Palestine Journal of Mathematics

Vol.15(3), (2026) , 660–676 © Palestine Polytechnic University-PPU 2026

A High-Accuracy Galerkin Scheme Using Shifted Fibonacci Polynomials for Solving the Time-Fractional Black–Scholes Equation

W. M. Abd-Elhameed, Y.H. Youssri and A. G. Atta

Communicated by Fathalla Ali Rihan

MSC 2010 Classifications: 35R11, 65N35, 40A05.

Fibonacci polynomials; shifted polynomials; Galekin method; error analysis

Abstract This paper employs a Galerkin approach to solve the time-fractional Black-Scholes model ($\mathcal{TFBSM}$), a crucial model in financial mathematics for option pricing under anomalous diffusion. The method designed utilizes modified shifted Fibonacci polynomials as basis functions. Particular integrals involving these polynomials are found in closed forms, enabling us to apply the Galerkin technique. The Galerkin method converts the time-fractional model into an algebraic system of equations, facilitating its effective numerical solution. Numerous numerical examples are provided to confirm the efficacy and validity of the error analysis of the polynomial expansion.

1 Introduction

Fractional differential equations (FDEs) extend the classical differential equations (DEs) by allowing differentiation with non-integer (fractional) orders. This expansion offers a robust tool for simulating diverse, intricate phenomena that display memory effects and anomalous diffusion. There are several fields where FDEs have become prominent in the last several decades, such as engineering, chemistry, and physics; see, for example, [1, 2]. Due to the lack of exact solutions to most FDEs, numerical solutions are necessary to address these equations. Here are a few numerical algorithms used to find numerical solutions for FDEs: homotopy analysis method [3], Adomian decomposition method [4], Reproducing kernel method [5], wavelets methods [6, 7], the collocation method [8], Galerkin method [9–11], and the tau method [12, 13].

Due to their unique properties, Fibonacci polynomials are highly useful and essential in various fields, including control theory, signal processing, and cryptography. The role of these polynomials has increased in numerical analysis and approximation theory. For example, the authors of [14] followed a numerical approach for solving FDEs using Fibonacci polynomials. These polynomials were used in [15] to handle other types of FDEs. Certain Abel's integral equations were solved via the Fibonacci polynomials in [16]. Fibonacci and Lucas polynomials were used in [17] to solve multi-dimensional advection-diffusion-reaction equations.

Studies of modified and generalized Fibonacci polynomials have also increased in the field of numerical analysis. Many authors have been interested in introducing modifications and generalizations of Fibonacci polynomials and utilizing them in various applications. The authors of [18] have presented the shifted Fibonacci polynomials and used them to address the time fractional Burgers equation. The authors of [19] introduced the convolved Fibonacci polynomials and employed them to address the Fitzhugh-Nagumo non-linear equation. In [20], Vieta-Fibonacci polynomials, which are linked to Fibonacci polynomials, were employed to solve stochastic FDEs. Using an operational matrix approach, the same polynomials were used in [21] to treat other FDEs. They were also used in other applications; see [22, 23]. In [24], the authors introduced fractional-order Fibonacci-hybrid functions to treat some delay DEs.

The importance of spectral methods [25, 26] as a family of numerical approaches [27–29] for solving significant problems that arise in several fields has recently come to light. These approaches offer several benefits comparable to those of other numerical approaches; see [30, 31]. These approaches produce accurate approximations that converge at exponential rates. Three main spectral approaches are available. There are benefits and applications for every approach.

The tau and collocation methods have fewer restrictions than the Galerkin method in choosing the basis functions; see [32–35]. According to [36–39], the Galerkin and Petrov-Galerkin techniques work well for linear and some specific non-linear problems.

The time-fractional Black-Scholes model ($\mathcal{TFBSM}$) is a generalization of the standard Black-Scholes model, which incorporates fractional calculus to accurately represent the irregular diffusion and memory effects observed in financial markets. Looking at it from a historical perspective, the Black-Scholes model (BSM) was created in 1973 by Black and Scholes [40] and Merton [41] to explain the predicted behavior of the underlying financial markets. The widespread use of this model by options traders has significantly contributed to the substantial increase in options trading. This may be attributed to the model's high level of accuracy and effectiveness in predicting options prices. This model and its variants were treated using different numerical approaches. In [42], the authors used a meshless method for solving numerically linear and nonlinear $\mathcal{TFBSM}$. Another numerical scheme is applied in [43] to treat the $\mathcal{TFBSM}$. In [44], a second-order numerical method was employed for solving the same types of equations. Based on Gegenbauer polynomials, the authors of [45] found approximate solutions for treating these polynomials. A difference scheme is applied in [46] to handle the tempered $\mathcal{TFBSM}$. The authors used fractional Jaiswal functions in [47] to solve the mixed $\mathcal{TFBSM}$. The authors of [48] found analytical and approximate solutions for the $\mathcal{TFBSM}$. A numerical scheme was applied in [49] for treating the same model. Another approach was followed in [50]. Based on the Laguerre neural network, the generalized Black-Scholes differential equation was handled in [51]. A sextic B-spline method was followed to treat the generalized $\mathcal{TFBSM}$ model in [52].

This paper aims to solve the $\mathcal{TFBSM}$ using the shifted Fibonacci polynomials as basis functions. The celebrated Galerkin method will be applied for this purpose. To our knowledge, this approach for solving such a model is new. The originality of this study lies in its use of modified shifted Fibonacci polynomials as basis functions to solve the time-fractional Black-Scholes model using the Galerkin technique. This novel method enhances the TFBSM's overall analytical tractability by providing a fresh perspective on managing fractional time derivatives in financial models and facilitating the computation of integrals in closed form.

The following is the outline of the paper. The Fibonacci polynomials and their shifted counterparts are introduced in Section 2, along with some basic information and properties. A Galerkin procedure for treating the $\mathcal{TFBSM}$ is followed with derivation in Section 3. A study of the error analysis is performed in Section 4. Some illustrative examples are presented in Section 5. Concluding remarks are reported in Section 6.

2 Some Essentials and Preliminaries

This section is dedicated to illustrating the principles of fractional calculus. Furthermore, an overview of Fibonacci polynomials and their shifted variants is provided.

2.1 Caputo's Fractional Derivative

Definition 1. [1], In Caputo's sense, the fractional-order derivative of $\xi \in C^p[0,s]$, where $p = \lceil \eta \rceil$ is defined as

$$\frac{d^\eta \xi(s)}{d\, s^\eta} = \frac{1}{\Gamma(p-\eta)} \int_0^s (s-t)^{p-\eta-1} \xi^{(p)}(t) dt, \quad \eta > 0, \quad s > 0,\ p-1 \le \eta < p, \quad p \in \mathbb{N}. \tag{2.1}$$

For $\frac{d^\eta}{d\, s^\eta}$ with $p-1 \leqslant \eta < p, \quad p \in \mathbb{N}$, the following identities are valid:

$$\frac{d^\eta\, C}{d\, s^\eta} = 0, \quad C \text{ is a constant}, \tag{2.2}$$

$$\frac{d^\eta\, s^p}{d\, s^\eta} = \begin{cases} 0, & \text{if } p \in \mathbb{N}_0 \quad and \quad p < \lceil \eta \rceil, \\ \frac{(p)!}{\Gamma(p-\eta+1)}\, s^{p-\eta}, & \text{if } p \in \mathbb{N}_0 \quad and \quad p \ge \lceil \eta \rceil, \end{cases} \tag{2.3}$$

where $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\lceil \eta \rceil$ is the ceiling function.

2.2 Some Properties of Fibonacci Polynomials and Their Shifted Counterparts

The following recursive formula can be used to generate Fibonacci polynomials [53]:

$$F_{j+2}(z) = z\,F_{j+1}(z) + F_j(z), \quad F_0(z) = 0,\ F_1(z) = 1, \quad j = 0, 1, \ldots, \tag{2.4}$$

and they can be written as

$$F_j(z) = \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j-r-1}{r} z^{j-2r-1}, \tag{2.5}$$

where $\lfloor z \rfloor$ is the well-known floor function.

In [18], the authors introduced the shifted Fibonacci polynomials defined as:

$$F_j^*(z) = F_j(2z-1). \tag{2.6}$$

In addition, they developed some fundamental formulas regarding these polynomials and utilized them to solve the fractional Burgers equation.
The following theorem presents the series representation of the shifted Fibonacci polynomials.

