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# COINCIDENCE AND MATCHING TYPE RESULTS FOR MAPS WITH UPPER SEMICONTINUOUS SELECTIONS

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**Abstract**. In this paper we present a number of coincidence results for general classes of compact or coercive maps. These results will then enable us to obtain new Ky Fan matching type theorems.

## 1 Introduction

In this paper we consider a very general class of maps, which include the Wu [15], $HLPY$ [8], Scalzo [14], $KLU$ [7], $Ad$ [5] and $PK$ [13] maps, and we present a variety of coincidence results between this general class and the $DKT$ [4] and $HLPY$ [8] maps where one of the maps considered is either compact or coercive. From these coincidence results we establish some Ky Fan matching theorems which are existence results between a convex hull and an intersection. The results in this paper extend and complement the theory in the literature; see [3–4, 10–13] and the references therein.

First we describe the maps considered in this paper. Let $H$ be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $K$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here $X$ is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the $q$–dimensional Čech homology group with compact carriers of $X$. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space $X$ is acyclic if $X$ is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let $X$, $Y$ and $\Gamma$ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). $p$ is a perfect map i.e. $p$ is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of $Y$). A pair $(p, q)$ of single valued continuous maps of the form $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of $\phi$ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). $p$ is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [5]. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of $Y$) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair $(p, q)$ of $\phi$. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here $Y$ is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of $Y$. Another example is an acyclic map which we now describe. Let $X$ and $Z$ be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. $F$ has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. Now

we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. $F$ has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. $F$ is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

Next we consider a general class of maps, namely the $PK$ maps of Park (which include $Kak$ and $Ad$ maps). Let $X$ and $Y$ be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of $Y$) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathcal{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of $F$.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i). $\mathcal{U}$ contains the class $C$ of single valued continuous functions;

(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, ....\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset $K$ of $X$ there is a $G \in \mathcal{U}_c(K, G(K))$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Now we present some properties of $PK$ maps.

(i). $PK$ is closed under compositions.

Let $X$, $Y$, $Z$ be Hausdorff topological spaces. Let $F_1 \in PK(X, Y)$ and $F_2 \in PK(Y, Z)$. Suppose $K$ is a compact subset of $X$. Then there exists a $G_1 \in \mathcal{U}_c(K, G_1(K))$ with $G_1(x) \subseteq F_1(x)$ for each $x \in K$. Note $G_1(K)$ is compact so there exists a $G_2 \in \mathcal{U}_c(G_1(K), G_2(G_1(K)))$ with $G_2(y) \subseteq F_2(y)$ for each $y \in G_1(K)$. As a result $G_2\, G_1(x) \subseteq F_2\, G_1(x) \subseteq F_2\, F_1(x)$ for $x \in K$ and note $G_2\, G_1 \in \mathcal{U}_c(K, G_2(G_1(K)))$. Let $G = G_2\, G_1$. Thus $F_2\, F_1 \in PK(X, Z)$.

(ii). Let $F \in PK(X, Y)$ and $Z \subseteq X$. Then $F \in PK(Z, Y)$.

This follows if we consider $F \circ i$ where $i : Z \to X$ is the inclusion. Alternatively, let $K$ be compact in $Z$. Then $K$ is compact in $X$ so there exists a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

(iii). Let $F \in PK(X, Y)$ and $F(X) \subseteq W \subseteq Y$. Then $F \in PK(X, W)$.

This is immediate from the definition of a $PK$ map.

Next we describe the better admissible class of maps $BPK$ due to Park [13]. Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a Hausdorff topological space. Now $F \in BPK(X, Y)$ if $F : X \to 2^Y$ and for any polytope $P$ in $X$ and any continuous map $f : F(P) \to P$ we have that $f\,(F|_P) : P \to 2^P$ has a fixed point.

We also note the following properties.

Let $C$ and $X$ be convex subsets of a Hausdorff topological vector space $E$ with $C \subseteq X$ and $Y$ a Hausdorff topological space.

(i). If $F \in BPK(X, Y)$ then $G \equiv F|_C \in BPK(C, Y)$.

Consider any polytope $P$ in $C$ and any continuous map $f : G(P) \to P$. Now since $P$ is a polytope in $X$ and (note $G(P) = F(P)$ since $P \subseteq C$) $f : F(p) \to P$ is a continuous map then since $F \in BPK(X, Y)$ there exists a $x \in P$ with $x \in f\, F|_P\,(x)$ i.e. $x \in f\, G(x)$ since $x \in P \subseteq C$. Thus $G = F|_C \in BPK(C, Y)$.

(ii). If $F \in BPK(X, Y)$ with $F(X) \subseteq Z \subseteq Y$ then $F \in BPK(X, Z)$.

Note $F : X \to 2^Y$ and let $G : X \to 2^Z$ be the map obtained by restricting the range of $F$. Consider any polytope $P$ in $X$ and any continuous map $f : G(P) \to P$. Now since $G(P) = F(P)$ then $f : F(P) \to P$ is continuous and since $F \in BPK(X, Y)$ there exists a $x \in P$ with $x \in f\, F|_P\,(x) = f\, G|_P\,(x)$. Thus $F \in BPK(X, Z)$.

(iii). If $F \in BPK(X, Y)$ and $f \in C(Y, X)$ then $f\, F \in BPK(X, X)$.

