-symmetric operator but it is not an $[2,C]$-symmetric operator.

Remark 1.4. It is clear that every quasi-$[m,C]$-symmetric operator is an n-quasi-$[m,C]$-symmetric operator for all $n\geq 2$. The converse is not true in general as shown in the following example.

Example 1.5. Let $T=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$ on $\mathbb{C}^3$, and $C:\mathbb{C}^3\longrightarrow\mathbb{C}^3$ such that $C(z_1,z_2,z_3)=(\overline{z_3},\overline{z_2},\overline{z_1})$, we get that $T^*(CTC-T)T\neq 0$ and so T is not quasi-$[1,C]$-symmetric operator. Since $T^{*^2}=T^2=0$, it follows that T is an 2-quasi-$[1,C]$-symmetric operator.

In this paper, we study various properties of n-quasi-$[m,C]$-symmetric operators. The perturbation of n-quasi-$[m,C]$-symmetric operators by nilpotent operators is obtained. We also study the product and tensor product of two n-quasi-$[m,C]$-symmetric operators.

2 Main results

In this section, we give some basic properties of n-quasi-$[m,C]$-symmetric operators.

In the following theorem, we show that by imposing certain condition on an n-quasi-$[m,C]$-symmetric operator for $n\geq 2$ it becomes a quasi-$[m,C]$-symmetric operator.

Theorem 2.1. *Let $T\in\mathcal{B}(\mathcal{H})$ be an n-quasi-$[m,C]$-symmetric operator for $n\geq 2$, where C is a conjugation on $\mathcal{H}$. If $\mathcal{N}(T^*)=\mathcal{N}\left(T^{*^2}\right)$, then T is a quasi-$[m,C]$-symmetric operator.*

Proof. By the assumption, we have that $\mathcal{N}(T^*) = \mathcal{N}\left(T^{*^n}\right)$.

Since

$$T^{*^n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}CT^j\right)T^n = 0,$$

then

$$T^*\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}CT^j\right)T^n = 0.$$

This means that

$$T^{*^n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(CT^{m-j}CT^j\right)^*\right)T = 0.$$

Using again the condition $\mathcal{N}(T^*) = \mathcal{N}\left(T^{*^n}\right)$, we obtain

$$T^*\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(CT^{m-j}CT^j\right)^*\right)T = 0.$$

Thus, we have

$$T^*\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}CT^j\right)T = 0.$$

Hence T is a quasi-$[m,C]$-symmetric operator and the proof is complete. □

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. If T is an invertible n-quasi-$[m,C]$-symmetric operator, then T^{-1} is an n-quasi-$[m,C]$-symmetric operator.*

Proof. Suppose that T is an invertible n-quasi-$[m,C]$-symmetric operator. Then T is an $[m,C]$-symmetric operator, by [6, Theorem 4.1.] so is T^{-1}. Hence T^{-1} is an n-quasi-$[m,C]$-symmetric operator. □

Theorem 2.3. *Let $C = C_1 \oplus C_2$ be a conjugation on $\mathcal{H}$ where C_1 and C_2 are conjugation on $\overline{\mathcal{R}(T^n)}$ and $\mathcal{N}\left(T^{*^n}\right)$ respectively. Assume that $\mathcal{R}(T^n)$ is not dense. Then, the following statements are equivalent:*

(1) T is n-quasi-$[m,C]$-symmetric operator.

(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}\left(T^{*^n}\right)$, where T_1 is an $[m,C_1]$-symmetric operator on $\overline{\mathcal{R}(T^n)}$, $T_3^n = 0$, and $\sigma(T) = \sigma(T_1) \cup \{0\}$, where $\sigma(T)$ is the spectrum of T.