Theorem 1. [18] $\mathbf{F}_{i+1}^*(t)$ can be expressed in the following form:

$$\mathbf{F}_{j+1}^*(t) = \sum_{\ell=0}^{j} B_{\ell,j}\, t^{\ell}, \quad j \geq 0. \tag{2.7}$$

where

$$B_{\ell,j} = (-1)^{j+\ell}\,(2)^{\ell} \binom{j}{\ell}\, {}_2F_1\left(\begin{matrix} \frac{1}{2}(-j+\ell), \frac{1}{2}(1-j+\ell) \\ -j \end{matrix} \middle| -4 \right). \tag{2.8}$$

3 Galerkin Procedure for the $\mathcal{TFBSM}$

We are interested in handling the following $\mathcal{TFBSM}$ [54–56]:

$$\frac{\partial^{\eta}\,\xi(\theta,t)}{\partial\, t^{\eta}} - z_1\,\frac{\partial^2\,\xi(\theta,t)}{\partial\,\theta^2} - z_2\,\frac{\partial\,\xi(\theta,t)}{\partial\,\theta} + z_3\,\xi(\theta,t) = S(\theta,t), \quad 0 < \eta < 1, \tag{3.1}$$

controlled by the conditions:

$$\xi(\theta,0) = \xi_1(\theta), \quad 0 < \theta \leq 1, \tag{3.2}$$

$$\xi(0,t) = \xi_2(t), \quad \xi(1,t) = \xi_3(t), \quad 0 < t \leq 1, \tag{3.3}$$

where $z_1 = \frac{1}{2}\,\sigma^2 > 0$, $z_2 = z_3 - z_1$, z_3 is nonnegative and $S(\theta,t)$ is the source term.
We provide the following transformation to go forward in our Galerkin scheme:

$$\phi(\theta,t) := \xi(\theta,t) + \hat{\xi}(\theta,t), \tag{3.4}$$

where

$$\hat{\xi}(\theta,t) = -\,(1-\theta)\;\xi_2(t) - \theta\,\xi_3(t), \tag{3.5}$$

and consequently, the $\mathcal{TFBSM}$ (3.1) governed by (3.2) and (3.3) is transformed into the following modified equation:

$$\frac{\partial^{\eta}\,\phi(\theta,t)}{\partial\, t^{\eta}} - z_1\,\frac{\partial^2\,\phi(\theta,t)}{\partial\,\theta^2} - z_2\,\frac{\partial\,\phi(\theta,t)}{\partial\,\theta} + z_3\,\phi(\theta,t) = g(\theta,t), \quad 0 < \eta < 1, \tag{3.6}$$

controlled by the following conditions:

$$\phi(\theta,0) = \xi_1(\theta) - (1-\theta)\;\xi_2(0) - \theta\,\xi_3(0), \quad 0 < \theta \leq 1, \tag{3.7}$$

$$\phi(0,t) = \phi(1,t) = 0, \quad 0 < t \leq 1, \tag{3.8}$$

where

$$g(\theta,t) = S(\theta,t) + \frac{\partial^{\eta}\,\hat{\xi}(\theta,t)}{\partial\, t^{\eta}} - z_1\,\frac{\partial^2\,\hat{\xi}(\theta,t)}{\partial\,\theta^2} - z_2\,\frac{\partial\,\hat{\xi}(\theta,t)}{\partial\,\theta} + z_3\,\hat{\xi}(\theta,t). \tag{3.9}$$

Thus, we can solve the modified equation (3.6), which is controlled by (3.7), and (3.8) instead of solving (3.1), which is governed by (3.2), (3.3).

3.1 Trial Functions

Our Galerkin approach can be applied using the following trial functions:

$$\varepsilon_i(\theta) = \theta\,(1-\theta)\,\mathbf{F}_i^*(\theta), \tag{3.10}$$

along with $\mathbf{F}_j^*(t)$ defined in (2.7). To proceed with our proposed numerical algorithm, some specific definite integrals involving the basis functions $\varepsilon_i(\theta)$ in (3.10). The following theorem exhibits these definite integrals.

Theorem 2. The following five integral formulas are valid:

$$(1)\quad a_{i,r} = \int_0^1 \varepsilon_i(\theta)\,\varepsilon_r(\theta)\,d\theta = \sum_{p=0}^{i}\sum_{k=0}^{r} \frac{2\,B_{p,i}\,B_{k,r}}{(k+p+3)\,(k+p+4)\,(k+p+5)}, \tag{3.11}$$

$$(2)\quad h_{i,r} = \int_0^1 \frac{d\,\varepsilon_i(\theta)}{d\,\theta}\,\varepsilon_r(\theta)\,d\theta = \sum_{p=0}^{i}\sum_{k=0}^{r} \frac{(p-k)\,B_{p,i}\,B_{k,r}}{(k+p+2)(k+p+3)(k+p+4)}, \tag{3.12}$$

$$(3)\quad b_{i,r} = \int_0^1 \frac{d^2\,\varepsilon_i(\theta)}{d\,\theta^2}\,\varepsilon_r(\theta)\,d\theta = \sum_{p=0}^{i}\sum_{k=0}^{r} \frac{-2\,(k+1)\,(p+1)\,B_{p,i}\,B_{k,r}}{(k+p+1)(k+p+2)(k+p+3)}, \tag{3.13}$$

$$(4)\quad v_{j,s} = \int_0^1 \mathbf{F}_i^*(t)\,\mathbf{F}_r^*(t)\,dt = \sum_{p=0}^{i}\sum_{k=0}^{r} \frac{B_{p,i}\,B_{k,r}}{k+p+1}, \tag{3.14}$$

$$(5)\quad k_{j,s} = \int_0^1 \frac{d^{\eta}\,\mathbf{F}_j^*(t)}{d\,t^{\eta}}\,\mathbf{F}_s^*(t)\,dt = \sum_{p=1}^{j}\sum_{n=0}^{s} \frac{p!\,B_{p,j}\,B_{n,s}}{\Gamma(p-\eta+1)\,(-\eta+n+p+1)}, \tag{3.15}$$

where the coefficients $B_{p,i}$ are as given in (2.8).

Proof. The proof of parts (1), (2), (3), (4) can be easily obtained after utilizing the power formula of $\mathbf{F}_{i+1}^*(t)$ in (2.7) along with the definition of basis functions in (3.10) and integrating the right-hand side to get the desired result. The proof of part (5) can be obtained after using Caputo's definition (2.3) and following similar steps as the previous parts. □

3.2 Galerkin Solution for the $\mathcal{TFBSM}$

Consider the following two space functions

$$\begin{aligned} \mathcal{Z}_N(\Omega) &= \text{span}\{\varepsilon_i(\theta)\,\mathbf{F}_j^*(t) : i,j = 0,1,\ldots,N\},\\ \mathcal{M}(\Omega) &= \{\phi(\theta,t) \in \mathcal{Z}_N(\Omega) : \phi(0,t) = \phi(1,t) = 0\}, \end{aligned} \tag{3.16}$$

where $\Omega = [0,1]^2$.
It is then easy to express any $\hat{\phi}(\theta,t) \in \mathcal{M}(\Omega)$ as

$$\hat{\phi}(\theta,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,\varepsilon_i(\theta)\,\mathbf{F}_j^*(t) = \boldsymbol{\varepsilon}\,\boldsymbol{C}\,\mathbf{F}^{*T}, \tag{3.17}$$

where

$$\boldsymbol{\varepsilon} = [\varepsilon_0(\theta),\varepsilon_1(\theta),\ldots,\varepsilon_N(\theta)], \quad \mathbf{F}^{*T} = [\mathbf{F}_0^*(t),\mathbf{F}_1^*(t),\ldots,\mathbf{F}_N^*(t)]^T,$$

and $\boldsymbol{C} = (c_{ij})_{0\leq i,\, j\leq N}$ is the matrix of unknowns has the order $(N+1)^2$.
The residual $\mathrm{R}(\theta, t)$ of Eq. (3.6) can now be calculated to give

$$\mathrm{R}(\theta,t) = \frac{\partial^{\eta}\,\hat{\phi}(\theta,t)}{\partial\, t^{\eta}} - z_1\,\frac{\partial^2\,\hat{\phi}(\theta,t)}{\partial\,\theta^2} - z_2\,\frac{\partial\,\hat{\phi}(\theta,t)}{\partial\,\theta} + z_3\,\hat{\phi}(\theta,t) - g(\theta,t). \tag{3.18}$$

Now, the Galerkin scheme enables us to write

$$\int_0^1\int_0^1 \mathrm{R}(\theta,t)\,\varepsilon_r(\theta)\,\mathbf{F}_s^*(t)\,d\theta\,dt = 0, \quad 0\leq r\leq N, \quad 0\leq s\leq N-1. \tag{3.19}$$