Note $f\, F : X \to 2^X$. Consider any polytope $P$ in $X$ and any continuous map $g : fF(P) \to P$. We must show there exists a $x \in P$ with $x \in g\,(f\, F)|_P(x)$. To see this note $g\, f\, F|_P = h\, F|_P : P \to 2^P$ where $h = g\, f : F(p) \to P$ is a continuous map (note $h(F(P)) = g\,(f\, F(P)) \subseteq P$). Since $F \in BPK(X, Y)$ there exists a $x \in P$ with $x \in h\, F|_P\,(x)$ i.e. $x \in (g\, f)\, F|_P(x) = g\,(f\, F)|_P(x)$. Thus $f\, F \in BPK(X, X)$.

Next we recall the following fixed point result for $BPK$ maps. Recall a nonempty subset $W$ of a Hausdorff topological vector space $E$ is said to be admissible if for any nonempty compact subset $K$ of $W$ and every neighborhood $V$ of 0 in $E$ there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of $E$ (for example every nonempty convex subset of a locally convex space is admissible). The following fixed point result can be found in [13].

**Theorem 1.1.** *Let $X$ be an admissible convex set in a Hausdorff topological vector space and $F \in BPK(X, X)$ be a closed compact map. Then $F$ has a fixed point in $X$.*

Next we describe the maps due to Wu [15]. Let $X$ and $Y$ be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X, Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1 there) for Wu maps.

**Theorem 1.2.** *Let $X$ be a paracompact subset of a Hausdorff topological vector space and $Y$ a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X, Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists a upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of $Y$) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

**Remark 1.3.** Let $X$ be paracompact and $Y$ a metrizable subset of a complete Hausdorff locally convex linear topological space $E$ and $\Phi \in W(X, Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [9] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.

Let $Z$ be a subset of a Hausdorff topological space $Y_1$ and $W$ a subset of a Hausdorff topological vector space $Y_2$ and $G$ a multifunction. We say $F \in HLPY(Z, W)$ [8] if $W$ is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of $W$)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{ int\, S^{-1}(w) : \ w \in W\}$; here $S^{-1}(w) = \{z \in Z : \ w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [8].

**Theorem 1.4.** *Let $X$ be a paracompact subset of a Hausdorff topological space, $Y$ a convex subset of a Hausdorff topological vector space and $F \in HLPY(X, Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup \{ int\, S^{-1}(w) : \ w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\, S(x) \subseteq F(x)$ for all $x \in X$.*

**Remark 1.5.** These maps are related to the $DKT$ maps in the literature and $F \in DKT(Z, W)$ [4] if $W$ is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in $Z$) for each $w \in W$. Note these maps were motivated from the $\Phi^\star$ maps. We say $G \in \Phi^\star(Z, W)$ [3] if $W$ is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in $Z$) for each $w \in W$.

We now note two properties for $HLPY$ maps (here $Y$ is convex).

(i). Let $F \in HLPY(X, Y)$ and $Z \subseteq X$. Then $F \in HLPY(Z, Y)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup \{ int\, S^{-1}(y) : \ y \in Y\}$. Let $S$ also denote the restriction of $S$ to $Z$. Notice

$$Z = Z \cap X = Z \cap \left( \bigcup \{ int\, S^{-1}(y) : \ y \in Y\} \right) = \bigcup \{ Z \cap int\, S^{-1}(y) : \ y \in Y\},$$

so $Z \subseteq \bigcup \{ int_Z\, S^{-1}(y) : \ y \in Y\}$ since for each $y \in Y$ we have that $Z \cap int\, S^{-1}(y)$ is open in $Z$. On the other hand clearly $\bigcup \{ int_Z\, S^{-1}(y) : \ y \in Y\} \subseteq Z$. Thus $Z = \bigcup \{ int_Z\, S^{-1}(y) : \ y \in Y\}$ so $F \in HLPY(Z, Y)$.

(ii). Let $F \in HLPY(X,Y)$ and $F(X) \subseteq W \subseteq Y$ with $W$ convex. Then $F \in HLPY(X,W)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup \{ int\, S^{-1}(y) : y \in Y\}$. Now note for any $x \in X$ there exists a $y \in Y$ with $x \in int\, S^{-1}(y) \subseteq S^{-1}(y)$ so $y \in S(x) \subseteq co\,(S(x)) \subseteq F(x) \subseteq W$. Thus $X = \bigcup \{ int\, S^{-1}(y) : y \in W\}$, so $F \in HLPY(X,W)$.

We also note two properties for $DKT$ maps (here $Y$ is convex).

(i). Let $F \in DKT(X,Y)$ and $Z \subseteq X$. Then $F \in DKT(Z,Y)$.

To see this note there exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$, $S(x) \neq \emptyset$ for each $x \in X$ and $S^{-1}(y)$ is open (in $X$) for each $y \in Y$. Let $S_0$ denote the restriction of $S$ to $Z$ and note for $x \in Z$ that $co\,(S_0(x)) = co\,(S(x)) \subseteq F(x)$ and $S_0(x) \neq \emptyset$. For $y \in Y$ (note $Z \subseteq X$) that

$$S_0^{-1}(y) = \{z \in Z : y \in S_0(z)\} = Z \cap \{z \in X : y \in S(z)\} = Z \cap S^{-1}(y),$$

which is open in $Z \cap X = Z$. Thus $F \in DKT(Z,Y)$.