Proof. (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}\left(T^{*^n}\right)$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(T^n)}$. Since T is an n-quasi-$[m,C]$-symmetric operator, it follows that

$$P\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C\,T^j\right)P = 0.$$

This means that

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}C_1T^{m-j}\,C_1T_1^j = 0.$$

Therefore, T_1 is an $[m, C_1]$-symmetric operator on $\overline{\mathcal{R}(T^n)}$.
Let $x = x_1 \oplus x_2 \in \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}\left(T^{*^n}\right) = \mathcal{H}$. If $x \in \mathcal{N}\left(T^{*^n}\right)$, then

$$\begin{aligned} \langle T_3^n x_2, x_2 \rangle &= \langle T^n (I-P) x, (I-P) x \rangle \\ &= \langle (I-P) x, T^{*^n} (I-P) x \rangle \\ &= 0. \end{aligned}$$

This implies that $T_3^n = 0$. So, we get that $\sigma(T) = \sigma(T_1) \cup \{0\}$.
(2) $\Rightarrow$ (1) Suppose that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}\left(T^{*^n}\right),$$

where $\overline{\mathcal{R}(T^n)}$ is the closure of $\mathcal{R}(T^n)$, such that T_1 is an $[m, C_1]$-symmetric operator and $T_3^n = 0$. Since

$$T^n = \begin{pmatrix} T_1^n & \sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \\ 0 & 0 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} & T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n \\ = & \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{m-j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{j} \right) \\ & \times \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{n} \\ = & \begin{pmatrix} T_1^{*^n} G T_1^n & T_1^{*^n} G \sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \\ \left(\sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \right)^* G T_1^n & \left(\sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \right)^* G \sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \end{pmatrix}, \end{aligned}$$

where

$$G = \sum_{j=0}^{m} (-1)^j \binom{m}{j} C_1 T_1^{m-j} \, C_1 T_1^j.$$

This implies that

$$T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} \, C T^j \right) T^n = 0$$

on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}\left(T^{*^n}\right)$. Thus, T is n-quasi-$[m, C]$-symmetric operator. □

Corollary 2.4. *If T is n-quasi-$[m, C]$-symmetric operator and $\mathcal{R}(T^n)$ is dense, then T is an $[m, C]$-symmetric operator.*

Corollary 2.5. *Under the same hypothesis as in Theorem 2.3, if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}\left(T^{*^n}\right)$ is such that T_1 is invertible, then T is similar to a direct sum of an $[m, C]$-symmetric operator and a nilpotent operator.*

Proof. Since T_1 is invertible, we have $\sigma(T_1)\cap\sigma(T_3)=\varnothing$. Then there exists an operator S such that $T_1S-ST_3=T_2$ by [15]. Hence

$$T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}=\begin{pmatrix} I & S\\ 0 & I\end{pmatrix}^{-1}\begin{pmatrix} T_1 & 0\\ 0 & T_3\end{pmatrix}\begin{pmatrix} I & S\\ 0 & I\end{pmatrix}$$

Consequently, the desired result follows from Theorem 2.3. □

Theorem 2.6. *Let $C=C_1\oplus C_2$ be a conjugation on $\mathcal{H}$, where $C_1$and C_2 are conjugation on $\overline{\mathcal{R}(T^n)}$ and $\mathcal{N}\left(T^{*^n}\right)$, respectively. If T is an n-quasi-$[m,C]$-symmetric operator, then T^k is also an n-quasi-$[m,C]$-symmetric operator for any $k\in\mathbb{N}$.*

Proof. If $\mathcal{R}(T^n)$ is dense, then T is an $[m,C]$-symmetric operator and so is T^k for any $k\in\mathbb{N}$ by [6, Theorem 4.1].

If $\mathcal{R}(T^n)$ is not dense, by Theorem 2.3, we write the matrix representation of T on $\mathcal{H}=\overline{\mathcal{R}(T^n)}\oplus\mathcal{N}(T^{*^n})$ as follows

$$T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}\text{ on }\mathcal{H}=\overline{\mathcal{R}(T^n)}\oplus\mathcal{N}\left(T^{*^n}\right),$$

where T_1 is an (m,C_1)-symmetric operator, and so is T_1^k. Since

$$T^k=\begin{pmatrix} T_1^k & \sum_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\\ 0 & T_3^k\end{pmatrix}\quad\text{on}\quad\mathcal{H}=\overline{\mathcal{R}(T^n)}\oplus\mathcal{N}\left(T^{*^n}\right).$$

Therefore, by Theorem 2.3, we obtain that T^k is an n-quasi-$[m,C]$-symmetric operator for every $k\in\mathbb{N}$. □

The converse of Theorem 2.3 is not true in general as shown in the following example.

