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MATHEMATICAL ANALYSIS OF AN INITIAL-BOUNDARY VALUE PROBLEM FOR GENERALIZED HEPATITIS B VIRUS (HBV) MODEL

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Abstract In this paper, we prove an existence result for solutions to an initial-boundary value problem for generalized hepatitis B virus (HBV) model. Our approach is based on fixed points for the sum of two operators. We then establish Ulam-Hyers stability for our problem. Some examples are provided to illustrate our main results.

1 Introduction

Hepatitis B virus infects liver cells (hepatocytes) and can cause both acute and chronic disease. Many mathematical models have been used to help understand the dynamics of this viral infection, see, for example [4], [5], [9], [14], [24], [25] and the references therein. In this paper, we investigate an initial-boundary value problem for the following generalized Hepatitis B Virus (HBV) mathematical model, see [15] and [27] :

$$\begin{aligned}
\frac{\partial u_1}{\partial t}(t,x) &= b - f(u_1(t,x),u_2(t,x),u_3(t,x))u_3(t,x) - \mu_1 u_1(t,x)\\
\frac{\partial u_2}{\partial t}(t,x) &= f(u_1(t-\tau_1,x),u_2(t-\tau_1,x),u_3(t-\tau_1,x))u_3(t-\tau_1,x)e^{-c_1\tau_1} - \mu_2 u_2(t,x)\\
\frac{\partial u_3}{\partial t}(t,x) &= a\frac{\partial^2 u_3}{\partial x^2}(t,x) + \gamma u_2(t-\tau_2,x)e^{-c_2\tau_2} - \mu_3 u_3(t,x), \quad (t,x)\in(0,\infty)\times[0,1],
\end{aligned} \tag{1.1}$$

where the general incidence function f is assumed to be merely continuous on $\mathbb{R}^3$ and satisfies some growth conditions that will be specified precisely later. The functions $u_1(t,x)$, $u_2(t,x)$, and $u_3(t,x)$ represent the densities of uninfected hepatocytes, infected hepatocytes, and free virus at time t and spatial position x, respectively. The uninfected hepatocytes are produced from a source at a constant rate b. The constants μ_1 and μ_2 represent the death rates of uninfected cells and infected cells, respectively, while μ_3 and γ represent the clearance rate of virions and the production rate of virions by infected cells, respectively.

The delay τ_1 corresponds to the time required for infected cells to begin producing virions following viral entry. The term $e^{-c_1\tau_1}$ represents the probability that these cells survive from time $t-\tau_1$ to time t, where c_1 is the death rate of infected cells that have not yet started producing viruses. The delay τ_2 represents the time required for newly produced virions to mature into infectious particles. The term $e^{-c_2\tau_2}$ describes the probability of immature virions surviving this maturation process, and the average lifetime of an immature virus is given by $\frac{1}{c_2}$. Finally, a denotes the diffusion coefficient of virions.

There are several works related to problem (1.1) :

The paper by Gourley et al. [12] formulates and examines the global dynamics of a basic hepatitis B virus (HBV) model using delay differential equations. In [7], the authors propose a model that combines logistic hepatocyte growth with a standard incidence function for viral infection, incorporating an explicit time delay in virus production. Gan et al. [11] investigate the existence of traveling waves in an HBV infection model with spatial diffusion, where the intracellular incubation period is represented by a discrete time delay. Yang et al. [31] focus on the stability and bifurcation analysis of a delay differential equation model for HBV infection.

Manna et al. [16] propose a non-standard finite difference (NSFD) scheme for a delayed diffusive HBV infection model that includes intracellular HBV DNA-containing capsids. In [17], the same author introduces and analyzes a delayed reaction-diffusion model for HBV infection, incorporating HBV DNA-containing capsids, the cytotoxic T lymphocyte (CTL) immune response, and spatial movement of capsids and virions. Danane et al. [6] address an optimal control problem for a time-delayed differential equation model describing the interactions between HBV, HBV DNA-containing capsids, hepatocytes, and the CTL immune response.

Tadmon et al. [29] analyze an initial boundary value problem for HBV infection. In [18], the authors develop and study a novel partial differential equation model to better understand the mechanisms and dynamics of HBV infection in vivo. This model includes intracellular HBV DNA-containing capsids, spatial diffusion of capsids and viruses, and adaptive immune responses mediated by cytotoxic T lymphocytes and antibodies. Tadmon et al. [30] propose a delay cell population model for HBV infection, incorporating spatial diffusion of free HBV particles and a Beddington-DeAngelis incidence function to describe viral infection. Finally, Foko et al. [8] employ the nonstandard finite difference method to discretize a nonlinear initial boundary value problem modeling the global dynamics of HBV infection with adaptive immunity.