Let us define

$$\boldsymbol{G} = (g_{r,s})_{(N+1)\times N}, \qquad g_{r,s} = \int_0^1\int_0^1 g(\theta,t)\,\varepsilon_r(\theta)\,\mathbf{F}_s^*(t)\,d\theta\,dt, \tag{3.20}$$

$$\boldsymbol{\mathcal{A}} = (a_{i,r})_{(N+1)\times(N+1)}, \qquad a_{i,r} = \int_0^1 \varepsilon_i(\theta)\,\varepsilon_r(\theta)\,d\theta, \tag{3.21}$$

$$\boldsymbol{\mathcal{H}} = (h_{i,r})_{(N+1)\times(N+1)}, \qquad h_{i,r} = \int_0^1 \frac{d\,\varepsilon_i(\theta)}{d\,\theta}\,\varepsilon_r(\theta)\,d\theta, \tag{3.22}$$

$$\boldsymbol{\mathcal{B}} = (b_{i,r})_{(N+1)\times(N+1)}, \qquad b_{i,r} = \int_0^1 \frac{d^2\,\varepsilon_i(\theta)}{d\,\theta^2}\,\varepsilon_r(\theta)\,d\theta, \tag{3.23}$$

$$\boldsymbol{\mathcal{V}} = (v_{j,s})_{(N+1)\times N}, \qquad v_{j,s} = \int_0^1 \mathbf{F}_j^*(t)\,\mathbf{F}_s^*(t)\,dt, \tag{3.24}$$

$$\boldsymbol{\mathcal{K}} = (k_{j,s})_{(N+1)\times N}, \qquad k_{j,s} = \int_0^1 \frac{d^{\eta}\,\mathbf{F}_j^*(t)}{d\,t^{\eta}}\,\mathbf{F}_s^*(t)\,dt. \tag{3.25}$$

As a result, we can rewrite Eq. (3.19) as

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,a_{i,r}\,k_{j,s} - z_1\,\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,b_{i,r}\,v_{j,s} - z_2\,\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,h_{i,r}\,v_{j,s} + z_3\,\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,a_{i,r}\,v_{j,s} = g_{r,s}. \tag{3.26}$$

Alternatively, in matrix form as

$$\boldsymbol{\mathcal{A}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{K}} - z_1\,\boldsymbol{\mathcal{B}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{V}} - z_2\,\boldsymbol{\mathcal{H}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{V}} + z_3\,\boldsymbol{\mathcal{A}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{V}} = \boldsymbol{G}. \tag{3.27}$$

Moreover, the initial condition in (3.7) leads to

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,a_{i,r}\,\mathbf{F}_j^*(0) = \int_0^1 \phi(\theta,0)\,\varepsilon_r(\theta)\,d\theta, \quad 0\leq r\leq N. \tag{3.28}$$

Currently, the Gauss elimination method may be used to solve the system of algebraic equations of order $(N+1)^2$ that consists of Eqs. (3.27) and (3.28).

Remark 1. The elements of matrices $a_{i,r}$, $h_{i,r}$, $b_{i,r}$, $v_{j,s}$ and $k_{j,s}$ are given in Theorem 2.

4 Error Bound

A comprehensive discussion of the suggested technique's error analysis is given in this section. Thus, we shall present and prove four theorems.

- An upper estimate for $\|\phi(\theta,t)-\hat{\phi}(\theta,t)\|_2$ is provided by the first theorem.
- An upper estimate for $\|\frac{\partial^m\,(\,\phi(\theta,t)-\hat{\phi}(\theta,t))}{\partial\,\theta^m}\|_2,\quad m\in\mathbb{N}$ is provided by the second theorem.
- An upper estimate for $\|D_t^{\eta}\,(\phi(\theta,t)-\hat{\phi}(\theta,t))\|_2,\quad 0<\eta<1$, is provided by the third theorem.
- An upper estimate for $\|\mathrm{R}(\theta,t)\|_2$ is provided by the final theorem.

Theorem 3. Assume that $\frac{\partial^{i+j}\,\phi(\theta,t)}{\partial\,\theta^i\,\partial\,t^j}\in\mathbf{C}(\Omega)$, $i=0,1,2,\ldots,N+3$, $j=0,1,2,\ldots,N+1$, and $\hat{\phi}(\theta,t)$ is the numerical solution belongs to $\mathcal{M}(\Omega)$ and

$$\mathcal{L}_N=\sup_{(\theta,t)\in\Omega}\left|\frac{\partial^{2\,(N+2)}\,\phi(\theta,t)}{\partial\,\theta^{N+3}\,\partial\,t^{N+1}}\right|. \tag{4.1}$$

Consequently, this estimate is valid:

$$\|\phi(\theta,t)-\hat{\phi}(\theta,t)\|_2\le\frac{\mathcal{L}_N}{(N+3)!\,(N+1)!\,\sqrt{(2\,N+7)}\,\sqrt{(2\,N+3)}}. \tag{4.2}$$

Proof. Assume that

$$\bar{v}(\theta,t)=\sum_{i=0}^{N+2}\sum_{j=0}^{N}\left(\frac{\partial^{i+j}\,\phi(\theta,t)}{\partial\,\theta^i\,\partial\,t^j}\right)_{(0,0)}\frac{\theta^i\,t^j}{i!\,j!}, \tag{4.3}$$

is the Taylor expansion of $\phi(\theta,t)$ of degree $(N+2)\times N$ about the point $(0,0)$, and

$$\phi(\theta,t)-\bar{v}(\theta,t)=\frac{\theta^{N+3}\,t^{N+1}\,\partial^{2\,(N+2)}\,\phi(a_1,a_2)}{(N+3)!\,(N+1)!\,\partial\,\theta^{N+3}\,\partial\,t^{N+1}},\quad (a_1,a_2)\in\Omega. \tag{4.4}$$

Based on the idea of the best approximation, since $\hat{\phi}(\theta,t)$ is the best approximate solution to $\phi(\theta,t)$, we obtain

$$\begin{aligned}\|\phi(\theta,t)-\hat{\phi}(\theta,t)\|_2^2&\le\|\phi(\theta,t)-\bar{v}(\theta,t)\|_2^2\\&=\int_0^1\int_0^1\frac{{\mathcal{L}_N}^2\,\theta^{2\,(N+3)}\,t^{2\,(N+1)}}{((N+3)!)^2\,((N+1)!)^2}\,d\,\theta\,d\,t\\&=\frac{{\mathcal{L}_N}^2}{((N+3)!)^2\,((N+1)!)^2\,(2\,N+7)\,(2\,N+3)}.\end{aligned} \tag{4.5}$$

Finally, the following estimation may be obtained

$$\|\phi(\theta,t)-\hat{\phi}(\theta,t)\|_2\le\frac{\mathcal{L}_N}{(N+3)!\,(N+1)!\,\sqrt{(2\,N+7)}\,\sqrt{(2\,N+3)}}. \tag{4.6}$$

We have now proved this theorem. □

Theorem 4. Suppose that $\phi(\theta,t)$, $\hat{\phi}(\theta,t)$ and $\frac{\partial^{i+j}\,\phi(\theta,t)}{\partial\,\theta^i\,\partial\,t^j}$ satisfy the condition of Theorem 3 and

$$\ell_{N,m}=\sup_{(\theta,t)\in\Omega}\left|\frac{\partial^{2\,N-m+4}\,\phi(\theta,t)}{\partial\,\theta^{N-m+3}\,\partial\,t^{N+1}}\right|,\quad m\in\mathbb{N}. \tag{4.7}$$

Consequently, this estimate is valid:

$$\left\|\frac{\partial^m\,(\,\phi(\theta,t)-\hat{\phi}(\theta,t))}{\partial\,\theta^m}\right\|_2\le\frac{\ell_{N,m}}{(N-m+3)!\,(N+1)!\,\sqrt{2\,(N-m)+7}\,\sqrt{2\,N+3}}. \tag{4.8}$$

Proof. Assume that $\frac{\partial^m \bar{v}(\theta,t)}{\partial \theta^m}$ is the Taylor expansion of $\frac{\partial^m \phi(\theta,t)}{\partial \theta^m}$ about the point $(0,0)$. Then the residual between $\frac{\partial^m \phi(\theta,t)}{\partial \theta^m}$ and $\frac{\partial^m \bar{v}(\theta,t)}{\partial \theta^m}$ can be written as

$$\frac{\partial^m \left(\phi(\theta,t)-\bar{v}(\theta,t)\right)}{\partial \theta^m}=\frac{\theta^{N-m+3}\, t^{N+1}\, \partial^{2N-m+4}\, \phi(a_1,a_2)}{(N-m+3)!\,(N+1)!\,\partial \theta^{N-m+3}\,\partial t^{N+1}},\quad (a_1,a_2)\in\Omega. \tag{4.9}$$

Since $\frac{\partial^m \hat{\phi}(\theta,t)}{\partial \theta^m}$ is the best approximate solution of $\frac{\partial^m \phi(\theta,t)}{\partial \theta^m}$, consequently, based on the concept of the best approximation, we obtain

$$\left\| \frac{\partial^m \left(\phi(\theta,t)-\hat{\phi}(\theta,t)\right)}{\partial \theta^m} \right\|_2 \leq \left\| \frac{\partial^m \left(\phi(\theta,t)-\bar{v}(\theta,t)\right)}{\partial \theta^m} \right\|_2. \tag{4.10}$$

The required result can now be attained using the same procedures as in Theorem 3. □

Theorem 5. Suppose that $\frac{\partial^\eta \phi(\theta,t)}{\partial t^\eta} \in \mathbf{C}(\Omega)$ satisfy the conditions of Theorem 3 and set

$$\mathcal{P}_{N,\eta}=\sup_{(\theta,t)\in\Omega}\left|\frac{\partial^{2N-\eta+4}\,\phi(\theta,t)}{\partial \theta^{N+3}\,\partial t^{N-\eta+1}}\right|,\quad 0<\eta<1. \tag{4.11}$$

Consequently, this estimate is valid:

$$\left\| \frac{\partial^\eta}{\partial t^\eta}\left(\phi(\theta,t)-\hat{\phi}(\theta,t)\right)\right\|_2 \leq \frac{\mathcal{P}_{N,\eta}}{(N+3)!\,\Gamma(N-\eta+2)\,\sqrt{2N+7}\,\sqrt{2(N-\eta)+3}}. \tag{4.12}$$

Proof. Using Caputo's formula (2.3) and the procedures outlined in Theorems 3 and 4, one may derive the proof of this theorem. □

Theorem 6. For large enough values of N, $\|\mathrm{R}(\theta,t)\|_2$ will be sufficiently small.