(ii). Let $F \in DKT(X,Y)$ and $F(X) \subseteq W \subseteq Y$ with $W$ convex. Then $F \in DKT(X,W)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$, $S(x) \neq \emptyset$ for each $x \in X$ and $S^{-1}(y)$ is open (in $X$) for each $y \in Y$. Let $S^\star : X \to 2^W$ be given by $S^\star(x) = S(x)$ for $x \in X$ (this makes sense since for $x \in X$ note $S(x) \subseteq co\,(S(x)) \subseteq F(x) \subseteq W$). Also note for $x \in X$ that $co\,(S^\star(x)) = co\,(S(x)) \subseteq F(x)$ and for $y \in Y$ note $(S^\star)^{-1}(y) = \{x \in X : y \in S^\star(x)\} = \{x \in X : y \in S(x)\}$ which is open in $X$.

Let $X$ be a subset of a Hausdorff topological space and $Y$ a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [7] at $x \in X$ if there exists an open set $U(x)$ in $X$ containing $x$ and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\,F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X,Y)$ if $T$ has the SCIP at every $x \in X$.

In this paper our map $T$ will be a compact map so $T$ has the SCIP is equivalent to $T$ has the CIP [6].

**Remark 1.6.** These maps contain as a special case the Scalzo maps [14] in the literature (see [7 pg12]).

Next we recall a selection theorem [7].

**Theorem 1.7.** *Let $X$ be a paracompact subset of a Hausdorff topological space, $Y$ a subset of a Hausdorff topological vector space and $T \in KLU(X,Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\,T(w)$ for all $w \in X$.*

In Section 2 we consider general classes of maps. Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological vector space. We say $F \in HYBPK(X,Y)$ if $F : X \to 2^Y$ and there exists a map $\Psi \in BPK(X,Y)$ with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X$. We say $F \in HYBPKU(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Psi \in BPK(X,Y)$ with compact values and with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X$.

Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological vector space. We say $F \in HYBPKC(X,Y)$ if $F : X \to 2^Y$ and there exists a map $\Psi \in BPK(X,Y)$ with $\Psi(x) \subseteq F(x)$ for $x \in X$. We say $F \in HYBPKUC(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Psi \in BPK(X,Y)$ with compact values and with $\Psi(x) \subseteq F(x)$ for $x \in X$.

## 2 Coincidence Theory

In this section we present some coincidence results between $HYBPK$ (or $HYBPKC$) and $HLPY$ (or $DKT$) maps.

**Theorem 2.1.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\,F$ a compact map and $G \in DKT(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap co\,F(x) \neq \emptyset$.*

*Proof.* Note there exists a compact map $\Phi \in BPK(X,Y)$ (note $\Phi$ is compact since $co\,F$ is a compact map) with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$. Let $K = \overline{\Phi(X)}$ and note (see Section 1) that $G \in DKT(K,X)$. Now [4] (note $G \in DKT(K,X)$ and $K$ is compact) guarantees that there exists a continuous single valued selection $g : K \to X$ (i.e. $g \in C(K,X)$) of $G$ and there exists a finite subset $A$ of $X$ with $g(K) \subseteq co\,(A)$, so $g \in C(K, co\,(A))$. Note (see Section 1) that $\Phi \in BPK(co\,(A),K)$ so $g\Phi \in BPK(co\,(A), co\,(A))$ (also note $co\,(A)$ is a compact convex subset in a finite dimensional subspace of $E$). Now from the definition of $BPK$ maps then there exists a $x \in co\,(A) \subseteq X$ with $x \in g\,(\Phi(x))$. Let $y \in \Phi(x)$ with $x = g(y)$. Then $y \in \Phi(x) \subseteq co\,(F(x))$ and $x = g(y) \subseteq G(y)$. □

**Remark 2.2.** Usually in applications $\Phi \in BPK(X,Y)$ in Theorem 2.1 would also be upper semicontinuous with compact values, so in this case one could apply Theorem 1.1 instead of the definition of a $BPK$ map. We also refer the reader to [10–12] for other types of results.

The same argument above immediately yields our next result.

**Theorem 2.3.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKC(X,Y)$ with $F$ a compact map and $G \in DKT(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

We now replace $DKT$ with $HLPY$ in Theorem 2.1 and Theorem 2.3.

**Theorem 2.4.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\,F$ a compact map and $G \in HLPY(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap co\,F(x) \neq \emptyset$.*

*Proof.* Let $\Phi$ and $K$ be as in the proof of Theorem 2.1. Also note (see Section 1) that $G \in HLPY(K,X)$ so [8] guarantees that there exists a continuous single valued selection $g : K \to X$ of $G$ and there exists a finite subset $A$ of $X$ with $g(K) \subseteq co\,(A)$, so $g \in C(K, co\,(A))$. The rest of the proof is the same as in Theorem 2.1. □

**Theorem 2.5.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKC(X,Y)$ with $F$ a compact map and $G \in HLPY(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Next we will replace the compactness assumption above with a coercive type condition (see [3]) i.e. a condition imposing some control on the map outside a compact set. Let $Z$ and $W$ be subsets of Hausdorff topological vector spaces. A multivalued map $Q : Z \to 2^W$ is said to be coercive if there exists a compact subset $K$ of $Z$ and a compact convex subset $C$ of $W$ such that $Q(x) \cap C \neq \emptyset$ for $x \in Z \backslash K$.