The article by Miao et al. [22] studies an HBV infection model that incorporates two nonlinear incidence functions and spatial diffusion in capsids, viruses, and the CTL immune response. In [10], the authors conduct a dynamical analysis of a general delayed HBV infection model that includes capsids and adaptive immune response, while also accounting for the presence of exposed infected hepatocytes.

For the model (1.1), the initial conditions are given by

$$u_1(t,x)=\phi_1(x), u_2(t,x)=\phi_2(x), \text{ and } u_3(t,x)=\phi_3(x), \quad (t,x)\in[-\tau,0]\times[0,1], \tag{1.2}$$

and the homogeneous Neumann type boundary conditions are given by

$$\frac{\partial u_3}{\partial x}(t,0)=\frac{\partial u_3}{\partial x}(t,1)=0, \quad t\in(0,\infty), \tag{1.3}$$

where $\tau=\max\{\tau_1,\tau_2\}$ with $\tau_1,\tau_2>0$. The following assumptions will stand throughout this paper:

(A1) $\tau_1,\tau_2>0$, $\tau=\max\{\tau_1,\tau_2\}$, b, μ_1, c_1, μ_2, a, γ, c_2, μ_3 are nonnegative constants that are less than or equal to a positive constant B, $f\in C(\mathbb{R}^3)$ is such that

$$\begin{aligned}|f(y_1,y_2,y_3)| \;\leq\; & d_0+\sum_{j=1}^{k_1} d_{jk_1}|y_1|^{p_{jk_1}}+\sum_{j=1}^{k_2} d_{jk_2}|y_2|^{p_{jk_2}}\\ & +\sum_{j=1}^{k_3} d_{jk_3}|y_3|^{p_{jk_3}},\ (y_1,y_2,y_3)\in\mathbb{R}^3,\end{aligned}$$

where

$$0\leq d_0,d_{jk_1},d_{ik_2},d_{lk_3}\leq B, \quad j\in\{1,\dots,k_1\}, \quad i\in\{1,\dots,k_2\}, \quad l\in\{1,\dots,k_3\},$$

and

$$0\leq p_{jk_1},p_{ik_2},p_{lk_3}, \quad j\in\{1,\dots,k_1\}, \quad i\in\{1,\dots,k_2\}, \quad l\in\{1,\dots,k_3\}.$$

The functions $\phi_1,\ \phi_2,\ \phi_3\in\mathcal{C}^2([0,1])$ are such that

$$0\leq\phi_j, \quad \left|\frac{\partial\phi_j}{\partial x}\right|, \quad \left|\frac{\partial^2\phi_j}{\partial x^2}\right|\leq B \quad \text{on} \quad [0,1], \quad \text{for } j=1,2,3.$$

The aim of this paper is twofold. First, under assumption $(A1)$, we establish the existence of at least one solution to the initial-boundary value problem (1.1)–(1.3). Notably, while prior work [15] addressed this problem under the restriction that the incidence function f is continuously differentiable, our analysis extends these results by requiring only continuity of f, thereby broadening the applicability of the model. Second, we prove the Ulam-Hyers stability of the system. This stability property ensures that for any approximate solution (or perturbed version) of the system, there exists an exact solution that remains "close" to the approximation. Such robustness is critical for practical applications, as it demonstrates the model's insensitivity to small perturbations or errors in initial data or parameters. For instance, even if the initial densities of cells or viruses are slightly misestimated, the model's predictions remain reliable at least locally during the early stages of infection.

Let $X_1 = C^1([-\tau,\infty), C^2([0,1]))$ be the space of continuously differentiable functions on $[-\tau,\infty)$ with values in the Banach space $C^2([0,1])$. The space X_1 will be endowed with the norm

$$\begin{aligned}\|u\|_1 \;=\; \max\{&\sup_{(t,x)\in[-\tau,\infty)\times[0,1]}|u(t,x)|, \quad \sup_{(t,x)\in[-\tau,\infty)\times[0,1]}|u_x(t,x)|,\\ &\sup_{(t,x)\in[-\tau,\infty)\times[0,1]}|u_{xx}(t,x)|, \quad \sup_{(t,x)\in[-\tau,\infty)\times[0,1]}|u_t(t,x)|\},\end{aligned}$$

provided it exists. Our main results are as follows.

Theorem 1.1. *Suppose that the condition* $(A1)$ *holds. Then the problem* (1.1)–(1.3) *has at least one bounded nonnegative solution in* X_1^3.

Theorem 1.2. *Suppose* $(A1)$ *and*

$$|f(u_1,u_2,u_3)u_3 - f(v_1,v_2,v_3)v_3| \le W|u_3 - v_3|, \quad (u_1,u_2,u_3),(v_1,v_2,v_3)\in\mathbb{R}^3, \tag{1.4}$$

for some nonnegative constant W. *Then the problem* (1.1)-(1.3) *is Ulam-Hyers stable in* $[-\tau,T]\times[0,1]$, $T>0$.

