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The graded secondary radicals of graded submodules

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Abstract Let G be a group with identity e. Let $\mathcal{A}$ be a G-graded commutative ring, $\mathcal{D}$ be a graded $\mathcal{A}$-module and C be a graded submodule of $\mathcal{D}$. In this paper, we introduce the concept of graded secondary radicals of a graded submodule. We define the graded secondary radicals of graded submodule C of $\mathcal{D}$ as the sum of all graded secondary submodule of $\mathcal{D}$ contained in C, denoted by $GSec^*(C)$, and discuss the related properties..

1 Introduction

Throughout this article, we assume that $\mathcal{A}$ is a commutative G-graded ring with identity and that $\mathcal{D}$ is a unitary graded $\mathcal{A}$-module. The concept of graded prime submodules was introduced and studied by Atani in [9]. The concept of graded second submodules was introduced and studied by Ansari-Toroghy and Farshadifar in [7]. In [10] Atani and Farzalipou introduced and studied the notion of graded secondary submodules. Recently, in [6], Refai and Abu-Dawwa introduced the notion of graded second radicals of graded submodules. In this paper, we introduce the concept of graded secondary radicals of graded submodules and present some properties related to this concept. For additional background, see [1, 5, 8, 14].

2 Preliminaries

In this section we will give the definitions and results which are required in the next section.

Definition 2.1. (a) Let G be a group with identity e and $\mathcal{A}$ be a commutative ring with identity $1_{\mathcal{A}}$. Then $\mathcal{A}$ is G-graded ring if there exist additive subgroups $\mathcal{A}_g$ of $\mathcal{A}$ indexed by the elements $g \in G$ such that $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$. The set of all homogeneous elements of $\mathcal{A}$ is denoted by $h(\mathcal{A})$, i.e. $h(\mathcal{A}) = \bigcup_{g\in G} \mathcal{A}_g$, see [14].

(b) Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be G-graded ring, an ideal P of $\mathcal{A}$ is called a graded ideal if $P = \sum_{h\in G} P \cap \mathcal{A}_h = \sum_{h\in G} P_h$, see [14].

(c) A left $\mathcal{A}$-module $\mathcal{D}$ is called a G-graded $\mathcal{A}$-module if $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ with $\mathcal{A}_g\mathcal{D}_h \subseteq \mathcal{D}_{gh}$ for all $g, h \in G$, where $\mathcal{D}_g$ is an additive subgroup of $\mathcal{D}$ for all $g \in G$. The set of all homogeneous elements of $\mathcal{D}$ is denoted by $h(\mathcal{D})$, i.e, $h(\mathcal{D}) = \bigcup_{g\in G} \mathcal{D}_g$. Note that $\mathcal{D}_h$ is an $\mathcal{A}_e$-module for every $h \in G$, see [14].

(d) A submodule S of $\mathcal{D}$ is called a graded submodule of $\mathcal{D}$ if $S = \oplus_{h\in G}(S \cap \mathcal{D}_h) := \oplus_{h\in G} S_h$, see [14].

(e) If S is graded submodule of $\mathcal{D}$, then $(S :_{\mathcal{A}} \mathcal{D}) = \{a \in \mathcal{A} \mid a\mathcal{D} \subseteq S\}$ is graded ideal of $\mathcal{A}$, (see [9]). Furthermore, the annihilator of S in $\mathcal{A}$ is denoted and defined by $Ann_{\mathcal{A}}(S) = \{a \in \mathcal{A} \mid aS = \{0\}\}$.

(f) A proper graded ideal K of $\mathcal{A}$ is called a graded prime ideal if whenever $r, s \in h(\mathcal{A})$ with $rs \in K$, we have $r \in K$ or $s \in K$. The graded radical of a graded ideal P, denoted by $Gr(P)$, is the set of all $t = \sum_{g\in G} t_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $t_g^{n_g} \in P$. Note that, if r is a homogeneous element, then $r \in Gr(P)$ if and only if $r^n \in P$ for some $n \in \mathbb{N}$, see [16].

(g) A proper graded submodule P of $\mathcal{D}$ is called a graded prime submodule if whenever $a \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$ with $am \in P$, then either $a \in (P :_{\mathcal{A}} \mathcal{D})$ or $m \in P$, see [9].

(h) A non-zero graded submodule N of a $\mathcal{D}$ is called graded second if for each $a \in h(\mathcal{A})$, the graded $\mathcal{A}$-homomorphism $\Psi : N \to N$ defined by $\Psi(x) = ax$ is either surjective or zero. In other words, N is a graded second submodule of $\mathcal{D}$ if $aN = N$ or $aN = 0$ for every $a \in h(\mathcal{A})$. Then $P = Ann_{\mathcal{A}}(N)$ is a graded prime ideal of $\mathcal{A}$ and N is called a P-graded second submodule. The graded second spectrum of $\mathcal{D}$, denoted by $GSpec^s(\mathcal{D})$, is the set of all graded second submodules of $\mathcal{D}$, see [7].

(i) A non-zero graded submodule N of $\mathcal{D}$ is called secondary if for every homogeneous element of $\mathcal{A}$ either divides N or is nilpotent on N. In other words, N is a graded secondary submodule of $\mathcal{D}$ if $aN = N$ or $a^n N = 0$ for some integer n, for every $a \in h(\mathcal{A})$, see [10].