We begin by presenting some coincidence results between $HYBPKU$ (or $HYBPKUC$) and $\Phi^\star$ maps. This will then enable us to consider coincidence results between $HYBPKU$ (or $HYBPKUC$) and $DKT$ maps.

**Theorem 2.6.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPKU(X,Y)$, $G \in \Phi^\star(Y,X)$ and suppose there exists a map $S : Y \to X$ with $S(y) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ and has convex values for each $y \in Y$, $S^{-1}(x)$ is open (in $Y$) for each $x \in X$ and there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $S(y) \cap C \neq \emptyset$ for $y \in Y \backslash K$. Then there exists an $x \in X$ with $G^{-1}(x) \cap co\,F(x) \neq \emptyset$.*

*Proof.* Now there exists an upper semicontinuous map $\Phi \in BPK(X,Y)$ with compact values and with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$. Now since $G \in \Phi^\star(Y,X)$ then (see Section 1) note $G \in \Phi^\star(K,X)$. Now $K$ is compact so [3] guarantees that there exists a selection $g \in C(K,X)$ of $G$ (in fact we have, see [3], $g(x) \subseteq S(x) \subseteq G(x)$ for $x \in K$) and a finite subset $B$ of

$X$ with $g(K) \subseteq co\,(B)$. Let $C_0 = co\,(co\,(B) \cup C)$ which is a convex compact [2] subset of $X$. Note since $\Phi \in BPK(X,Y)$ is an upper semicontinuous map with compact values then (see Section 1) we have that $\Phi \in BPK(C_0,Y)$ is an upper semicontinuous compact map with compact values. Let $G^\star(y) = G(y) \cap C_0$ for $y \in Y$ and we claim $G^\star \in \Phi^\star(Y,C_0)$. Let $S^{\star\star}(y) = S(y) \cap C_0$ for $y \in Y$. If $y \in Y\backslash K$ then $S^{\star\star}(y) = S(y) \cap C_0 \neq \emptyset$ since $S(y) \cap C \neq \emptyset$ and $C \subseteq C_0$ whereas if $y \in K$ then since $g(K) \subseteq S(K)$ and $g(K) \subseteq co\,(B) \subseteq C_0$ we have $S^{\star\star}(y) = S(y) \cap C_0 \neq \emptyset$. Next if $y \in Y$ then $S^{\star\star}(y) = S(y) \cap C_0 \subseteq G(y) \cap C_0 = G^\star(y)$. Finally note if $x \in C_0$ then

$$(S^{\star\star})^{-1}(x) = \{y \in Y :\ x \in S^{\star\star}(y)\} = \{y \in Y :\ x \in S(y) \cap C_0\} = \{y \in Y :\ x \in S(y)\}$$

which is open in $Y$. Thus $G^\star \in \Phi^\star(Y,C_0)$ (in fact, also a compact map) and from the above $\Phi \in BPK(C_0,Y)$ is an upper semicontinuous compact map with compact values. From the proof of Theorem 2.1 there exists a $x \in C_0$ with $\Phi(x) \cap (G^\star)^{-1}(x) \neq \emptyset$ i.e. there exists a $y$ with $y \in \Phi(x)$ and $y \in (G^\star)^{-1}(x)$. Thus $y \in \Phi(x)$ and $x \in G^\star(y) = G(y) \cap C_0$ i.e. $y \in co\,F(x)$, $x \in C_0$ and $x \in G(y)$ i.e. $x \in C_0$ and $G^{-1}(x) \cap co\,F(x) \neq \emptyset$. □

The same argument above immediately yields our next result.

**Theorem 2.7.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKUC(X,Y)$, $G \in \Phi^\star(Y,X)$ and suppose there exists a map $S : Y \to X$ with $S(y) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ and has convex values for each $y \in Y$, $S^{-1}(x)$ is open (in $Y$) for each $x \in X$ and there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $S(y) \cap C \neq \emptyset$ for $y \in Y\backslash K$. Then there exists an $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Now we replace the $\Phi^\star$ maps in Theorem 2.6 and Theorem 2.7 with $DKT$ maps.

**Theorem 2.8.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPKU(X,Y)$, $G \in DKT(Y,X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for $y \in Y$, $T(y) \neq \emptyset$ and has convex values for each $y \in Y$, $T^{-1}(x)$ is open (in $Y$) for each $x \in X$ and there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $T(y) \cap C \neq \emptyset$ (or, alternatively $co\,(T(y)) \cap Z \neq \emptyset$) for $y \in Y\backslash K$. Then there exists an $x \in X$ with $G^{-1}(x) \cap co\,F(x) \neq \emptyset$.*

*Proof.* Let $S(y) = co\,(T(y))$ for $y \in Y$. First note $S(y)$ has convex values for each $y \in Y$ and note $S(y) \subseteq G(y)$ for $y \in Y$. In addition from [16, Lemma 5.1] we have that $S^{-1}(w)$ is open (in $Y$) for each $w \in X$. Let $y \in Y\backslash K$. We claim $S(y) \cap Z \neq \emptyset$ and for this we only need to consider the case when $T(y) \cap Z \neq \emptyset$, but here $\emptyset \neq T(y) \cap Z \subseteq co\,(T(y)) \cap Z = S(y) \cap Z$. Thus $G \in \Phi^\star(Y,X)$. Now apply Theorem 2.6. □

**Theorem 2.9.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKUC(X,Y)$, $G \in DKT(Y,X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for $y \in Y$, $T(y) \neq \emptyset$ and has convex values for each $y \in Y$, $T^{-1}(x)$ is open (in $Y$) for each $x \in X$ and there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $T(y) \cap C \neq \emptyset$ (or, alternatively $co\,(T(y)) \cap Z \neq \emptyset$) for $y \in Y\backslash K$. Then there exists an $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Now we replace the $\Phi^\star$ maps in Theorem 2.6 and Theorem 2.7 with $HLPY$ maps.