Proof. Eqs. (3.6) and (3.18) enable us to write $\mathrm{R}(\theta,t)$ as

$$\begin{aligned}\mathrm{R}(\theta,t)&=\frac{\partial^\eta \hat{\phi}(\theta,t)}{\partial t^\eta}-z_1\,\frac{\partial^2 \hat{\phi}(\theta,t)}{\partial \theta^2}-z_2\,\frac{\partial \hat{\phi}(\theta,t)}{\partial \theta}+z_3\,\hat{\phi}(\theta,t)-g(\theta,t)\\&=\frac{\partial^\eta}{\partial t^\eta}\left(\hat{\phi}(\theta,t)-\phi(\theta,t)\right)-z_1\,\frac{\partial^2}{\partial \theta^2}\left(\hat{\phi}(\theta,t)-\phi(\theta,t)\right)-z_2\,\frac{\partial}{\partial \theta}\left(\hat{\phi}(\theta,t)-\phi(\theta,t)\right)\\&+z_3\left(\hat{\phi}(\theta,t)-\phi(\theta,t)\right).\end{aligned} \tag{4.13}$$

Taking $\|.\|_2$ and using Theorems 4 and 5, one has

$$\begin{aligned}\|\mathrm{R}(\theta,t)\|_2 \leq\ &\frac{\mathcal{P}_{N,\eta}}{(N+3)!\,\Gamma(N-\eta+2)\,\sqrt{2N+7}\,\sqrt{2(N-\eta)+3}}\\&+z_1\,\frac{\ell_{N,2}}{(N+1)!\,(N+1)!\,\sqrt{2N+3}\,\sqrt{2N+3}}\\&+z_2\,\frac{\ell_{N,1}}{(N+2)!\,(N+1)!\,\sqrt{2N+5}\,\sqrt{2N+3}}\\&+z_3\,\frac{\mathcal{L}_N}{(N+3)!\,(N+1)!\,\sqrt{(2N+7)}\,\sqrt{(2N+3)}}.\end{aligned} \tag{4.14}$$

It is shown in Eq. (4.14) that for large enough values of N, $\|\mathrm{R}(\theta,t)\|_2$ will be sufficiently small. Thus, the proof of the theorem is complete. □

5 Illustrative Examples

This section provides five test problems to demonstrate the efficiency and practicality of our suggested algorithm.

Example 1. [54,55] Consider the $\mathcal{TFBSM}$ of the form

$$\frac{\partial^{\eta}\xi(\theta,t)}{\partial t^{\eta}} - z_1\frac{\partial^2\xi(\theta,t)}{\partial\theta^2} - z_2\frac{\partial\xi(\theta,t)}{\partial\theta} + z_3\,\xi(\theta,t) = S(\theta,t), \tag{5.1}$$

subject to

$$\begin{aligned}&\xi(\theta,0) = \theta^3+\theta^2+1, \quad 0<\theta\le 1,\\ &\xi(0,t) = (t+1)^2, \quad \xi(1,t) = 3(t+1)^2, \quad 0<t\le 1,\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}S(\theta,t) = &\left(\theta^3+\theta^2+1\right)\left(\frac{2t^{1-\eta}}{\Gamma(2-\eta)}+\frac{2t^{2-\eta}}{\Gamma(3-\eta)}\right)\\ &+(t+1)^2\left[\left(\theta^3+\theta^2+1\right)z_3 - 2(3\theta+1)z_1 - \theta(3\theta+2)z_2\right].\end{aligned}$$

In addition, $\xi(\theta,t) = (t+1)^2\left(\theta^3+\theta^2+1\right)$ is the exact solution of this problem.

This problem is solved for the two cases corresponding to

(i) $z_1 = 1,\ z_2 = -\frac{1}{2},\ z_3 = \frac{1}{2}$

(ii) $z_1 = 0.125,\ z_2 = 0.075,\ z_3 = 0.3.$

- For $z_1 = 1,\ z_2 = -\frac{1}{2},\ z_3 = \frac{1}{2}$, Table 1 shows a comparison of the L^{∞} error between our method and the method in [54] at different values of η. Figure 1 illustrates the absolute errors (AEs) at different values of η when $N = 3$.
- For $z_1 = 0.125,\ z_2 = 0.075,\ z_3 = 0.3$, Table 2 shows a comparison of the AEs between our method and the method in [55] at different values of η. Figure 2 illustrates the AEs (left) and approximate solution (right) at $\eta = 0.9$ and $N = 3$.

Table 1: Comparison of the L^{∞} error for Example 1.

	method in [54] at $N=5$		Our method at $N=3$
η	M	L^{∞} error	L^{∞} error
0.2	10	4.40574×10^{-4}	2.13718×10^{-14}
	20	1.35249×10^{-4}	
	40	4.10200×10^{-5}	
	80	1.23246×10^{-4}	
	160	3.67548×10^{-6}	
	320	1.08954×10^{-4}	
0.7	10	5.46782×10^{-3}	2.08722×10^{-14}
	20	2.23846×10^{-3}	
	40	9.13453×10^{-4}	
	80	3.72052×10^{-4}	
	160	1.51366×10^{-4}	
	320	6.15399×10^{-5}	
0.9	15	7.73898×10^{-3}	2.13718×10^{-14}
	30	3.61517×10^{-3}	
	60	1.68767×10^{-3}	
	120	7.87598×10^{-4}	
	240	3.67494×10^{-4}	

Table 2: Comparison of the AEs for Example 1 at $t = 0.4$.

	Method in [55]		Our method at $N = 3$	
θ	$\eta = 0.47$	$\eta = 0.81$	$\eta = 0.47$	$\eta = 0.81$
0.1	7.28×10^{-6}	5.41×10^{-9}	6.93889×10^{-15}	7.71605×10^{-15}
0.2	6.87×10^{-6}	5.10×10^{-9}	3.9968×10^{-15}	1.22125×10^{-15}
0.3	6.43×10^{-6}	4.78×10^{-9}	1.11022×10^{-16}	5.21805×10^{-15}
0.4	5.97×10^{-6}	4.43×10^{-9}	1.9984×10^{-15}	5.9952×10^{-15}
0.5	5.49×10^{-6}	4.08×10^{-9}	1.11022×10^{-15}	1.55431×10^{-15}
0.6	4.99×10^{-6}	3.70×10^{-9}	0	3.55271×10^{-15}
0.7	4.47×10^{-6}	3.32×10^{-9}	1.33227×10^{-15}	4.66294×10^{-15}
0.8	3.94×10^{-6}	2.92×10^{-9}	0	9.99201×10^{-16}
0.9	3.39×10^{-6}	2.51×10^{-9}	1.33227×10^{-15}	4.77396×10^{-15}

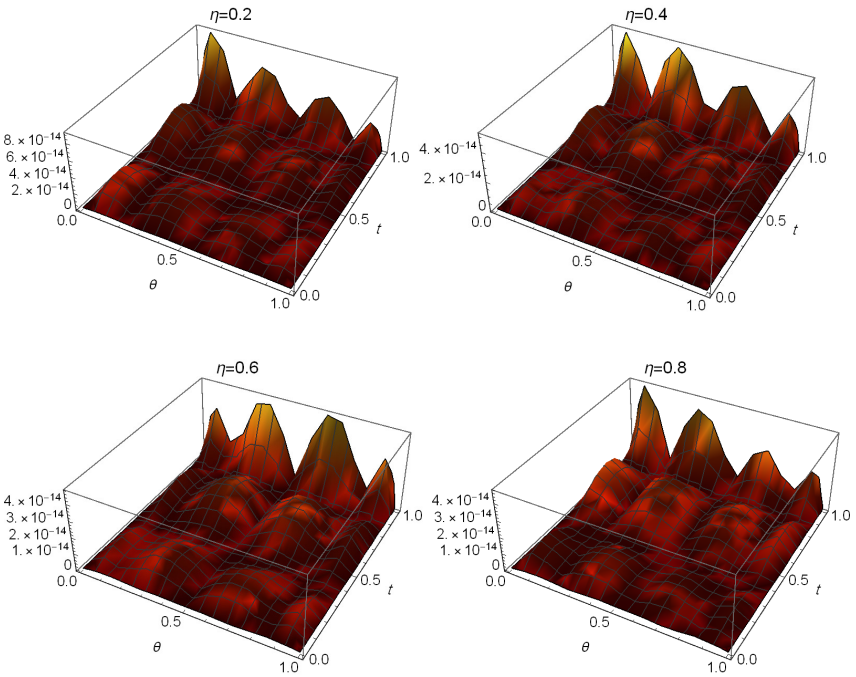


Figure 1: The AE of Example 1 at different values of η.