The condition (1.4) means the infection rate changes smoothly with the virus level, keeping the model stable and realistic. The constant W limits how sensitive the system is to viral fluctuations. Theorem 1.2 establishes that solutions in $[-\tau,T]\times[0,1]$, $T>0$, exhibit robustness against small perturbations, a property of paramount importance for ensuring the reliability of numerical simulations and practical applications.

The remainder of this paper is structured as follows: Section 2 presents essential auxiliary results. In Sections 3 and 4, we provide the proofs of Theorems 1.1 and 1.2 respectively, along with illustrative examples to demonstrate their applicability.

2 Auxiliary Results

Let $X = X_1^5$ be endowed with the norm

$$\|u\| = \max_{j\in\{1,\dots,5\}}\|u_j\|_1,$$

provided it exists. For $u=(u_1,u_2,u_3,u_4,u_5)\in X$, define the operators:

$$F_1u(t,x) = \begin{cases}\phi_1(x) + bt - \displaystyle\int_0^t [f(u_1(s,x),u_2(s,x),u_3(s,x))u_3(s,x) + \mu_1u_1(s,x)]\,ds,\\ \qquad (t,x)\in(0,\infty)\times[0,1],\\ \phi_1(x), \quad (t,x)\in[-\tau,0]\times[0,1],\end{cases}$$

$$F_2u(t,x) = \begin{cases}\phi_2(x) + e^{-c_1\tau_1}\displaystyle\int_0^t f(u_1(s-\tau_1,x),u_2(s-\tau_1,x),u_3(s-\tau_1,x))u_3(s-\tau_1,x)ds\\ \qquad -\mu_2\displaystyle\int_0^t u_2(s,x)ds, \quad (t,x)\in(0,\infty)\times[0,1],\\ \phi_2(x), \quad (t,x)\in[-\tau,0]\times[0,1],\end{cases}$$

$$F_3u(t,x)=\begin{cases}\phi_3(x)+\displaystyle\int_0^t\left[a\frac{\partial^2u_3}{\partial x^2}(s,x)+\gamma u_2(s-\tau_2,x)e^{-c_2\tau_2}-\mu_3u_3(s,x)\right]ds,\\ \qquad (t,x)\in(0,\infty)\times[0,1],\\ \phi_3(x),\quad (t,x)\in[-\tau,0]\times[0,1],\end{cases}$$

$$F_4u(t,x)=\begin{cases}\frac{\partial u_3}{\partial x}(t,0)+u_4(t,x), & (t,x)\in(0,\infty)\times[0,1],\\ \frac{\partial u_3}{\partial x}(0,0)+u_4(t,x), & (t,x)\in[-\tau,0]\times[0,1],\end{cases}$$

$$F_5u(t,x)=\begin{cases}\frac{\partial u_3}{\partial x}(t,1)+u_5(t,x), & (t,x)\in(0,\infty)\times[0,1],\\ \frac{\partial u_3}{\partial x}(0,1)+u_5(t,x), & (t,x)\in[-\tau,0]\times[0,1],\end{cases}$$

and let

$$Fu(t,x)=(F_1u(t,x),F_2u(t,x),F_3u(t,x),F_4u(t,x),F_5u(t,x)),\quad (t,x)\in[-\tau,\infty)\times[0,1].$$

Lemma 2.1. *Suppose that* $(A1)$ *holds. Let* $u=(u_1,u_2,u_3,u_4,u_5)\in X$ *be a fixed point of the operator* F. *Then* (u_1,u_2,u_3) *is a solution to the IBVP* (1.1)–(1.3).

Proof. Let $u=(u_1,u_2,u_3,u_4,u_5)\in X$ be a fixed point of the operator F, i.e., $u=Fu$. Then, for $(t,x)\in(0,\infty)\times[0,1]$, we have

$$u_1(t,x)=\phi_1(x)+bt-\int_0^t f(u_1(s,x),u_2(s,x),u_3(s,x))u_3(s,x)\,ds-\mu_1\int_0^t u_1(s,x)\,ds,$$

$$u_2(t,x)=\phi_2(x)+e^{-c_1\tau_1}\int_0^t f(u_1(s-\tau_1,x),u_2(s-\tau_1,x),u_3(s-\tau_1,x))u_3(s-\tau_1,x)\,ds$$
$$-\mu_2\int_0^t u_2(s,x)\,ds,$$

$$u_3(t,x)=\phi_3(x)+\int_0^t\left(a\frac{\partial^2u_3}{\partial x^2}(s,x)+\gamma u_2(s-\tau_2,x)e^{-c_2\tau_2}-\mu_3u_3(s,x)\right)ds,$$

$$u_4(t,x)=\frac{\partial u_3}{\partial x}(t,0)+u_4(t,x),$$

$$u_5(t,x)=\frac{\partial u_3}{\partial x}(t,1)+u_5(t,x).$$

From the equations for u_4 and u_5, it immediately follows that:

$$\frac{\partial u_3}{\partial x}(t,0)=0\quad\text{and}\quad\frac{\partial u_3}{\partial x}(t,1)=0,\quad\text{for } t\in(0,\infty).$$