**Theorem 2.10.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPKU(X,Y)$, $G \in HLPY(Y,X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for $y \in Y$, $Y = \bigcup\{int\,T^{-1}(w) :\ w \in X\}$ and there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $R(y) \cap Z \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap Z \neq \emptyset$) for $y \in Y\backslash K$ where $R : Y \to X$ is given by $R(y) = \{z :\ y \in int\,T^{-1}(z)\}$ for $y \in Y$. Then there exists an $x \in X$ with $G^{-1}(x) \cap co\,F(x) \neq \emptyset$.*

*Proof.* Let $S(y) = co\,(R(y))$ for $y \in Y$. First note that $S(y)$ has convex values for each $y \in Y$ and also note that $R(y) \subseteq T(y)$ for $y \in Y$ since if $z \in R(y)$ then $y \in int\, T^{-1}(z) \subseteq T^{-1}(z) = \{w \in Y : \ z \in T(w)\}$ so $z \in T(y)$ and putting this together yields $R(y) \subseteq T(y)$ for $y \in Y$. Thus we have $S(y) = co\,(R(y)) \subseteq co\,(T(y)) \subseteq G(y)$ for $y \in Y$.

Now notice for $x \in X$ that $R^{-1}(x) = \{z : \ x \in R(z)\} = int\, T^{-1}(x)$ so $R^{-1}(x)$ is open (in $Y$) and so from [16, Lemma 5.1] we have that $S^{-1}(x)$ is open (in $Y$).

Now let $y \in Y$. Since $Y = \bigcup\{ int\, T^{-1}(w) : \ w \in X\}$ there exists a $y \in int\, T^{-1}(w)$ for some $w \in X$ and so $w \in R(y)$ i.e. $R(y) \neq \emptyset$ and as a result $\emptyset \neq R(y) \subseteq co\,(R(y)) = S(y)$.

Let $y \in Y\backslash K$. We claim $S(y) \cap Z \neq \emptyset$. We need only consider the case when $R(y) \cap Z \neq \emptyset$, but here $\emptyset \neq R(y) \cap Z \subseteq co\,(R(y)) \cap Z = S(y) \cap Z$. Now apply Theorem 2.6. □

The same argument above immediately yields our next result.

**Theorem 2.11.** *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKUC(X,Y)$, $G \in HLPY(Y,X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for $y \in Y$,*

$$Y = \bigcup\{ int\, T^{-1}(w) : \ w \in X\}$$

*and there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $R(y) \cap Z \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap Z \neq \emptyset$) for $y \in Y\backslash K$ where $R : Y \to X$ is given by $R(y) = \{z : y \in int\, T^{-1}(z)\}$ for $y \in Y$. Then there exists an $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

**Remark 2.12.** Note in the proof of Theorem 2.10 and Theorem 2.11 we show that $R(y) \subseteq T(y)$ for $y \in Y$. If say in Theorem 2.10 and Theorem 2.11 we assume $T^{-1}(z)$ is open (in $Y$) for $z \in T(Y\backslash K)$. Then $R(y) = T(y)$ for $y \in Y\backslash K$. To see this fix $y \in Y\backslash K$. Now if $z \in T(y)$ then $y \in T^{-1}(z) = int\, T^{-1}(z)$ so $z \in R(y)$. Thus $T(y) \subseteq R(y)$ for $y \in Y\backslash K$.

**Remark 2.13.** Let $X, Y, E$ and $F$ be as in Theorem 2.10 or Theorem 2.11. Let $G \in HLPY(Y,X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for $y \in Y$ and $Y = \bigcup\{ int\, T^{-1}(w) : \ w \in X\}$. Assume there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $\cap_{x \in Z} \left(Y\backslash int\, T^{-1}(x)\right) \subseteq K$.

We now claim that if $y \in Y\backslash K$ then $R(y) \cap Z \neq \emptyset$ where $R(y) = \{z : \ y \in int\, T^{-1}(z)\}$ for $y \in Y$. To see this let $y \in Y\backslash K$. Then since

$$\cap_{x \in Z} \left(Y\backslash int\, T^{-1}(x)\right) \ \left(= Y\backslash \left(\cup_{x \in Z}\, int\, T^{-1}(x)\right)\right) \subseteq K$$

we have

$$y \in Y\backslash K \subseteq Y\backslash \left(Y\backslash \left(\cup_{x \in Z}\, int\, T^{-1}(x)\right)\right) = \cup_{x \in Z}\, int\, T^{-1}(x).$$

Thus there exists a $x \in Z$ with $y \in int\, T^{-1}(x)$ i.e. $x \in Z$ and $x \in R(y)$ i.e. $R(y) \cap Z \neq \emptyset$.

Finally we present some Ky Fan matching type theorems for $HYBPK$ and $HYBPYC$ maps.