Definition 2.2. (a) A graded $\mathcal{A}$-module $\mathcal{D}$ is called a graded finitely cogenerated if for every set $\{\mathcal{D}_\lambda\}_{\lambda\in\Delta}$ of graded submodules of $\mathcal{D}$ with $\cap_{\lambda\in\Delta}\mathcal{D}_\lambda = 0$, then $\cap_{j=1}^{m}\mathcal{D}_{\lambda_j} = 0$, for some positive integer m, see [15].

(b) A proper graded submodule K of $\mathcal{D}$ is called a completely graded irreducible if $K = \cap_{j\in J}K_j$, where $\{K_j\}_{j\in J}$ is a family of graded submodule of $\mathcal{D}$, then $K = K_j$ for some $j \in J$. Every graded submodule of $\mathcal{D}$ is intersection of all completely graded irreducible submodules containing it. See [2].

(c) Let $\Omega(\mathcal{D})$ be the set of all graded completely irreducible $\mathcal{A}$-submodules of $\mathcal{D}$. Assume that P is a graded prime ideal of $\mathcal{A}$ and K is a graded $\mathcal{A}$-submodule of $\mathcal{D}$. Then we define $I_P^{\mathcal{D}}(K) = \cap_{L\in\Omega(\mathcal{D})}\{L : rK \subseteq L \text{ for some } r \in h(\mathcal{A}) - P\}$. , see [6].

(d) Let $\mathcal{A}$ be a graded integral domain and $\mathcal{D}$ be a graded $\mathcal{A}$-module. We say that $\mathcal{D}$ is gr-cotorsion-free if $I_0^{\mathcal{D}}(\mathcal{D}) = \mathcal{D}$ and is gr-cotorsion if $I_0^{\mathcal{D}}(\mathcal{D}) = 0$.

(e) A graded $\mathcal{A}$-module $\mathcal{D}$ is called graded comultiplication module (gr-comultiplication module) if for every graded submodule K of $\mathcal{D}$, there exists a graded ideal J of $\mathcal{A}$ with $K = (0 :_{\mathcal{D}} J)$, equivalently, for every graded submodule K of $\mathcal{D}$, $K = (0 :_{\mathcal{D}} Ann_{\mathcal{A}}(K))$, see [4, 3].

(f) A non-zero graded submodule K of a graded $\mathcal{A}$-module $\mathcal{D}$ is said to be gr-essential (or gr-large) if for every non-zero graded submodule H of $\mathcal{D}$, we have $K \cap H \neq 0$, see [4].

(g) Let $\mathcal{D}$ be graded $\mathcal{A}$-module. The graded Socle of $\mathcal{D}$ is defined as the sum of all graded minimal submodules of $\mathcal{D}$, it will denoted by $Soc^{gr}(\mathcal{D})$, see [4].

Also,

$$Soc^{gr}(\mathcal{D}) = \cap\{L \subset \mathcal{D} | \text{ L is graded essential submodule of } \mathcal{D}\},$$

see [17].

(h) A graded $\mathcal{A}$-module $\mathcal{D}$ is called gr-Artinian if $\mathcal{D}$ satisfies the descending chain condition for graded submodules, see [4].

(i) Let $\mathcal{D}$ be a graded $\mathcal{A}$-module let K be graded submodule of $\mathcal{D}$. We say that K is gr-copure submodule if $(K :_{\mathcal{D}} I) = K + (0 :_{\mathcal{D}} I)$ for every graded ideal I of $\mathcal{A}$.

(j) We say that a graded $\mathcal{A}$-module $\mathcal{D}$ is fully copure if every graded submodule of $\mathcal{D}$ is graded copure.

(k) We say that a graded $\mathcal{A}$-module $\mathcal{D}$ is gr-atomic if every non-zero graded submodule of $\mathcal{D}$ contains a graded minimal submodule.

(l) Let $\mathcal{D}$ be a graded $\mathcal{A}$-module, we say that a family $\{K_j\}_{j\in J}$ of graded submodules is inverse family of graded submodules of $\mathcal{D}$ if the intersection of two of its graded submodules again contains a graded module in $\{K_j\}_{j\in J}$. Also, we say a graded $\mathcal{A}$-module $\mathcal{D}$ satisfies the property $AB5^*$ if for every graded submodule H of $\mathcal{D}$ and every graded inverse family $\{K_j\}_{j\in J}$ of graded submodules of $\mathcal{D}$, $H + \cap_{j\in J} K_j = \cap_{j\in J}(H + K_j)$.

(m) For a graded $\mathcal{A}$-submodule C of $\mathcal{D}$, the graded second radical of C is defined as the sum of all graded second $\mathcal{A}$-submodules of $\mathcal{D}$ contained in C, and its denoted by $GSec(C)$. If C does not contain any graded second $\mathcal{A}$-submodule, then $GSec(C) = \{0\}$. The graded second spectrum of $\mathcal{D}$ is the collection of all graded second $\mathcal{A}$-submodules, and it is represented by the symbol $GSpec^s(\mathcal{D})$. On the other hand, the set of all graded prime $\mathcal{A}$-submodules of $\mathcal{D}$ is called the graded spectrum of $\mathcal{D}$, and is denoted by $GSpec(\mathcal{D})$. The map $\phi : GSpec^s(\mathcal{D}) \to GSpec(\mathcal{A}/Ann_R(\mathcal{D}))$ defined by $\phi(C) = Ann_{\mathcal{A}}(C)/Ann_{\mathcal{A}}(\mathcal{D})$ is called the natural map of $GSpec^s(\mathcal{D})$, see [6].