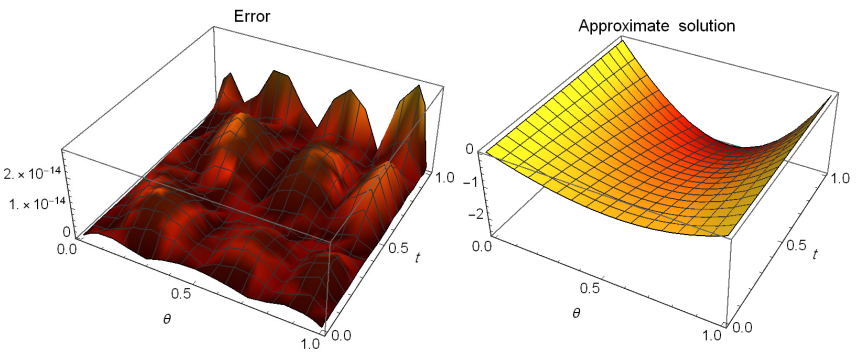


Figure 2: The AE (left) and approximate solution (right) of Example 1 at $\eta = 0.9$.

Example 2. Consider the $\mathcal{TFBSM}$ of the form

$$\frac{\partial^{\eta}\xi(\theta,t)}{\partial t^{\eta}} - z_1\frac{\partial^2\xi(\theta,t)}{\partial\theta^2} - z_2\frac{\partial\xi(\theta,t)}{\partial\theta} + z_3\xi(\theta,t) = S(\theta,t), \tag{5.3}$$

subject to

$$\begin{aligned}&\xi(\theta,0) = \cos(\pi\theta), \quad 0<\theta\le 1,\\ &\xi(0,t) = t^6+1, \quad \xi(1,t) = -t^6-1, \quad 0<t\le 1,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}S(\theta,t) =& \sin\left(\pi\theta+\frac{\pi}{2}\right)\frac{6!\,t^{6-\eta}}{\Gamma(7-\eta)} + \left(t^6+1\right)\times\\ &\left(\pi z_2\sin(\pi\theta) + \pi^2 z_1\cos(\pi\theta) + z_3\cos(\pi\theta)\right).\end{aligned}$$

In addition, the exact solution is:

$$\xi(\theta,t) = \left(t^6+1\right)\sin\left(\pi\theta+\frac{\pi}{2}\right).$$

This problem is solved for $z_1 = 0.03125$, $z_2 = 0.01875$, $z_3 = 0.05$. Table 3 presents the AEs at different values of η when $N = 7$. Table 4 presents the L^{∞}- errors at different values of η and N. Figure 3 illustrates the AEs (left) and approximate solution (right) at $\eta = 0.9$ and $N = 7$. These results verify that the suggested approach reduces errors consistently throughout the domain and shows a good agreement with the approximate solution.

Table 3: The AEs of Example 2.

(θ,t)	$\eta = 0.1$	$\eta = 0.4$	$\eta = 0.7$
(0.1,0.1)	5.52207×10^{-9}	1.2838×10^{-8}	9.85669×10^{-9}
(0.2,0.2)	1.1416×10^{-9}	9.08322×10^{-9}	2.74923×10^{-9}
(0.3,0.3)	3.0439×10^{-9}	2.02895×10^{-8}	1.20803×10^{-8}
(0.4,0.4)	3.59456×10^{-9}	2.16704×10^{-8}	1.1453×10^{-8}
(0.5,0.5)	2.61664×10^{-9}	5.8103×10^{-9}	4.96725×10^{-9}
(0.6,0.6)	1.77082×10^{-8}	2.35313×10^{-8}	2.70439×10^{-8}
(0.7,0.7)	2.92816×10^{-9}	7.5222×10^{-9}	2.25159×10^{-9}
(0.8,0.8)	1.53919×10^{-8}	2.21698×10^{-8}	2.31351×10^{-8}
(0.9,0.9)	1.40077×10^{-8}	3.01869×10^{-8}	1.57675×10^{-8}

Table 4: The L^{∞}- errors of Example 2.

N	$\eta = 0.3$	$\eta = 0.6$	$\eta = 0.9$
4	1.31966×10^{-3}	1.39526×10^{-3}	1.60631×10^{-3}
5	120079×10^{-4}	1.18002×10^{-4}	1.23533×10^{-4}
6	1.12834×10^{-6}	1.11072×10^{-6}	1.06901×10^{-6}
7	2.82342×10^{-8}	1.20619×10^{-8}	5.36803×10^{-8}

Example 3. [54, 55] Consider the $\mathcal{TFBSM}$ of the form

$$\frac{\partial^{\eta}\xi(\theta,t)}{\partial t^{\eta}} - z_1\frac{\partial^2\xi(\theta,t)}{\partial\theta^2} - z_2\frac{\partial\xi(\theta,t)}{\partial\theta} + z_3\xi(\theta,t) = S(\theta,t), \tag{5.5}$$

subject to

$$\begin{aligned}&\xi(\theta,0) = (1-\theta)\,\theta^2, \quad 0<\theta\le 1,\\ &\xi(0,t) = \xi(1,t) = 0, \quad 0<t\le 1,\end{aligned} \tag{5.6}$$

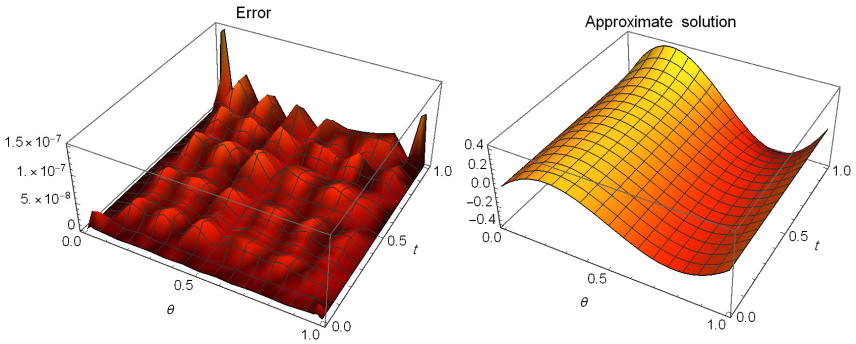


Figure 3: The AE (left) and approximate solution (right) of Example 2 at $\eta = 0.9$.

where

$$S(\theta,t) = (1-\theta)\,\theta^2\left(\frac{2t^{1-\eta}}{\Gamma(2-\eta)} + \frac{2t^{2-\eta}}{\Gamma(3-\eta)}\right) + (t+1)^2\left[(6\theta-2)z_1 + \theta\left((3\theta-2)z_2 - (\theta-1)\theta z_3\right)\right].$$

In addition, the exact solution to this problem is

$$\xi(\theta,t) = (t+1)^2\,\theta^2\,(1-\theta).$$

This problem is solved for the two cases corresponding to $z_1 = 0.03125$, $z_2 = 0.01875$, $z_3 = 0.05$ and $z_1 = 0.125$, $z_2 = 0.075$, $z_3 = 0.3$.

- For $z_1 = 0.03125$, $z_2 = 0.01875$, $z_3 = 0.05$, Table 5 shows a comparison of the L^∞ and L^2 errors between our method and the method in [54] at different values of η. Figure 4 illustrates the AE (left) and approximate solution (right) at $\eta = 0.5$ and $N = 3$.
- For $z_1 = 0.125$, $z_2 = 0.075$, $z_3 = 0.3$, Table 6 shows a comparison of AEs between our method and the method in [55] at different values of η. Figure 5 illustrates the AEs at different values of η when $N = 3$.