**Theorem 2.14.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\, F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq co\, F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $\overline{\Phi(X)} \subseteq T(X)$ and $T(x)$ is open (in $Y$) for each $x \in X$. Then there exists a finite set $M \subseteq X$ with $\Phi(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset$ so*

$$co\,(F(\, co\,(M))) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset.$$

*Proof.* Let $\Phi$ be as in the statement of Theorem 2.14 and let $G(w) = co\,(\overline{T^{-1}(w)})$ for $w \in Y$ (it may have empty values). We claim $G \in HLPY(K,X)$ where $K = \overline{\Phi(X)}$. Note $T^{-1} : K \to 2^X$ since if $y \in K$ then from $K \subseteq T(X)$ there exists a $w \in X$ with $y \in T(w)$ (i.e. $w \in T^{-1}(y)$). Thus $G : K \to 2^X$ and $G(z)$ is convex for each $z \in K$. We will now show $K = \cup_{x \in X}\, int_K\, G^{-1}(x)$ which will then imply $G \in HLPY(K,X)$. Note $\cup_{x \in X}\, int_K\, G^{-1}(x) \subseteq K$ so we need only show $K \subseteq \cup_{x \in X}\, int_K\, G^{-1}(x)$. Let $y \in K$ and (as above) there exists a $w \in X$

with $y \in T(w)$. Fix $\eta \in T(w)$ and note $w \in T^{-1}(\eta) \subseteq co\,(T^{-1}(\eta)) = G(\eta)$ i.e. $\eta \in G^{-1}(w)$. Consequently $T(w) \subseteq G^{-1}(w)$. Thus if $y \in K$ there exists a $w \in X$ with $y \in T(w)$ and from the above $y \in T(w) \subseteq G^{-1}(w)$, and since $T(w)$ is open we have $y \in T(w)\,(= int\,T(w)) \subseteq G^{-1}(w)$ i.e. $y \in int\,G^{-1}(w)$. Thus $K \subseteq \cup_{x\in X}\, int\,G^{-1}(x)$. Next note that

$$K \subseteq K \cap \left(\cup_{x\in X}\, int\,G^{-1}(x)\right) = \cup_{x\in X}\,(K \cap\, int\,G^{-1}(x))$$

and since for each $x \in X$ we have that $K \cap int\,G^{-1}(x)$ is open in $K$ then $K \subseteq \cup_{x\in X}\, int_K\,G^{-1}(x)$. Thus $G \in HLPY(K,X)$ (here the selection of $G$ chosen is $G$ itself i.e. $S(y) = G(y)$ for $y \in K$ so $co\,S(y) = co\,G(y) = G(y)$ for $y \in K$) so from the proof of Theorem 2.4 there exists a $x \in X$ with $G^{-1}(x) \cap \Phi(x) \neq \emptyset$ i.e. there exists a $y \in G^{-1}(x) \cap \Phi(x)$ i.e. $x \in G(y)$ and $y \in \Phi(x)$. Since $x \in G(y) = co\,(T^{-1}(y))$ then from the definition of $co$ there exists a $M = \{z_1, ..., z_n\} \subseteq X$ with $M \subseteq T^{-1}(y)$ such that $x \in co\,(M)$. Fix $i \in \{1, ..., n\}$ and note $z_i \in T^{-1}(y)$ so $y \in T(z_i)$ i.e. $y \in T(z_i)$ for all $i \in \{1, ..., n\}$ i.e. $y \in \cap_{i=1}^n T(z_i)$. Also $y \in \Phi(x)$ and $x \in co\,(M)$ so $y \in \Phi(\,co\,(M))$. Thus $y \in \Phi(\,co\,(M)) \cap (\cap_{i=1}^n T(z_i))$, so $y \in co\,(F(\,co\,(M)))\ \cap\ (\cap_{i=1}^n T(z_i))$. □

The same argument above immediately yields our next result.

**Theorem 2.15.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKC(X,Y)$ with $F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $\overline{\Phi(X)} \subseteq T(X)$ and $T(x)$ is open (in $Y$) for each $x \in X$. Then there exists a finite set $M \subseteq X$ with $\Phi(\,co\,(M))\ \cap\ (\cap_{x\in M}\ T(x)) \neq \emptyset$ so*

$$F(\,co\,(M))\ \cap\ (\cap_{x\in M}\ T(x)) \neq \emptyset.$$

Also we have the following two results.

**Theorem 2.16.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\,F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq co\,F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $T(X) = Y$ and $T(x)$ is open (in $Y$) for each $x \in X$. Then there exists a finite set $M \subseteq X$ with $\Phi(\,co\,(M))\ \cap\ (\cap_{x\in M}\ T(x)) \neq \emptyset$ so*

$$co\,(F(\,co\,(M)))\ \cap\ (\cap_{x\in M}\ T(x)) \neq \emptyset.$$

*Proof.* Let $G(w) = co\,(T^{-1}(w))$ for $w \in Y$ and note the argument (easier in this case) in Theorem 2.14 guarantees that $Y = \cup_{x\in X}\, int\,G^{-1}(x)$, so $G \in HLPY(Y,X)$. From the proof of Theorem 2.4 there exists a $x \in X$ with $G^{-1}(x) \cap \Phi(x) \neq \emptyset$. A similar argument as in Theorem 2.14 completes the proof. □

The same argument above immediately yields our next result.