Remark 2.3. Let K and H are two graded submodule of an graded $\mathcal{A}$-module. To prove $K \subseteq H$, its enough to show that if J is a completely graded irreducible submodule of $\mathcal{D}$ with $H \subseteq J$, then $K \subseteq J$, see [2].

3 RESULTS

Definition 3.1. Let $\mathcal{D}$ be a graded $\mathcal{A}$-module and K be a graded submodule of $\mathcal{D}$. We define the graded secondary radical of K as the sum of all graded secondary submodule of $\mathcal{D}$ contained in K, denoted by $GSec^*(K)$. If K does not contain any graded secondary submodule, we put $GSec^*(K) = \{0\}$. Moreover, K is called graded secondary radical submodule of $\mathcal{D}$ if $K \neq 0$ and $GSec^*(K) = K$.

Theorem 3.2. *Let K and H be two graded submodules of a graded $\mathcal{A}$-module $\mathcal{D}$. Then the following hold:*

(a) $GSec(K) \subseteq GSec^*(K) \subseteq K$.

(b) *If $K \subseteq H$, then* $GSec^*(K) \subseteq GSec^*(H)$.

(c) $GSec^*(GSec^*(K)) = GSec^*(K)$

(d) $GSec^*(K) + GSec^*(H) \subset GSec^*(K + H)$.

(e) $GSec^*(K \cap H) = GSec^*(GSec^*(K) \cap GSec^*(H))$.

(f) *If S is a graded P-secondary submodule of $\mathcal{D}$ with $S \subseteq K + H$ and $Ann_{\mathcal{A}}(K) \nsubseteq P$, then $S \subseteq H$.*

(g) *If $K + H = GSec^*(K) + GSec^*(H)$, then $GSec^*(K + H) = K + H$.*

Proof. (a) Let S be graded second submodule contained in K, then S is graded secondary submodule in K. So $GSec(K) \subseteq GSec^*(K)$. By defintion of $GSec^*(K)$, we have $GSec^*(K) \subseteq K$.

(b) This is straightforward.

(c) By part (a) and (b), we have $GSec^*(GSec^*(K)) \subseteq GSec^*(K)$. For the converse inclusion, let S be graded secondary submodule of $\mathcal{D}$ with $S \subseteq K$. Since $GSec^*(K)$ is the sum of all graded secondary submodule S of $\mathcal{D}$ such taht $S \subseteq K$, then $S \subseteq GSec^*(K)$, so $GSec^*(K) \subseteq GSec^*(GSec^*(K))$.

(d) Let S be graded secondary submodule of $\mathcal{D}$ contained in K, then $S \subseteq K \subseteq K + H$. Thus $GSec^*(K) \subset GSec^*(K + H)$. Similarly, if S be graded secondary submodule of $\mathcal{D}$ contained in H, then $S \subseteq H \subseteq K + H$. So, $GSec^*(H) \subset GSec^*(K + H)$. Therefore, $GSec^*(K) + GSec^*(H) \subset GSec^*(K + H)$.

(e) This follows from part (a), (b) and (c).

(f) Since $Ann_{\mathcal{A}}(K) \nsubseteq P = Gr(Ann_{\mathcal{A}}(S))$, there exists $r_g \in h(\mathcal{A})$ such that $r_g K = 0$ but $r_g \notin P$. As S is graded P-secondary submodule of $\mathcal{D}$, then we have $r_g S = S$, by [10]. Since $r_g S \subseteq r_g K + r_g H$ and H is submodule of $\mathcal{A}$, then we have $S \subseteq r_g H \subseteq H$. Therefore, $S \subseteq H$.

(g)By part (d) $K+H = GSec^*(K)+GSec^*(H) \subseteq GSec^*(K+H)$ and $GSec^*(K+H) \subseteq K+H$. □

Let $\mathcal{A}$ be G-graded ring, Graded dimension $\mathcal{A}$ is defined by supermum of all numbers k for which there exists a chain of graded prime ideals $I_0 \subset I_1 \subset ... \subset I_k$ in $\mathcal{A}$, and it is denoted by $GdimR$ [11].

Theorem 3.3. *Let $\mathcal{D}$ be graded comultiplication $\mathcal{A}$-module. Then we have the following:*

(a) $GSec^*(\mathcal{D}) = 0$ *if and only if* $\mathcal{D} = 0$.

(b) *If K and H are two graded submodules of $\mathcal{D}$, then $GSec^*(K) \cap GSec^*(H) = 0$ if and only if $K \cap H = 0$.*

Proof. (a) Assume that $GSec^*(\mathcal{D}) = 0$. By (Theorem 3.9 [4]), $\mathcal{D}$ contains a graded minimal submodule, and every graded minimal submodule is graded second by (Lemma 2.3 [7]), hence is graded secondary submodule. Therefore $\mathcal{D} = 0$. The converse is trivial.