Example 2.7. Let $T=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ on $\mathbb{C}^3$, and $C:\mathbb{C}^3\longrightarrow\mathbb{C}^3$ such that $C(z_1,z_2,z_3)=(\overline{z_3},\overline{z_2},\overline{z_1})$. A simple calculation shows that $T^{*^2}\left(CT^2C-T^2\right)T^2=0$ and $T^*(CTC-T)T\neq 0$. So, we obtain that T^2 is a quasi-$[1,C]$-symmetric operator, but T is not a quasi-$[1,C]$-symmetric operator.

The following lemma gives a condition for which an n-quasi-$[m,C]$-symmetric operator with a conjugation C becames an n-quasi-$[k,C]$-symmetric operator for every positive integer $k\geq m$.

Lemma 2.8. *Let $T\in\mathcal{B}(\mathcal{H})$ be an n-quasi-$[m,C]$-symmetric operator, where C is a conjugation on $\mathcal{H}$. If $T^*(CTC)=(CTC)T^*$, then T is an n-quasi-$[k,C]$-symmetric operator for every positive integer $k\geq m$.*

Proof. It is well known that from [6]

$$\delta_{m+1}(T,C)=CTC\delta_m(T,C)-\delta_m(T,C)T$$

Under the assumptions that T is an n-quasi-$[m,C]$-symmetric operator and satisfies $T^*(CTC)=(CTC)T^*$, it follows

$$T^{*^n}\delta_{m+1}(T,C)T^n=CTC\,T^{*n}\,\delta_m(T,C)T^n-T^{*^n}\delta_m(T,C)T^{n+1}.$$

Therefore, T is an n-quasi-$[m+1,C]$-symmetric operator, this implies that T is an n-quasi-$[k,C]$-symmetric operator for every $k\geq m$. □

In the general case, we give the following proposition.

Proposition 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $C = C_1 \oplus C_2$ be a conjugation on $\mathcal{H}$, where C_1 and C_2 are conjugations on $\overline{\mathcal{R}(T^n)}$ and $\mathcal{N}(T^{*^n})$, respectively. If T is an n-quasi-$[m, C]$-symmetric operator, then T is an l-quasi-$[k, C]$-symmetric operator for every positive integer $k \geq m$ and $l \geq n$.*

Proof. If $\mathcal{R}(T^n)$ is dense, then T is an $[m, C]$-symmetric operator and hence T is an $[k, C]$-symmetric operator for every positive integer $k \geq m$.

If $\mathcal{R}(T^n)$ is not dense, by Theorem 2.3 we write the matrix representation of T on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*^n})$ as follows $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where $T_1 = T_{/\overline{\mathcal{R}(T^n)}}$ is an $[m, C_1]$-symmetric operator and $T_3^n = 0$. Obviously that T_1 is an $[k, C_1]$-symmetric operator for every integer $k \geq m$. The conclusion follows from the statement (2) of Theorem 2.3. □

Next, we need the following lemma.

Lemma 2.10. *[6, Lemma 4.5] Let $T, S \in \mathcal{B}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$ such that $[T, S] = [T, CSC] = 0$. Then*

$$\delta_m(T + S, C) = \sum_{k=0}^{m} \binom{m}{k} \delta_k(T, C)\, \delta_{m-k}(S, C).$$

In the following theorem we investigate the nilpotent perturbations of an n-quasi-$[m, C]$-symmetric operator.

Theorem 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ be an n-quasi-$[m, C]$-symmetric operator and let $Q \in \mathcal{B}(\mathcal{H})$ be an nilpotent operator of order p. If T and Q satisfying $[T, Q] = [T, Q^*] = [T, CQC] = [T, CQ^*C] = [T^*, CTC] = 0$, then $T + Q$ is an $(n + p)$-quasi-$[m + 2p - 2, C]$-symmetric operator .*

Proof. We need to show that

$$(T + Q)^{*^\alpha} \delta_{m+2p-2}(T + Q, C)\, (T + Q)^\alpha = 0, \text{ where } \alpha = n + p.$$