**Theorem 2.17.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKC(X,Y)$ with $F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $T(X) = Y$ and $T(x)$ is open (in $Y$) for each $x \in X$. Then there exists a finite set $M \subseteq X$ with $\Phi(\,co\,(M))\ \cap\ (\cap_{x\in M}\ T(x)) \neq \emptyset$ so*

$$F(\,co\,(M))\ \cap\ (\cap_{x\in M}\ T(x)) \neq \emptyset.$$

Note one could replace the compactness assumption in the matching theorems with a coercive type condition. We will illustrate this with two results modelled off Theorem 2.16 and Theorem 2.17.

**Theorem 2.18.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\,F$ a compact*

*map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq co\, F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $T(X) = Y$ and $T(x)$ is open (in $Y$) for each $x \in X$. In addition assume there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $R(y) \cap Z \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap Z \neq \emptyset$) for $y \in Y \backslash K$ where $R : Y \to X$ is given by $R(y) = \{z : \ y \in int\, G^{-1}(z)\}$ for $y \in Y$ where $G(w) = co\,(T^{-1}(w))$ for $w \in Y$. Then there exists a finite set $M \subseteq X$ with $\Phi(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset$ so*

$$co\,(F(\, co\,(M))) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset.$$

*Proof.* As in Theorem 2.16 we have $Y = \cup_{x \in X}\, int\, G^{-1}(x)$, so $G \in HLPY(Y,X)$ (here the selection of $G$ chosen is $G$ itself i.e. $S(y) = G(y)$ for $y \in Y$ so $co\, S(y) = co\, G(y) = G(y)$ for $y \in Y$). From the proof of Theorem 2.11 there exists a $x \in X$ with $G^{-1}(x) \cap \Phi(x) \neq \emptyset$. A similar argument as in Theorem 2.14 completes the proof. □

The same argument above immediately yields our next result.

**Theorem 2.19.** *t Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKUC(X,Y)$ (so in particular there exists an upper semicontinuous $\Phi \in BPK(X,Y)$ with compact values and with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $T(X) = Y$ and $T(x)$ is open (in $Y$) for each $x \in X$. In addition assume there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $R(y) \cap Z \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap Z \neq \emptyset$) for $y \in Y \backslash K$ where $R : Y \to X$ is given by $R(y) = \{z : \ y \in int\, G^{-1}(z)\}$ for $y \in Y$ where $G(w) = co\,(T^{-1}(w))$ for $w \in Y$. Then there exists a finite set $M \subseteq X$ with $\Phi(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset$ so*

$$F(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset.$$

**Remark 2.20.** Assume there is a compact subset $K$ of $Y$ and a convex compact subset $Z$ of $X$ with $\cap_{x \in Z}\ \left(Y \backslash int\, G^{-1}(x)\right) \subseteq K$. If $y \in Y \backslash K$ then from Remark 2.13 we have $R(y) \cap Z \neq \emptyset$.

In the matching theorems above the map $G$ was of $HLPY$ type. One could change the assumption on $T$ (i.e. change $T(x)$ is open (in $Y$) for each $x \in X$) so that $G$ is of $DKT$ or $\Phi^\star$ type. To illustrate the idea we will present three results modelled off Theorem 2.16, Theorem 2.19 and Theorem 2.14. There is also an obvious analogue of Theorem 2.15, Theorem 2.17 and Theorem 2.18.

**Theorem 2.21.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\, F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq co\, F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $T(X) = Y$. Let $G(w) = co\,(T^{-1}(w))$ for $w \in Y$ and assume $G^{-1}(w)$ is open (in $Y$) for $w \in X$. Then there exists a finite set $M \subseteq X$ with $\Phi(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset$ so*

$$co\,(F(\, co\,(M))) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset.$$

*Proof.* Note $T^{-1} : Y \to X$ and $G \in DKT(Y,X)$ (here the selection of $G$ chosen is $G$ itself i.e. $S(y) = G(y)$ for $y \in Y$ so $co\, S(y) = co\, G(y) = G(y)$ for $y \in Y$). From the proof of Theorem 2.1 there exists a $x \in X$ with $G^{-1}(x) \cap \Phi(x) \neq \emptyset$. A similar argument as in Theorem 2.14 completes the proof. □

**Theorem 2.22.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKUC(X,Y)$ (so in particular there exists an upper semicontinuous map $\Phi \in BPK(X,Y)$ with compact values and with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $T(X) = Y$. Let $G(w) = co\,(T^{-1}(w))$ for $w \in Y$ and suppose $G^{-1}(w)$ is open (in $Y$) for $w \in X$. In addition assume there is a compact subset $K$ of $Y$ and a convex compact subset $C$ of $X$ with $G(y) \cap C \neq \emptyset$ for $y \in Y \backslash K$. Then there exists a finite set $M \subseteq X$ with $\Phi(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset$ so*

$$F(\, co\,(M)) \ \cap \ (\cap_{x \in M}\ T(x)) \neq \emptyset.$$

*Proof.* Note $G \in DKT(Y,X)$ (here the selection of $G$ chosen is $G$ itself) so from the proof of Theorem 2.7 (or Theorem 2.9) there exists a $x \in X$ with $G^{-1}(x) \cap \Phi(x) \neq \emptyset$. A similar argument as in Theorem 2.14 completes the proof. □

**Remark 2.23.** Note if $(T^{-1})^{-1}(w)$ is open (in $Y$) for each $w \in X$ then from [16, Lemma 5.1] we have that $G^{-1}(w)$ is open (in $Y$) for $w \in X$.