(b) This follows from part (a) and Theorem 3.2(e). □

Theorem 3.4. *Let K and H are two graded submodules of graded $\mathcal{A}$-module $\mathcal{D}$ such that whenever $S \subseteq K+H$ we have $S \subseteq K$ or $S \subseteq H$, for every graded secondary submodule S of $\mathcal{D}$. Then $GSec^*(K+H) = GSec^*(K)+GSec^*(H)$.*

Proof. If $GSec^*(K+H) = 0$, then clearly $GSec^*(K) = GSec^*(H) = 0$ and so $GSec^*(K+H) = GSec^*(K)+GSec^*(H)$. If $GSec^*(K+H) \neq 0$, then there exists a graded secondary submodule Q of $\mathcal{D}$ such that $Q \subseteq K+H$, by assumption $Q \subseteq K$ or $Q \subseteq H$. Thus $Q \subseteq GSec^*(K)$ or $Q \subseteq GSec^*(H)$. As this is true for all graded secondary submodules contained in $K+H$, $GSec^*(K+H) \subseteq GSec^*(K)+GSec^*(H)$. The reverse inclusion is clear. □

Theorem 3.5. *Let $\mathcal{D}$ be graded $\mathcal{A}$-module. If K and H are two graded submodules of $\mathcal{D}$ such that $Gr(Ann_{\mathcal{A}}(K))$ and $Gr(Ann_{\mathcal{A}}(H))$ are comaximal, then $GSec^*(K+H) = GSec^*(K)+GSec^*(H)$.*

Proof. It is clear that $GSec^*(K)+GSec^*(H) \subseteq GSec^*(K+H)$. Now, If Q is graded secondary submodule contained in $K+H$, then

$$\begin{aligned} Ann_{\mathcal{A}}(K)Ann_{\mathcal{A}}(H) &\subseteq Ann_{\mathcal{A}}(K) \cap Ann_{\mathcal{A}}(H) \\ &\subseteq Ann_{\mathcal{A}}(Q) \subseteq Gr(Ann_{\mathcal{A}}(Q)). \end{aligned}$$

As $Gr(Ann_{\mathcal{A}}(Q))$ is graded prime ideal of $\mathcal{A}$, then either $Ann_{\mathcal{A}}(K) \subseteq Gr(Ann_{\mathcal{A}}(Q))$ or $Ann_{\mathcal{A}}(H) \subseteq Gr(Ann_{\mathcal{A}}(Q))$. Without loss of generality we may assume that $Ann_{\mathcal{A}}(K) \subseteq Gr(Ann_{\mathcal{A}}(Q))$. Then $Ann_{\mathcal{A}}(H) \nsubseteq Gr(Ann_{\mathcal{A}}(Q))$, otherwise it contradicts comaximality. We show that $Q \subseteq K$. Assume that L is a graded completely irreducible submodule of $\mathcal{D}$ such that $K \subseteq L$ and let $r_g \in Ann_{\mathcal{A}}(H) - Gr(Ann_{\mathcal{A}}(Q))$. As $H \subseteq (L :_{\mathcal{D}} r_g)$ and $K \subseteq L \subseteq (L :_{\mathcal{D}} r_g)$, then $Q \subseteq K+H \subseteq (L :_{\mathcal{D}} r_g)$. Thus $r_g Q \subseteq L$. Since Q is graded secondary, it follows that $Q \subseteq L$. This implies $Q \subseteq K$, and so $Q \subseteq GSec^*(K) \subseteq GSec^*(K)+GSec^*(H)$. Hence $GSec^*(K+H) \subseteq GSec^*(K)+GSec^*(H)$. □

The following corollary follows immediately from Theorem 3.5.

Corollary 3.6. *Let $N_1, N_2, ..., N_n$ be graded submodules of graded $\mathcal{A}$-module $\mathcal{D}$ such that $Gr(Ann_{\mathcal{A}}(N_i)$ are pairwise comaximal. Then*

$$GSec^*(N_1+N_2+...+N_n) = GSec^*(N_1)+GSec^*(N_2)+....+GSec^*(N_n).$$

Definition 3.7. Let $\mathcal{D}$ be graded $\mathcal{A}$-module. We call the collection of all graded secondary $\mathcal{A}$-submodules of $\mathcal{D}$ the graded secondary spectrum of $\mathcal{D}$ and denote by $GSpec^{s*}(\mathcal{D})$. On the other hand, the set of all graded prime $\mathcal{A}$-submodules of $\mathcal{D}$ is called the graded spectrum of $\mathcal{D}$, and is denoted by $GSpec(\mathcal{D})$. The map $\phi : GSpec^{s*}(\mathcal{D}) \to GSpec(\mathcal{A}/Ann_{\mathcal{A}}(\mathcal{D}))$ defined by $\phi(S) = Gr(Ann_{\mathcal{A}}(S))/Ann_{\mathcal{A}}(\mathcal{D})$ for every $S \in GSpec^{s*}(\mathcal{D})$, is called the natural map of $GSpec^{s*}(\mathcal{D})$.

Theorem 3.8. *Let $\mathcal{D}$ be graded $\mathcal{A}$-module, let K be graded submodule of $\mathcal{D}$ such that the natural map ϕ of $GSpec^{s*}(K)$ is surjective. Then $Gr(Ann_{\mathcal{A}}(GSec^*(K))) = Gr(Ann_{\mathcal{A}}(K))$.*

Proof. If $K = 0$, the theorem is trivially true. Assume that $K \neq 0$, since $GSec^*(K) \subseteq K$, then $Gr(Ann_{\mathcal{A}}(K)) \subseteq Gr(Ann_{\mathcal{A}}(GSec^*(K)))$. Now let $Gr(Ann_{\mathcal{A}}(K)) = \cap_i P_i$, where P_i is graded prime ideal of $\mathcal{A}$ containing $Ann_{\mathcal{A}}(K)$. As ϕ is surjective, for each such P_i there exists $Q_i \in GSpec^{s*}(K)$ such that $Gr(Ann_{\mathcal{A}}(Q_i)) = P_i$. Thus $\sum_i Q_i \subseteq GSec^*(K)$. Therefore

$$Gr(Ann_{\mathcal{A}}(GSec^*(K))) \subseteq Gr(Ann_{\mathcal{A}}(\sum_i Q_i)) = Gr(\cap_i Ann_{\mathcal{A}}(Q_i))$$
$$\subseteq \cap_i Gr(Ann_{\mathcal{A}}(Q_i)) = \cap_i P_i = Gr(Ann_{\mathcal{A}}(K)),$$

as needed. □

Definition 3.9. We define the graded secondary submodule dimension of graded $\mathcal{A}$-module $\mathcal{D}$ to be the supremum of the length of chains of graded secondary submodules of $\mathcal{D}$ if $GSpec^{s*}(\mathcal{D}) \neq \emptyset$ and -1 otherwise, and it is denoted by $Gdim^{s*}\mathcal{D}$.