By using Lemma 2.10, we obtain

$$\delta_{m+2p-2}(T + Q, C) = \sum_{k=0}^{m+2p-2} \binom{m + 2p - 2}{k} \delta_k(T, C)\, \delta_{m+2p-2-k}(Q, C).$$

Moreover, we have

$$\begin{aligned} & (T + Q)^{*^\alpha} \delta_{m+2p-2}(T + Q, C)\, (T + Q)^\alpha \\ = & \left(\sum_{r=0}^{\alpha} \binom{\alpha}{r} T^{*(\alpha-r)} Q^{*r} \right) \left(\sum_{k=0}^{m+2p-2} \binom{m + 2p - 2}{k} \delta_k(T, C)\, \delta_{m+2p-2-k}(Q, C) \right) \\ & \times \left(\sum_{s=0}^{\alpha} \binom{\alpha}{s} T^{\alpha-s} Q^s \right). \end{aligned}$$

Since $[T, Q] = [T, Q^*] = [T, CQC] = [T, CQ^*C] = [T^*, CTC] = 0$ and $Q^k = 0$ for $k \geq p$, we get

$$\begin{aligned} & (T + Q)^{*^\alpha} \delta_{m+2p-2}(T + Q, C)\, (T + Q)^\alpha \\ = & \left(\sum_{r=0}^{p-1} \binom{\alpha}{r} T^{*(\alpha-r)} Q^{*r} \right) \left(\sum_{k=0}^{m+2p-2} \binom{m + 2p - 2}{k} \delta_k(T, C)\ \delta_{m+2p-2-k}(Q, C) \right) \\ & \times \left(\sum_{s=0}^{p-1} \binom{\alpha}{s} T^{\alpha-s} Q^s \right). \end{aligned}$$

By using Lemma 2.8, we obtain that

$$T^{*^{\alpha-r}}\delta_k(T,C)\,T^{\alpha-s}=0 \text{ for } k\geq m, \text{ and } r,s\in\{0,...,p-1\}.$$

However, if $0\leq k\leq m-1$, then $2p-1\leq m+2p-2-k$. Hence,

$$\delta_{m+2p-2-k}(Q,C)=0.$$

Combining the above arguments, we obtain

$$(T+Q)^{*^{n+p}}\delta_{m+2p-2}(T+Q,C)\,(T+Q)^{n+p}=0.$$

Therefore, $T+Q$ is an $(n+p)$-quasi-$[m+2p-2,C]$-symmetric operator. □

Theorem 2.12. *Let $T,S\in\mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$ such that $[T,S]=[T,S^*]=[T,CSC]=[T,CS^*C]=[S^*,CSC]=0$. If T is n_1-quasi-$[m,C]$-symmetric operator and S is n_2-quasi-$[k,C]$-symmetric operator, then TS is an $n_3=max\{n_1,n_2\}$-quasi-$[m+k-1,C]$-symmetric operator.*

Proof. We need to prove that

$$(TS)^{*^{n_3}}\delta_{m+k-1}(TS,C)\,(TS)^{n_3}=0.$$

Under the assumptions $[T,S]=[T,CSC]=0$, and by taking into account [6, Lemma 4.10], we have

$$\delta_{m+k-1}(TS,C)=\sum_{j=0}^{m+k-1}\binom{m+k-1}{j}\delta_j(T,C)\,T^{m+k-1-j}CS^jC\delta_{m+k-1-j}(S,C).$$

Furthermore, since $[T,S]=[T,S^*]=[T,CSC]=[T,CS^*C]=[S^*,CSC]=0$, then

$$\begin{aligned}&(TS)^{*^{n_3}}\delta_{m+k-1}(TS,C)(TS)^{n_3}\\ =&\sum_{j=0}^{m+k-1}\binom{m+k-1}{j}T^{*^{n_3}}\delta_j(T,C)\,T^{n_3}T^{m+k-1-j}CS^jCS^{*^{n_3}}\delta_{m+k-1-j}(S,C)\,S^{n_3}.\end{aligned}$$

Since T is an n_1-quasi-$[m,C]$-symmetric operator, it follows from Proposition 2.9 that

$$T^{*^{n_3}}\delta_j(T,C)\,T^{n_3}=0, \text{ for } j\geqslant m \text{ and } n_3\geqslant n_1.$$