**Theorem 2.24.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological vector space. Suppose $F \in HYBPK(X,Y)$ with $co\,F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq co\,F(x)$ for $x \in X$). Also assume there exists a map $T : X \to 2^Y$ such that $\overline{\Phi(X)} \subseteq T(X)$. Let $G(w) = co\,(T^{-1}(w))$ for $w \in Y$ and suppose $G^{-1}(w)$ is open (in $Y$) for $w \in X$. Then there exists a finite set $M \subseteq X$ with $\Phi(co\,(M)) \cap (\cap_{x\in M}\, T(x)) \neq \emptyset$ so*

$$co\,(F(co\,(M))) \cap (\cap_{x\in M}\, T(x)) \neq \emptyset.$$

*Proof.* Let $K = \overline{\Phi(X)}$ and note (see Theorem 2.14) that $T^{-1} : K \to 2^X$. Let $G_0$ denote the restriction of $G$ to $K$. For $w \in X$ note that

$$G_0^{-1}(w) = \{z \in K :\, w \in G_0(z)\} = K \cap \{z \in Y :\, w \in G(z)\} = K \cap G^{-1}(w),$$

which is open in $K \cap Y = K$, so $G_0 \in DKT(K,X)$. From the proof of Theorem 2.1 there exists a $x \in X$ with $G_0^{-1}(x) \cap \Phi(x) \neq \emptyset$. A similar argument as in Theorem 2.14 completes the proof. □

All the above matching theorems have contrapositive reformulations. We will just consider Theorem 2.15 and Theorem 2.17 to illustrate the ideas. There is also obvious analogues of the other ones.

**Theorem 2.25.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKC(X,Y)$ with $F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$) and $S : X \to 2^Y$ with $S(x)$ closed (in $Y$) for each $x \in X$. In addition assume either (a). $\Phi(co\,(A)) \subseteq S(A)$ for each finite subset $A$ of $X$, or (b). $F(co\,(A)) \subseteq S(A)$ for each finite subset $A$ of $X$, occurs. Then $\overline{\Phi(X)} \bigcap \left(\bigcap_{x\in X}\, S(x)\right) \neq \emptyset$ and $\overline{F(X)} \bigcap \left(\bigcap_{x\in X}\, S(x)\right) \neq \emptyset$.*

*Proof.* Let $T(x) = Y \backslash S(x)$ for $x \in X$. Now assume $\overline{\Phi(X)} \bigcap \left(\bigcap_{x\in X}\, S(x)\right) = \emptyset$ (note if we assumed $\overline{F(X)} \bigcap \left(\bigcap_{x\in X}\, S(x)\right) = \emptyset$ then also $\overline{\Phi(X)} \bigcap \left(\bigcap_{x\in X}\, S(x)\right) = \emptyset$ would be true). Then $\overline{\Phi(X)} \subseteq T(X)$ since $Y \setminus \left(\bigcap_{x\in X}\, S(x)\right) = \bigcup_{x\in X} (Y \backslash S(x)) = \bigcup_{x\in X} T(x) = T(X)$. Now Theorem 2.15 guarantee the existence of a finite set $M \subseteq X$ with $\Phi(co\,(M)) \bigcap \left(\bigcap_{x\in M}\, T(x)\right) \neq \emptyset$ and $F(co\,(M)) \bigcap \left(\bigcap_{x\in M}\, T(x)\right) \neq \emptyset$. Then (a) and (b) are not true when $A = M$ since $\bigcap_{x\in M}\, T(x) = \bigcap_{x\in M}\, (Y \backslash S(x)) = Y \setminus \left(\bigcup_{x\in M} S(x)\right) = Y \backslash S(M)$. Thus

$$\overline{\Phi(X)} \bigcap \left( \bigcap_{x\in X} S(x) \right) \neq \emptyset.$$

□

**Theorem 2.26.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a subset of a Hausdorff topological space. Suppose $F \in HYBPKC(X,Y)$ with $F$ a compact map (so in particular there exists a compact $\Phi \in BPK(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$) and $S : X \to 2^Y$ with $S(x)$ closed (in $Y$) for each $x \in X$. In addition assume either (a). $\Phi(co\,(A)) \subseteq S(A)$ for each finite subset $A$ of $X$, or (b). $F(co\,(A)) \subseteq S(A)$ for each finite subset $A$ of $X$, occurs. Then $\bigcap_{x\in X}\, S(x) \neq \emptyset$.*

*Proof.* Let $T(x) = Y \backslash S(x)$ for $x \in X$. Assume $\bigcap_{x\in X}\, S(x) = \emptyset$. Then $T(X) = Y$ since $Y = Y \setminus \left(\bigcap_{x\in X}\, S(x)\right) = \bigcup_{x\in X} (Y \backslash S(x)) = \bigcup_{x\in X} T(x) = T(X)$. Now Theorem 2.17

guarantee the existence of a finite set $M \subseteq X$ with $\Phi(\,co\,(M)) \bigcap \left(\bigcap_{x\in M}\ T(x)\right) \neq \emptyset$ and $F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M}\ T(x)\right) \neq \emptyset$. Then (a) and (b) are not true when $A = M$. Thus

$$\bigcap_{x\in X} S(x) \neq \emptyset.$$

□

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