Lemma 3.10. *Let $\mathcal{D}$ be graded $\mathcal{A}$-module and K be graded submodule of $\mathcal{D}$, then*

(a) $Soc^{gr}(K) = K \cap Soc^{gr}(\mathcal{D})$.

(b) *If $Soc^{gr}(\mathcal{D})$ is graded essential submodule of $\mathcal{D}$, then every non-zero graded submodule of $\mathcal{D}$ contains a graded minimal submodule.*

Proof. (a) Let $f : K \to \mathcal{D}$ be the inclusion map, one can see that $Soc^{gr}(K) \subseteq Soc^{gr}(\mathcal{D})$ and clearly it's in K, so $Soc^{gr}(K) \subseteq K \cap Soc^{gr}(\mathcal{D})$. Now let $S \subseteq K$ and S is graded minimal submodule of $\mathcal{D}$, so S is graded minimal submodule of K and so $S \subseteq Soc^{gr}(K)$. Therefore $K \cap Soc^{gr}(\mathcal{D}) \subseteq Soc^{gr}(K)$.

(b) Let K be non-zero graded submodule of $\mathcal{D}$, by part (a), $Soc^{gr}(K) = K \cap Soc^{gr}(\mathcal{D})$. Since $Soc^{gr}(\mathcal{D})$ is graded essential submodule of $\mathcal{D}$ and $K \neq 0$, then $Soc^{gr}(K) \neq 0$. Thus K contains a graded minimal submodule. □

Lemma 3.11. *Let $\mathcal{D}$ be graded $\mathcal{A}$-module. The intersection of two (finitely) graded essential submodules is a graded essential submodule of $\mathcal{D}$.*

Proof. Let $H \neq 0$ be a graded submodule of $\mathcal{D}$ and let K_1 and K_2 be two graded essential submodules of $\mathcal{D}$. Then $H \cap K_1 \neq 0$ and $H \cap K_2 \neq 0$, so $H \cap (K_1 \cap K_2) = (H \cap K_1) \cap K_2 \neq 0$ since $H \cap K_1$ is non-zero graded submodule and K_2 is graded essential submodule. Therefore $K_1 \cap K_2$ is graded essential submodule of $\mathcal{D}$. □

Theorem 3.12. *If $\mathcal{D}$ is graded finitely cogenerated $\mathcal{A}$-module, then the $Soc^{gr}(\mathcal{D})$ is graded finitely cogenerated and essential submodule.*

Proof. If $\mathcal{D}$ is graded finitely cogenerated, then every graded submodule of $\mathcal{D}$ is graded finitely cogenerated. We show that $Soc^{gr}(\mathcal{D})$ is graded essential submodule of $\mathcal{D}$. Let K be garded submodule of $\mathcal{D}$ such that $K \cap Soc^{gr}(\mathcal{D}) = 0$. Since $\mathcal{D}$ is graded finitely cogenerated there exists graded essential submodules $L_1, L_2, ..., L_n$ of $\mathcal{D}$ with $K \cap L_1 \cap ... \cap L_n = 0$. But by Lemma 3.11, $L_1 \cap ... \cap L_n$ is graded essential submodule of $\mathcal{D}$, so $K = 0$. Hence $Soc^{gr}(\mathcal{D})$ is graded essential submodule of $\mathcal{D}$. □

Corollary 3.13. *If $\mathcal{D}$ is graded finitely cogenerated $\mathcal{A}$-module, then every non-zero graded submodule of $\mathcal{D}$ contains a graded minimal submodule.*

Proof. This follows from Theorem 3.12 and Lemma 3.10 (b). □

Theorem 3.14. *If $\mathcal{A}$ is graded one dimensional domain and $\mathcal{D}$ is graded finitely cogenerated cotorsion $\mathcal{A}$-module with $Gdim^{s*}\mathcal{D} = 1$, then the following are equivalent:*

(a) *$\mathcal{D}$ is graded secondary module.*

(b) *$S_1 + S_2 = \mathcal{D}$ for any distinct graded secondary submodule S_1 and S_2.*

(c) *Every proper graded submodule contains exactly one graded secondary submodule.*

(d) *Every proper graded secondary submodule is minimal.*

Proof. (a) $\Rightarrow$ (b). As $\mathcal{D}$ is graded finitely cogenerated cotorsion, $Ann_{\mathcal{A}}(\mathcal{D}) \neq 0$. Since $\mathcal{D}$ is graded secondary module, then $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ is graded prime ideal of $\mathcal{A}$. Thus if Q is any graded secondary submodule of $\mathcal{D}$, then we have $Gr(Ann_{\mathcal{A}}(\mathcal{D})) = Gr(Ann_{\mathcal{A}}(Q))$ because $GdimR = 1$. Thus for any distinct graded secondary submodules Q_1 and Q_2 of $\mathcal{D}$, $Gr(Ann_{\mathcal{A}}(\mathcal{D})) = Gr(Ann_{\mathcal{A}}(Q_1)) = Gr(Ann_{\mathcal{A}}(Q_2))$ so that $Q_1 + Q_2$ is graded secondary. Therefore $Q_1 + Q_2 = \mathcal{D}$ because $Gdim^{s*}\mathcal{D} = 1$.