Table 5: Comparison of the L^∞ and L^2 errors for Example 3.

		method in [54]		Our method at $N = 3$	
η	M	L^∞ error at $N = 5$	L^2 error at $N = 5$	L^∞ error	L^2 error
0.2	100	5.5524×10^{-6}	0.00001192	4.06272×10^{-15}	1.74498×10^{-15}
	200	1.6558×10^{-6}	3.5563×10^{-6}		
	400	4.9081×10^{-7}	1.0542×10^{-6}		
	800	1.4477×10^{-7}	3.1097×10^{-6}		
	1600	4.2531×10^{-8}	9.1361×10^{-8}		
0.7	100	0.00019528	0.00041772	3.91527×10^{-15}	2.10077×10^{-15}
	200	0.00007960	0.00017030		
	400	0.00003240	0.00006932		
	800	0.00001318	0.00002820		
	1600	5.3555×10^{-6}	0.00001146		

Example 4. Consider the $\mathcal{TFBSM}$ of the form

$$\frac{\partial^\eta \xi(\theta,t)}{\partial t^\eta} - z_1\,\frac{\partial^2 \xi(\theta,t)}{\partial \theta^2} - z_2\,\frac{\partial \xi(\theta,t)}{\partial \theta} + z_3\,\xi(\theta,t) = S(\theta,t), \tag{5.7}$$

Table 6: Comparison of the AEs for Example 3 at $t = 0.4$.

	Method in [55]		Our method at $N = 3$	
θ	$\eta = 0.47$	$\eta = 0.81$	$\eta = 0.47$	$\eta = 0.81$
0.1	8.97×10^{-6}	6.67×10^{-9}	1.32186×10^{-15}	5.16948×10^{-16}
0.2	8.76×10^{-6}	6.51×10^{-9}	2.77556×10^{-16}	4.996×10^{-16}
0.3	8.52×10^{-6}	6.33×10^{-9}	1.63758×10^{-15}	1.02696×10^{-15}
0.4	8.24×10^{-6}	6.13×10^{-9}	1.58207×10^{-15}	6.38378×10^{-16}
0.5	7.93×10^{-6}	5.90×10^{-9}	3.60822×10^{-16}	3.60822×10^{-16}
0.6	7.59×10^{-6}	5.64×10^{-9}	9.4369×10^{-16}	1.11022×10^{-15}
0.7	7.22×10^{-6}	5.37×10^{-9}	1.44329×10^{-15}	9.4369×10^{-16}
0.8	6.83×10^{-6}	5.07×10^{-9}	6.10623×10^{-16}	2.77556×10^{-16}
0.9	6.41×10^{-6}	4.76×10^{-9}	8.32667×10^{-16}	1.47105×10^{-15}

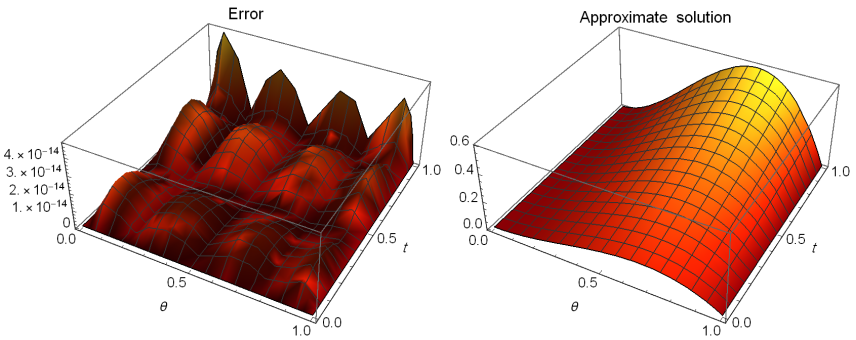


Figure 4: The AE (left) and approximate solution (right) of Example 3 at $\eta = 0.5$.

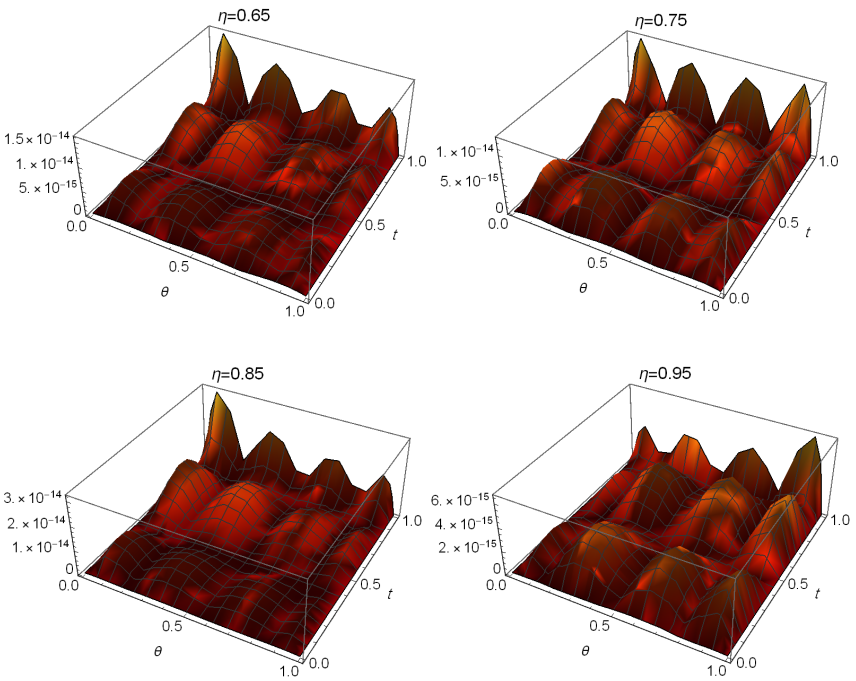


Figure 5: The AE of Example 3 at different values of η.

subject to

$$\begin{aligned} &\xi(\theta, 0) = \theta^3 + \theta^2 + 1, \quad 0 < \theta \leq 1, \\ &\xi(0, t) = \left(t^{\eta+2} + 1\right), \quad \xi(1, t) = 3\left(t^{\eta+2} + 1\right), \quad 0 < t \leq 1, \end{aligned} \tag{5.8}$$

where

$$S(\theta,t)=\left(t^{\eta+2}+1\right)\left(\left(\theta^3+\theta^2+1\right)z_3-2(3\theta+1)z_1-\theta(3\theta+2)z_2\right)+\frac{\Gamma(\eta+3)}{2}\left(\theta^3+\theta^2+1\right)t^2. \tag{5.9}$$

In addition, $\xi(\theta,t)=\left(\theta^3+\theta^2+1\right)\left(t^{\eta+2}+1\right)$ is the exact solution of this problem.
This problem is solved at $z_1=0.125$, $z_2=0.375$, $z_3=0.5$. Table 7 presents the AEs at different values of t when $\eta=0.5$ and $N=8$. Figure 6 illustrates the AEs at different values of η when $N=8$. Table 8 shows the maximum AEs at different values of N and η.

Table 7: The AEs for Example 4 at $\eta=0.5$.

θ	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$
0.1	1.74294×10^{-6}	1.15666×10^{-6}	7.41114×10^{-7}	1.54928×10^{-7}
0.2	3.25077×10^{-6}	2.13995×10^{-6}	1.39365×10^{-7}	2.99025×10^{-7}
0.3	4.42008×10^{-6}	2.87229×10^{-6}	1.78657×10^{-7}	5.80688×10^{-7}
0.4	5.22969×10^{-6}	3.3239×10^{-6}	2.00792×10^{-7}	8.99845×10^{-7}
0.5	5.66154×10^{-6}	3.5596×10^{-6}	2.13855×10^{-7}	1.06439×10^{-6}
0.6	5.62518×10^{-6}	3.51337×10^{-6}	2.04667×10^{-7}	1.18241×10^{-6}
0.7	5.05822×10^{-6}	3.08058×10^{-6}	1.70002×10^{-7}	1.32389×10^{-6}
0.8	3.96386×10^{-6}	2.34226×10^{-6}	1.24619×10^{-7}	1.2266×10^{-6}
0.9	2.29733×10^{-6}	1.35592×10^{-6}	6.83135×10^{-7}	7.64908×10^{-7}

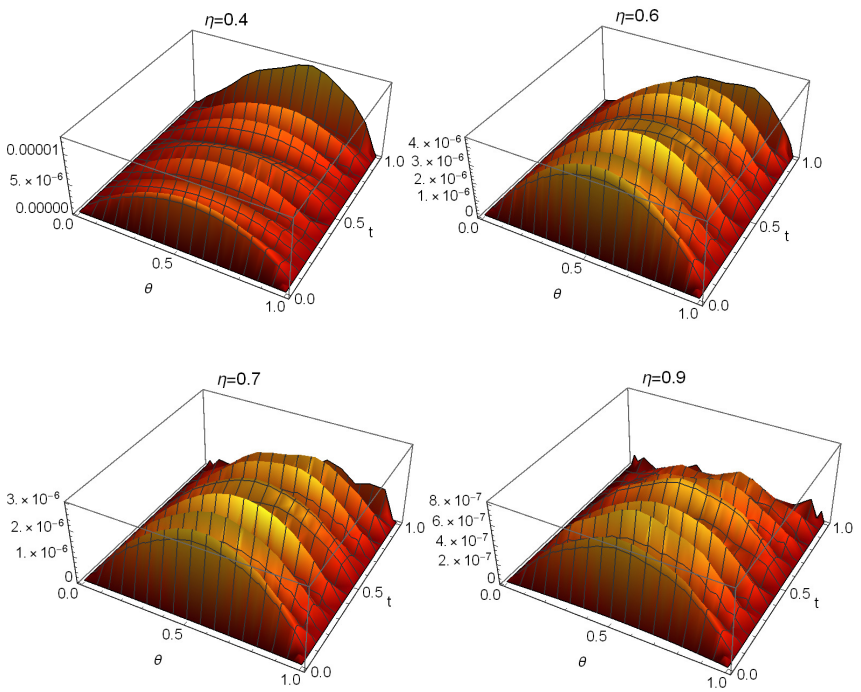


Figure 6: The AE of Example 4 at different values of η.