On the other hand, if $j\leqslant m-1$, then $m+k-1-j\geqslant m+k-1-m+1=k$, and $n_3\geq n_2$. So by Proposition 2.9 again, it follows that

$$S^{*^{n_3}}\delta_{m+k-1-j}(S,C)\,S^{n_3}=0.$$

Combining the above cases, we obtain

$$(TS)^{*^{n_3}}\delta_{m+k-1}(TS,C)(TS)^{n_3}=0.$$

Therefore, TS is an n_3-quasi-$[m+k-1,C]$-symmetric operator. □

Corollary 2.13. *Let $T,S\in\mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$ such that $[T,S]=[T,S^*]=[T,CSC]=[T,CS^*C]=[S^*,CSC]=0$. If T is an n_1-quasi-$[m,C]$-symmetric operator and S is an n_2-quasi-$[k,C]$-symmetric operator, then TS^q is an $n_3=max\{n_1,n_2\}$-quasi-$[m+k-1,C]$-symmetric operator for some positive integer q.*

Proof. In view of Theorem 2.6 we have that S^q is an n_2-quasi-$[k,C]$-symmetric operator with conjugation C. However,

$$[T,CS^qC]=[S^{*q},CS^qC]=[S^{*q},CTC]=0.$$

Moreover, T and S^q satisfy the conditions of Theorem 2.12. Hence TS^q is an n_3-quasi-$[m+k-1,C]$-symmetric operator. □

The following theorem shows that the class of n-quasi-$[m,C]$-symmetric operators is a closed subset of $\mathcal{B}(\mathcal{H})$ equipped with the uniform operator (norm) topology.

Theorem 2.14. *Let C be a conjugation on $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$. If $\{T_k\}$ is a sequence of n-quasi-$[m,C]$-symmetric operators such that $\lim_{k\to\infty} \|T_k - T\| = 0$, then T is also n-quasi-$[m,C]$-symmetric operator.*

Proof. Suppose that $\{T_k\}$ is a sequence of n-quasi-$[m,C]$-symmetric operators such that $\lim_{k\to\infty} \|T_k - T\| = 0$. Since $\|C\| = 1$, we get

$$\begin{aligned}
&\left\| T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T_k^{m-j} C T_k^j \right) T_k^n - T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n \right\| \\
&\leq \left\| T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T_k^{m-j} C T_k^j \right) T_k^n - T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n \right\| \\
&+ \left\| T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n - T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n \right\| \\
&\leq \left\| T_k^{*^n} \right\| \left\| \sum_{j=0}^{m} (-1)^j \binom{m}{j} C \left(T_k^{m-j} C T_k^{j+n} - T^{m-j} C T^{j+n} \right) \right\| \\
&+ \left\| T_k^{*^n} - T^{*^n} \right\| \left\| \sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^{j+n} \right\| \\
&\leq \sum_{j=0}^{m} \binom{m}{j} \left\| T_k^{*^n} \right\| \left\| T_k^{m-j} C T_k^{j+n} - T^{m-j} C T^{j+n} \right\| + \sum_{j=0}^{m} \binom{m}{j} \left\| T_k^{*^n} - T^{*^n} \right\| \left\| T^{m-j} C T^{j+n} \right\| \to 0.
\end{aligned}$$

This implies that

$$\left\| T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T_k^{m-j} C T_k^j \right) T_k^n - T^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n \right\| \to 0.$$

Since $\{T_k\}$ is an n-quasi-$[m,C]$-symmetric operator, i.e.,

$$T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T_k^{m-j} C T_k^j \right) T_k^n = 0.$$

Then, we deduce that

$$T_k^{*^n} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C T^j \right) T^n = 0.$$

This means that, T is an n-quasi-$[m,C]$-symmetric operator. □

Let $\mathcal{H} \bar{\otimes} \mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$ of $\mathcal{H}$ and $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, $T \otimes S \in \mathcal{B}(\mathcal{H} \bar{\otimes} \mathcal{H})$ denote the tensor product operator defined by T and S. In [9, Lemma 6], it was proved that if C and D are conjugations on $\mathcal{H}$, then $C \otimes D$ is a conjugation on $\mathcal{H} \bar{\otimes} \mathcal{H}$.