(b) $\Rightarrow$ (c). As $\mathcal{D}$ is graded finitely cogenerated, by Corollary 3.13, every non-zero graded submodule of $\mathcal{D}$ contains a graded minimal submodule which is graded secondary. Now if K is proper graded submodule of $\mathcal{D}$ and Q_1 and Q_2 are distinct graded secondary submodules of $\mathcal{D}$ with $Q_1 \subseteq K$ and $Q_2 \subseteq K$, then $\mathcal{D} \subseteq K$ by part (b), which is a contradiction.

(c) $\Rightarrow$ (d). Let Q be graded proper secondary submodule of $\mathcal{D}$. By Corollary 3.13, Q contains a graded minimal submodule H. But every graded minimal submodule is graded secondary. Hence $Q = H$ by part (c).

(d) $\Rightarrow$ (a). As $Gdim^{s*}\mathcal{D} = 1$, there must exists a chain of graded secondary submodules $Q_2 \subset Q_1$. By part (d), Q_1 must be $\mathcal{D}$. □

Corollary 3.15. *If $\mathcal{A}$ is graded one dimensional domain and $\mathcal{D}$ is graded secondary finitely cogenerated cotorsion $\mathcal{A}$-module with $Gdim^{s*}\mathcal{D} = 1$, then $GSec^*(K)$ is graded secondary for every non-zero graded submodule K of $\mathcal{D}$.*

Proof. This follows from Theorem 3.14. □

Lemma 3.16. *If $\mathcal{A}$ is graded integral domain and $\mathcal{D}$ is graded artinian $\mathcal{A}$-module such that $I_0^{\mathcal{D}}(\mathcal{D}) \neq 0$, then $I_0^{\mathcal{D}}(\mathcal{D})$ is graded (0)-second submodule of $\mathcal{D}$.*

Proof. Since $\mathcal{D}$ is graded artinian module, by Theorem 9.4.3 [12], there exists finite number of completely graded irreducible submodules $L_1, L_2, ..., L_n$ of $\mathcal{D}$ such that $I_0^{\mathcal{D}}(\mathcal{D}) = \cap_i^n L_i$ and $r_i\mathcal{D} \subseteq L_i$, for some $0 \neq r_i \in h(\mathcal{A})$, $i = 1, 2, ..., n$. Set $r = r_1r_2...r_n \in h(\mathcal{A})$, then $r\mathcal{D} \subseteq I_0^{\mathcal{D}}(\mathcal{D})$. Now assume that $tI_0^{\mathcal{D}}(\mathcal{D}) \subseteq L$ for some $t \in h(\mathcal{A})$ and completely graded irreducible submodule L of $\mathcal{D}$ and that $I_0^{\mathcal{D}}(\mathcal{D}) \nsubseteq L$. As $I_0^{\mathcal{D}}(\mathcal{D}) \nsubseteq L$, we have $Ann_{\mathcal{A}}(\mathcal{D}/L) = 0$. Now we have $rt\mathcal{D} \subseteq tI_0^{\mathcal{D}}(\mathcal{D}) \subseteq L$. Since $\mathcal{A}$ is graded integral domain and $r \neq 0$, we must have $t = 0$. Thus by (Lemma 2.5, [6]), $I_0^{\mathcal{D}}(\mathcal{D})$ is graded second submodule of $\mathcal{D}$. Suppose $s \in Ann_{\mathcal{A}}(I_0^{\mathcal{D}}(\mathcal{D}))$. As $I_0^{\mathcal{D}}(\mathcal{D}) \neq 0$, there exists completely graded irreducible submodule J of $\mathcal{D}$ such that $I_0^{\mathcal{D}}(\mathcal{D}) \nsubseteq J$. Thus $Ann_{\mathcal{A}}(\mathcal{D}/J) = 0$ so that $Ann_{\mathcal{A}}(\mathcal{D}) = 0$. Therefore $sr\mathcal{D} \subseteq sI_0^{\mathcal{D}}(\mathcal{D}) = 0$ implies that $s = 0$. Hence $Ann_{\mathcal{A}}(I_0^{\mathcal{D}}(\mathcal{D})) = 0$. □

Theorem 3.17. *Let $\mathcal{A}$ be graded integral domain and $\mathcal{D}$ be graded $\mathcal{A}$-module with $Gdim^{s*}\mathcal{D} = 1$. If $\mathcal{D}$ is graded artinian cotorsion-free, then $GSec^*(K) = Soc^{gr}(K)$ for any proper graded submodule K of $\mathcal{D}$.*

Proof. Since $\mathcal{D}$ is graded artinian cotorsion-free, $I_0^{\mathcal{D}}(\mathcal{D}) = \mathcal{D} \neq 0$ is graded second submodule by Lemma 3.16 and so graded secondary submodule and as $\mathcal{D}$ is graded artinian, then every non-zero graded submodule of $\mathcal{D}$ contains a garded minimal submodule. As $Gdim^{s*}\mathcal{D} = 1$, then every proper graded secondary submodule of $\mathcal{D}$ is graded minimal. Thus $GSec^*(K) \subseteq Soc^{gr}(K)$, where K is a graded submodule of M. The reverse inclusion is clear since every graded minimal submodule of $\mathcal{D}$ is graded second submodule and so graded secondary submodule. □