Example 5. Consider the $\mathcal{TFBSM}$ of the form

$$\frac{\partial^{\eta}\xi(\theta,t)}{\partial t^{\eta}}-z_1\frac{\partial^2\xi(\theta,t)}{\partial\theta^2}-z_2\frac{\partial\xi(\theta,t)}{\partial\theta}+z_3\xi(\theta,t)=S(\theta,t), \tag{5.10}$$

subject to

$$\begin{aligned}&\xi(\theta,0)=e^{\theta},\quad 0<\theta\le1,\\&\xi(0,t)=(t+1)^{\eta+1},\quad \xi(1,t)=(t+1)^{\eta+1}e,\quad 0<t\le1,\end{aligned} \tag{5.11}$$

Table 8: The maximum AEs for Example 4.

η	$N=2$	$N=4$	$N=6$	$N=8$
0.95	2.62079×10^{-2}	2.65068×10^{-5}	1.86217×10^{-6}	3.62195×10^{-7}
0.85	2.25118×10^{-2}	7.99425×10^{-5}	5.99393×10^{-6}	9.18844×10^{-7}
0.75	1.89615×10^{-2}	1.29232×10^{-4}	1.02781×10^{-5}	1.72109×10^{-6}

where $S(\theta,t)$ is chosen such that the exact solution of this problem is $\xi(\theta,t)=e^{\theta}\,(t+1)^{\eta+1}$. This problem is solved at $z_1=0.125$, $z_2=0.375$, $z_3=0.5$. Table 9 presents the maximum AEs at different values of N and η. Figure 7 illustrates the AE (left) and approximate solution (right) at $\eta=0.9$ and $N=5$. These results verify that the suggested approach reduces errors consistently throughout the domain and shows a good agreement with the approximate solution.

Table 9: The maximum AEs for Example 5.

η	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
0.6	1.23648×10^{-2}	2.39206×10^{-4}	1.90014×10^{-5}	1.49511×10^{-6}	1.48662×10^{-7}
0.7	1.65524×10^{-2}	2.2529×10^{-4}	1.84795×10^{-5}	1.29518×10^{-6}	1.24058×10^{-7}
0.8	2.18189×10^{-2}	1.75414×10^{-4}	1.63198×10^{-5}	9.36178×10^{-7}	9.41644×10^{-8}
0.9	2.84808×10^{-2}	1.44883×10^{-4}	115326×10^{-5}	4.40199×10^{-7}	5.48541×10^{-8}

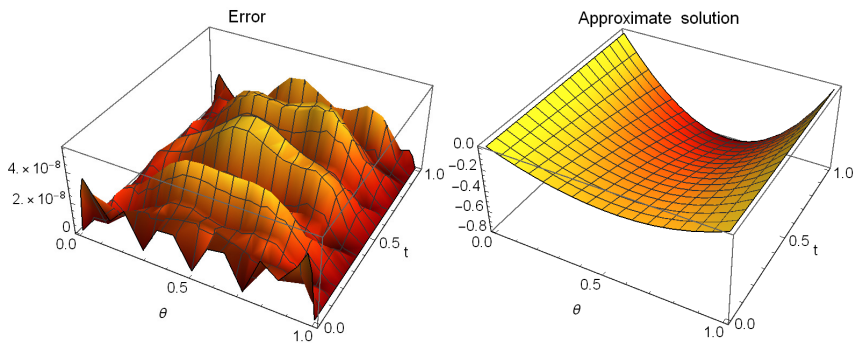


Figure 7: The AE (left) and approximate solution (right) of Example 5 at $\eta=0.9$.

6 Concluding Remarks

This paper introduces a numerical algorithm that employs the shifted Fibonacci polynomials to solve the $\mathcal{TFBSM}$. The Galerkin method is applied to obtain the proposed approximate solutions. This approach successfully transformed the fractional equation into a system of solvable algebraic equations. Furthermore, it has been demonstrated that the present algorithm compares favorably with exact ones, as shown by presenting numerical examples, and is also advantageous when compared with other algorithms in the literature. Finally, the presented algorithm can be applied to other real-life problems.

References

[1] Podlubny, I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. *Elsevier*, 1998.

[2] Oldham, K., Spanier, J. The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. *Elsevier*, 1974.

[3] Dehghan, M., Manafian, J., Saadatmandi, A. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer. Methods Partial Differ. Equ.*, 26(2):448–479, 2010.

[4] Ali, A., Gul, Z., Khan, W.A., Ahmad, S., Zeb, S. Investigation of fractional order sine-Gordon equation using Laplace Adomian decomposition method. *Fractals*, 29(05):2150121, 2021.

[5] Li, Z., Chen, Q., Wang, Y., Li, X. Solving two-sided fractional super-diffusive partial differential equations with variable coefficients in a class of new reproducing kernel spaces. *Fractal Fract.*, 6(9):492, 2022.

[6] Maleknejad, K., Rashidinia, J., Eftekhari, T. Numerical solutions of distributed order fractional differential equations in the time domain using the Müntz–Legendre wavelets approach. *Numer. Methods Partial Differ. Equ.*, 37(1):707–731, 2021.

[7] Kumar, Y., Singh, V.K. Computational approach based on wavelets for financial mathematical model governed by distributed order fractional differential equation. *Math. Comput. Simul.*, 190:531–569, 2021.

[8] Zayernouri, M., Karniadakis, G.E. Fractional spectral collocation method. *SIAM J. Sci. Comput.*, 36(1):A40–A62, 2014.

[9] Atta, A.G., Abd-Elhameed, W.M., Youssri, Y.H. Shifted fifth-kind Chebyshev polynomials Galerkin-based procedure for treating fractional diffusion-wave equation. *Int. J. Mod. Phys. C*, 33(08):2250102, 2022.

[10] Alsuyuti, M.M., Doha, E.H., Bayoumi, B.I., Ezz-Eldien, S.S. Robust spectral treatment for time-fractional delay partial differential equations. *Comput. Appl. Math.*, 42(4):159, 2023.

[11] Abd-Elhameed, W.M., Al-Sady, A.M., Alqubori, O.M., Atta, A.G. Numerical treatment of the fractional Rayleigh–Stokes problem using some orthogonal combinations of Chebyshev polynomials. *AIMS Math.*, 9(9):25457–25481, 2024.

[12] Ahmed, H.F., Hashem, W.A. A fully spectral tau method for a class of linear and nonlinear variable-order time-fractional partial differential equations in multi-dimensions. *Math. Comput. Simul.*, 214:388–408, 2023.

[13] Abdelghany, E.M., Abd-Elhameed, W.M., Moatimid, G.M., Youssri, Y.H., Atta, A.G. A tau approach for solving time-fractional heat equation based on the shifted sixth-kind Chebyshev polynomials. *Symmetry*, 15(3):594, 2023.

[14] Postavaru, O. An efficient numerical method based on Fibonacci polynomials to solve fractional differential equations. *Math. Comput. Simul.*, 212:406–422, 2023.

[15] Heydari, M.H., Avazzadeh, Z. Fibonacci polynomials for the numerical solution of variable-order space-time fractional Burgers–Huxley equation. *Math. Meth. Appl. Sci.*, 44(8):6774–6786, 2021.

[16] Deilami, H. The Fibonacci polynomials solution for Abel's integral equation of second kind. *Iran. J. Numer. Anal. Optim.*, 10(1):63–79, 2020.

[17] Ali, I., Haq, S., Nisar, K.S., Arifeen, S.U. Numerical study of 1D and 2D advection-diffusion-reaction equations using Lucas and Fibonacci polynomials. *Arab. J. Math.*, 10(3):513–526, 2021.

[18] Alharbi, M.H., Abu Sunayh, A.F., Atta, A.G., Abd-Elhameed, W.M. Novel approach by shifted Fibonacci polynomials for solving the fractional Burgers equation. *Fractal Fract.*, 8(7):427, 2024.