Lemma 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ and let C and D be two conjugations on $\mathcal{H}$, respectively. Then T is an n-quasi-$[m,C]$-symmetric operator if and only if the tensor product $T \otimes I$ (resp. $I \otimes T$) is an n-quasi-$[m, C \otimes D]$-symmetric operator .*

Proof. We have

$$\begin{aligned}
&(T\otimes I)^{*^n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}(C\otimes D)\ (T\otimes I)^{m-j}(C\otimes D)\ (T\otimes I)^j\right)(T\otimes I)^n\\
=\ & T^{*^n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C\,T^j\right)T^n\otimes I\\
=\ & T^{*^n}\delta_m(T,C)\,T^n\otimes I.
\end{aligned}$$

Hence, T is an n-quasi-$[m, C]$-symmetric operator if and only if $T \otimes I$ is an n-quasi-$[m, C \otimes D]$-symmetric operator, and the same with $I \otimes T$. □

Theorem 2.16. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If T is an n_1-quasi-$[m, C]$-symmetric operator and S is an n_2-quasi-$[k, D]$-symmetric operator, where C and D are conjugations on $\mathcal{H}$, respectively such that $[S^*, DSD] = 0$, then $T \otimes S$ is an $n_3 = max\{n_1, n_2\}$-quasi-$[m + k - 1, C \otimes D]$-symmetric operator.*

Proof. Since T is an n_1-quasi-$[m, C]$-symmetric operator, and S is an n_2-quasi-$[k, D]$-symmetric operator, then by Lemma 2.15 we obtain $T \otimes I$ is an n_1-quasi-$[m, C \otimes D]$-symmetric operator, $I \otimes S$ is an n_2-quasi-$[k, C \otimes D]$-symmetric operator. We observe that

$$T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I).$$

On the other hand, note that $T \otimes I$ and $I \otimes S$ satisfy all conditions required in Theorem 2.12 as

$$[T \otimes I, I \otimes S] = [T \otimes I, (I \otimes S)^*] = 0,$$

$$[T \otimes I, (C \otimes D)(I \otimes S)(C \otimes D)] = [T \otimes I, (C \otimes D)(I \otimes S)^*(C \otimes D)] = 0$$

and

$$[(I \otimes S)^*, (C \otimes D)(I \otimes S)(C \otimes D)] = 0.$$

Therefore, $T \otimes S$ is an n_3-quasi-$[m + k - 1, C \otimes D]$-symmetric operator. □

Proposition 2.17. *Let $T, N \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. If T is an n-quasi-$[m, C]$-symmetric operator and N is a nilpotent operator of order p, then $T \otimes I + I \otimes N$ is an $(n + p)$-quasi-$[m + 2p - 2, C \otimes C]$-symmetric operator*

Proof. Since T is an n-quasi-$[m, C]$-symmetric operator, then by Lemma 2.15 we obtain $T \otimes I$ is an n-quasi-$[m, C \otimes C]$-symmetric operator, since N is a nilpotent of order p, then $I \otimes N$ is a nilpotent of order p as well. We have

$$T \otimes N = (T \otimes I)(I \otimes N) = (I \otimes N)(T \otimes I).$$

On the other hand, note that $T \otimes I$ and $I \otimes N$ satisfy all conditions required in Theorem 2.12. Therefore, $T \otimes I + I \otimes N$ is an $(n + p)$-quasi-$[m + 2p - 2, C \otimes C]$-symmetric operator. □

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Author information

Imene Djaber, Faculty of Exact Sciences, Department of Mathematics, Operators Theory and PDE Foundations and Applications Laboratory Echahid Hamma Lakhdar University, 39000 El Oued, Algeria.
E-mail: imenedjaber5@gmail.com

Messaoud Guesba, Faculty of Exact Sciences, Department of Mathematics, Operators Theory and PDE Foundations and Applications Laboratory Echahid Hamma Lakhdar University, 39000 El Oued, Algeria.
E-mail: guesbamessaoud2@gmail.com, guesba-messaoud@univ-eloued.dz

Received: 2025-10-02
Accepted: 2026-01-04