Definition 3.18. Let $\mathcal{D}$ be graded $\mathcal{A}$-module. We say that a non-zero graded submodule K of $\mathcal{D}$ is graded secondary cocompactly packed if for each family $\{Q_i\}_{i\in I}$ of graded secondary submodules of $\mathcal{D}$ with $\cap_{i\in I}Q_i \subseteq K$, we have $Q_i \subseteq K$ for some $i \in I$. A graded module $\mathcal{D}$ is called secondary cocompactly packed if every graded submodule of $\mathcal{D}$ is graded secondary cocompactly packed .

Theorem 3.19. *Let $f : \mathcal{D} \to \mathcal{D}'$ be a graded monomorphism of graded $\mathcal{A}$-modules. Then the following hold:*

(a) *A graded submodule K of $\mathcal{D}$ is graded secondary submodule of $\mathcal{D}$ if and only if $f(K)$ is graded secondary submodule of $\mathcal{D}'$.*

(b) *If K' is graded secondary submodule of $\mathcal{D}'$ with $K' \subseteq Im(f)$, then $f^{-1}(K')$ is graded secondary submodule of $\mathcal{D}$.*

(c) *If $\mathcal{D}'$ is graded secondary cocompactly packed, then $\mathcal{D}$ is so. The reverse is true if $GSec^*(\mathcal{D}') \subseteq Im(f)$.*

Proof. (a) Let $r_g \in h(\mathcal{A})$, then $r_gK = K$ or $r_g^mK = 0$ for some positive integer m. Thus $r_gf(K) = f(K)$ or $r_g^mf(K) = 0$, and so $f(K)$ is graded secondary submodule of $\mathcal{D}'$. Now, let $f(K)$ is graded secondary submodule of $\mathcal{D}'$ and $t_h \in h(\mathcal{A})$, then $t_hf(K) = f(K)$ or $t_h^nf(K) = 0$ for some positive integer n. Hence $t_hf^{-1}(f(K)) = f^{-1}(f(K))$ or $t_h^nf^{-1}(f(K)) = 0$, since f is injective we have $t_hK = K$ or $t_h^nK = 0$, as needed.

(b) Let K' be a graded secondary submodule of $\mathcal{D}'$ with $K' \subseteq Im(f)$ and let $r_g \in h(\mathcal{A})$, then $r_gK' = K'$ or $r_g^mK' = 0$ for some positive integer m. Therefore $r_gf^{-1}(K') = f^{-1}(K')$ or $r_g^mf^{-1}(K') = 0$, as required.

(c) Let K be non-zero graded submodule of $\mathcal{D}$ and let $\{Q_i\}_{i\in I}$ be family of graded secondary submodule of $\mathcal{D}$ with $\cap_{i\in I}Q_i \subseteq K$. Then $\{f(Q_i)\}_{i\in I}$ is family of graded secondary submodule of $\mathcal{D}'$ and
$\cap_{i\in I}f(Q_i) = f(\cap_{i\in I}Q_i) \subseteq f(K)$. As $\mathcal{D}'$ is graded secondary cocompactly packed, then there exists $i \in I$ such that $f(Q_i) \subseteq f(K)$. Therefore, $Q_i = f^{-1}(f(Q_i) \subseteq f^{-1}(f(K)) = K$, as needed. Conversely, let K' be graded submodule of $\mathcal{D}$, let $\{Q_i'\}_{i\in I}$ be family of graded secondary submodule of $\mathcal{D}'$ with $\cap_{i\in I}Q_i' \subseteq K'$. Since $Q_i' \subseteq GSec^*(\mathcal{D}') \subseteq Im(f)$, hence by part (b), $\{f^{-1}(Q_i')\}_{i\in I}$ is family of graded secondary submodule of $\mathcal{D}$ and $\cap_{i\in I}f^{-1}(Q_i') = f^{-1}(\cap_{i\in I}(Q_i')) \subseteq f^{-1}(K')$, so there exists $i \in I$ such that $f^{-1}(Q_i') \subseteq f^{-1}(K')$. Thus $f(\mathcal{D}) \cap Q_i' = f(f^{-1}(Q_i')) \subseteq f(f^{-1}(K')) \subseteq K'$ and so $Q_i' \subseteq K'$, as required. □

Lemma 3.20. *Let $\mathcal{D}$ be graded secondary cocompactly packed $\mathcal{A}$-module. Then for each non-zero graded submodule K of $\mathcal{D}$ there exists a completely graded irreducible submodule L of $\mathcal{D}$ such that $GSec^*(K) = GSec^*(L)$.*

Proof. Let K be a non-zero graded submodule of $\mathcal{D}$. Then there exists a family $\{L_\lambda\}_{\lambda\in\Delta}$ of completely graded irreducible submodules of $\mathcal{D}$ such that $K = \cap_{\lambda\in\Delta}L_\lambda$. It is clear that $GSec^*(K) \subseteq GSec^*(L_\lambda)$ for each $\lambda \in \Delta$. Assume that $GSec^*(L_\lambda) \nsubseteq GSec^*(K)$ for each $\lambda \in \Delta$. Then for each $\lambda \in \Delta$, there exists a graded secondary submodule Q_λ such that $Q_\lambda \nsubseteq K$ and $Q_\lambda \subseteq L_\lambda$. Now $\cap_{\lambda\in\Delta}Q_\lambda \subseteq K$, hence $\mathcal{D}$ is not a graded secondary cocompactly packed module, which is a contradiction. □