[19] Abd-Elhameed, W.M., Al-Harbi, M.S., Atta, A.G. New convolved Fibonacci collocation procedure for the Fitzhugh–Nagumo non-linear equation. *Nonlinear Eng.*, 13:20220332, 2024.

[20] Gupta, R., Saha, R.S. A new effective coherent numerical technique based on shifted Vieta–Fibonacci polynomials for solving stochastic fractional integro-differential equation. *Comput. Appl. Math.*, 42(6):256, 2023.

[21] Sivalingam, S.M., Kumar, P., Govindaraj, V., Qahiti, R.A., Hamali, W., Mutum, Z.M. An operational matrix approach with Vieta–Fibonacci polynomial for solving generalized Caputo fractal-fractional differential equations. *Ain Shams Eng. J.*, 15(5):102678, 2024.

[22] Moumen, A., Mennouni, A., Bouye, M. A novel Vieta–Fibonacci projection method for solving a system of fractional integro-differential equations. *Mathematics*, 11(18):3985, 2023.

[23] Sadri, K., Hosseini, K., Baleanu, D., Salahshour, S., Park, C. Designing a matrix collocation method for fractional delay integro-differential equations with weakly singular kernels based on Vieta–Fibonacci polynomials. *Fractal Fract.*, 6(1):2, 2021.

[24] Sabermahani, S., Ordokhani, Y., Yousefi, S.A. Fractional-order Fibonacci-hybrid functions approach for solving fractional delay differential equations. *Eng. Comput.*, 36:795–806, 2020.

[25] Doha, E.H., Abdelkawy, M.A., Amin, A.Z., Baleanu, D. Shifted Jacobi spectral collocation method with convergence analysis for solving integro-differential equations and system of integro-differential equations. *Nonlinear Anal. Model. Control*, 24(3):332–352, 2019.

[26] Bhrawy, A.H., Abdelkawy, M.A., Mallawi, F. An accurate Chebyshev pseudospectral scheme for multi-dimensional parabolic problems with time delays. *Bound. Value Probl.*, 2015(1):103, 2015.

[27] Taema, M., Dagher, M., Youssri, Y. Spectral collocation method via Fermat polynomials for Fredholm–Volterra integral equations with singular kernels and fractional differential equations. *Palestine Journal of Mathematics*, 14(2), 481–492 (2025).

[28] El Ghabi, M., Alaa, H., Aqel, F., Alaa, N.E. Nonlinear boundary periodic equation with arbitrary growth nonlinearity and data measure: Mathematical analysis and numerical simulation by LBM method. *Palestine Journal of Mathematics*, 13(4), 2024.

[29] Lateli, A., Boutaghou, A., Meddour, L. Numerical solution of the convection diffusion equation with a source term via the spectral method. *Palestine Journal of Mathematics*, 14(1), 2025.

[30] Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A. Spectral Methods in Fluid Dynamics. *Springer-Verlag*, 1988.

[31] Shen, J., Tang, T., Wang, L.L. Spectral Methods: Algorithms, Analysis and Applications, vol. 41 of *Springer Ser. Comput. Math. Springer*, 2011.

[32] A. G. Atta, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri. Shifted fifth-kind Chebyshev Galerkin treatment for linear hyperbolic first-order partial differential equations. *Appl. Numer. Math.*, 167:237–256, 2021.

[33] Shah, K., Amin, R., Abdeljawad, T. Utilization of Haar wavelet collocation technique for fractal-fractional order problem. *Heliyon*, 9(6), 2023.

[34] Ahmed, H.F., Hashem, W.A. Improved Gegenbauer spectral tau algorithms for distributed-order time-fractional telegraph models in multi-dimensions. *Numer. Algorithms*, 93(3):1013–1043, 2023.

[35] Atta, A.G., Abd-Elhameed, W.M., Moatimid, G.M., Youssri, Y.H. Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem. *Math. Sci.*, 17(4):415–429, 2023.

[36] Youssri, Y.H., Atta, A.G. Petrov-Galerkin Lucas polynomials procedure for the time-fractional diffusion equation. *Contemp. Math.*, 4(2): 230–248, 2023.

[37] Doha, E.H., Bhrawy, A.H., Abd-Elhameed, W.M. Jacobi spectral Galerkin method for elliptic Neumann problems, *Numer. Algorithms*, **50** (2009), 67–91.

[38] Youssri, Y.H., Hafez, R.M., Atta, A.G. An innovative pseudo-spectral Galerkin algorithm for the time-fractional Tricomi-type equation. *Phys. Scr.*, 99(10):105238, 2024.

[39] Abd-Elhameed, W.M., Youssri, Y.H., Amin, A.K., Atta, A.G. Eighth-kind Chebyshev polynomials collocation algorithm for the nonlinear time-fractional generalized Kawahara equation, *Fract. Fract.*, **7** (2023), 652.

[40] Black, F., Scholes, M. The pricing of options and corporate liabilities. *J. Polit. Econ.*, 81(3):637–654, 1973.

[41] Merton, R.C. Theory of rational option pricing. *Bell J. Econ. Manag. Sci.*, 4(1):141–183, 1973.

[42] Ahmad, H., Khan, M.N., Ahmad, I., Omri, M., Alotaibi, M.F. A meshless method for numerical solutions of linear and nonlinear time-fractional Black-Scholes models. *AIMS Math.*, 8(8):19677–19698, 2023.

[43] Kaur, J., Natesan, S. A novel numerical scheme for time-fractional Black–Scholes PDE governing European options in mathematical finance. *Numer. Algorithms*, 94(4):1519–1549, 2023.

[44] Kazmi, K. A second order numerical method for the time-fractional Black–Scholes European option pricing model. *J. Comput. Appl. Math.*, 418:114647, 2023.

[45] Aghdam, Y.E., Mesgarani, H., Amin, A., Gómez-Aguilar, J.F. An efficient numerical scheme to approach the time fractional Black–Scholes model using orthogonal Gegenbauer polynomials. *Comput. Econ.*, 4: 211–224, 2024.

[46] Zhou, J., Gu, X.M., Zhao, Y.L., Li, H. A fast compact difference scheme with unequal time-steps for the tempered time-fractional Black–Scholes model. *Int. J. Comput. Math.*, 101: 989-1011, 2024.

[47] Alazemi, F., Alsenafi, A., Najafi, A. A spectral approach using fractional Jaiswal functions to solve the mixed time-fractional Black-Scholes European option pricing model with error analysis. *Numer. Algorithms*, (98):347–371, 2025

[48] Mohapatra, J., Santra, S., Ramos, H. Analytical and numerical solution for the time fractional Black–Scholes model under jump-diffusion. *Comput. Econ.*, 63(5):1853–1878, 2024.

[49] Aghdam, Y.E., Mesgarani, H., Amin, A., Gómez-Aguilar, J.F. An efficient numerical scheme to approach the time fractional Black–Scholes model using orthogonal Gegenbauer polynomials. *Comput. Econ.*, 64(1):211–224, 2024.

[50] Ahmadian, D., Ballestra, L.V., Karimi, N. An extremely efficient numerical method for pricing options in the Black–Scholes model with jumps. *Math. Meth. Appl. Sci.*, 44(2):1843–1862, 2021.

[51] Chen, Y., Yu, H., Meng, X., Xie, X., Hou, M., Chevallier, J. Numerical solving of the generalized Black–Scholes differential equation using Laguerre neural network. *Digit. Signal Process.*, 112:103003, 2021.

[52] Roul, P., Goura, V.M.K.P. A sixth order numerical method and its convergence for generalized Black–Scholes PDE. *J. Comput. Appl. Math.*, 377:112881, 2020.

[53] Koshy, T. Fibonacci and Lucas Numbers With Applications, volume 51. *John Wiley & Sons*, 2011.

[54] Mesgarani, H., Ahanj, S., Aghdam, Y.E. Numerical investigation of the time-fractional Black–Scholes equation with barrier choice of regulating European option. *J. Math. Model.*, 10(1):1–10, 2022.

[55] Qayyum, M., Ahmad, E. New solutions of time- and space-fractional Black–Scholes European option pricing model via fractional extension of He–Aboodh algorithm. *J. Math.*, 2024:1–10, 2024.

[56] An, X., Liu, F., Zheng, M., Anh, V.V., Turner, I.W. A space-time spectral method for time-fractional Black–Scholes equation. *Appl. Numer. Math.*, 165:152–166, 2021.

Author information

W. M. Abd-Elhameed, Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt.
E-mail: waleed@cu.edu.eg

Y.H. Youssri, Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt.
E-mail: youssri@cu.edu.eg

A. G. Atta, Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11341, Cairo, Egypt.
E-mail: ahmed_gamal@edu.asu.edu.eg

Received: 2025-09-28
Accepted: 2025-12-30