Lemma 3.21. *Let $\mathcal{D}$ be graded $\mathcal{A}$-module satisfying $AB5^*$ property, if $\mathcal{D}$ is graded fully copure, then every non-zero graded submodule of $\mathcal{D}$ is graded socle submodule of $\mathcal{D}$ (that is. $GSec(K) = K$).*

Proof. Let K be a non-zero graded submodule of $\mathcal{D}$, then there exists a completely graded irreducible submodule L of $\mathcal{D}$ such that $K \nsubseteq L$. For each completely graded irreducible submodule L of $\mathcal{D}$ with $K \nsubseteq L$ set $\Omega_L = \{H|H \subseteq K \text{ and } H \nsubseteq L\}$.Then $\Omega_L \neq \emptyset$ since $K \in \Omega_L$. Order Ω_L by inclusion, i.e. for $Q, S \in \Omega_L$ then $Q \leqslant S$ if $Q \subseteq S$. Clearly, $(\Omega_L, \leqslant)$ is a partially order set. Let $\{H_\alpha\}_{\alpha\in\Delta}$ be any chain in Ω_L. Since $\mathcal{D}$ satisfies $AB5^*$ property, by Zorn's Lemma, $\cap_{\alpha\in\Delta}H_\alpha$ is a lower bound of $\{H_\alpha\}_{\alpha\in\Delta}$ in Ω_L. Thus Ω_L has a minimal element, say H_L. Clearly by Remark 2.3, $K = \sum_L H_L$. Set $T_L = L \cap H_L$. Then H_L is a local graded $\mathcal{A}$-module with graded maximal submodule T_L. Let $P = (T_L :_{\mathcal{A}} H_L)$. Now since $\mathcal{D}$ is graded fully copure $T_L + (0 :_{H_L} P) = (T_L :_{H_L} P)$, and so $PH_L = 0$. Now if $r \in h(\mathcal{A})$ such that $rH_L \neq H_L$, then $rH_L \subseteq T_L$ and hence $rH_L = 0$. Thus H_L is graded second submodule of $\mathcal{D}$ for each completely graded irreducible submodule L of $\mathcal{D}$ with $K \nsubseteq L$ and hence K is graded socle submodule of $\mathcal{D}$. □

Theorem 3.22. *Let $\mathcal{D}$ be a graded fully copure $\mathcal{A}$-module which satisfies the $AB5^*$ property. If $\mathcal{D}$ is graded secondary cocompactly packed module, then every non-zero graded submodule of $\mathcal{D}$ is graded completely irreducible.*

Proof. Let K be a proper graded submodule of $\mathcal{D}$. Then by Lemma 3.20, there a completely graded irreducible submodule L of $\mathcal{D}$ with $K \subseteq L$ such that $GSec^*(K) = GSec^*(L)$. But $K = GSec^*(K)$ for each non-zero graded submodule K of $\mathcal{D}$ by Lemma 3.21. Therefore $K = L$. □

Example 3.23. Let $\mathcal{A} = \mathbb{Z}$ and $G = \mathbb{Z}_2$, then $\mathcal{A}$ is a G-graded ring with $\mathcal{A}_0 = \mathbb{Z}$ and $\mathcal{A}_1 = \{0\}$. Let $\mathcal{D} = \mathbb{Z}_6$, then $\mathcal{D}$ is G-graded $\mathcal{A}$-module with $\mathcal{D}_0 = \mathbb{Z}_6$ and $\mathcal{D}_1 = \{\bar{0}\}$. Then $\mathcal{D}$ is gr-fully copure $\mathbb{Z}$-module which satisfies the $AB5^*$ property. Also, $\mathcal{D}$ is graded secondary cocompactly packed module. Hence by Theorem 3.22, every non-zero graded submodule of $\mathcal{D}$ is graded completely irreducible.

Theorem 3.24. *Let $\mathcal{D}$ be a graded $\mathcal{A}$-module. Then the following are equivalent:*

(a) *$\mathcal{D}$ is graded secondary cocompactly packed module.*

(b) *For each non-zero graded submodule K of $\mathcal{D}$, if $\{K_i\}_{i\in I}$ is a family of graded submodules of $\mathcal{D}$ with $\cap_{i\in I}K_i \subseteq K$, then $GSec^*(K_i) \subseteq K$ for some $i \in I$.*

(c) *For each non-zero graded submodule K of $\mathcal{D}$, if $\{K_i\}_{i\in I}$ is a family of graded secondary radical submodules of $\mathcal{D}$ with $\cap_{i\in I}K_i \subseteq K$, then $K_i \subseteq K$ for some $i \in I$.*

Proof. (a) ⇒ (b). Let K be a non-zero graded submodule of $\mathcal{D}$ and $GSec^*(K_i) \nsubseteq K$ for each $i \in I$. Then for each $i \in I$ there exists a graded secondary submodule Q_i such that $Q_i \subseteq K_i$ and $Q_i \nsubseteq K$. Thus $\cap_{i\in I}Q_i \subseteq K$, this implies that $\mathcal{D}$ is not graded secondary cocompactly packed module, which is a contradiction.

(b) ⇒ (c). Clear.

(c) ⇒ (a). This follows from the fact that each graded secondary submodule is a graded secondary radical submodule.

□

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