Differentiating the first three equations with respect to t recovers the original system (1.1). For $(t,x)\in[-\tau,0]\times[0,1]$, we have $u_j(t,x)=\phi_j(x)$ for $j=1,2,3$, which are the initial conditions. This completes the proof. □

Let $\widetilde{B}=\max\left\{B+2B^2+B\left(\sum_{j=1}^{k_1}B^{p_{jk_1}+1}+\sum_{j=1}^{k_2}B^{p_{jk_2}+1}+\sum_{j=1}^{k_3}B^{p_{jk_3}+1}\right),3B^2,2B\right\}$, where B is the constant from assumption $(A1)$.

Lemma 2.2. *Suppose that* $(A1)$ *holds. If* $u\in X$ *and* $\|u\|\le B$, *then*

$$|F_ju(t,x)|\le\widetilde{B}(1+|t|),\quad (t,x)\in[-\tau,\infty)\times[0,1],\quad j\in\{1,\dots,5\}.$$

Proof. For $(t,x)\in[-\tau,\infty)\times[0,1]$, we have

$$|f(u_1(t,x),u_2(t,x),u_3(t,x))|\ \le\ d_0+\sum_{j=1}^{k_1}d_{jk_1}|u_1(t,x)|^{p_{jk_1}}+\sum_{j=1}^{k_2}d_{jk_2}|u_2(t,x)|^{p_{jk_2}}$$
$$+\sum_{j=1}^{k_3}d_{jk_3}|u_3(t,x)|^{p_{jk_3}}$$

$$\leq B + \sum_{j=1}^{k_1} B^{p_{jk_1}+1} + \sum_{j=1}^{k_2} B^{p_{jk_2}+1} + \sum_{j=1}^{k_3} B^{p_{jk_3}+1}.$$

Let $(t,x) \in (0,\infty) \times [0,1]$. Then

$$\begin{aligned}
|F_1u(t,x)| &\leq \phi_1(x) + bt + \int_0^t |f(u_1(s,x),u_2(s,x),u_3(s,x))||u_3(s,x)|ds + \mu_1 \int_0^t |u_1(s,x)|ds \\
&\leq B + Bt + \left(B + \sum_{j=1}^{k_1} B^{p_{jk_1}+1} + \sum_{j=1}^{k_2} B^{p_{jk_2}+1} + \sum_{j=1}^{k_3} B^{p_{jk_3}+1} \right) Bt + B^2 t \\
&= B + \left(B + 2B^2 + B \left(\sum_{j=1}^{k_1} B^{p_{jk_1}+1} + \sum_{j=1}^{k_2} B^{p_{jk_2}+1} + \sum_{j=1}^{k_3} B^{p_{jk_3}+1} \right) \right) t \\
&\leq \widetilde{B}(1+t)
\end{aligned}$$

and

$$\begin{aligned}
|F_2u(t,x)| &\leq \phi_2(x) + e^{-c_1\tau_1} \int_0^t |f(u_1(s-\tau_1,x),u_2(s-\tau_1,x),u_3(s-\tau_1,x))||u_3(s-\tau_1,x)|ds \\
&\quad + \mu_2 \int_0^t |u_2(s,x)|ds \\
&\leq B + \left(B + \sum_{j=1}^{k_1} B^{p_{jk_1}+1} + \sum_{j=1}^{k_2} B^{p_{jk_2}+1} + \sum_{j=1}^{k_3} B^{p_{jk_3}+1} \right) Bt + B^2 t \\
&= B + \left(2B^2 + B \left(\sum_{j=1}^{k_1} B^{p_{jk_1}+1} + \sum_{j=1}^{k_2} B^{p_{jk_2}+1} + \sum_{j=1}^{k_3} B^{p_{jk_3}+1} \right) \right) t \\
&\leq \widetilde{B}(1+t),
\end{aligned}$$

and

$$\begin{aligned}
|F_3u(t,x)| &\leq |\phi_3(x)| + \int_0^t \left(a \left| \frac{\partial^2 u_3}{\partial x^2}(s,x) \right| + \gamma |u_2(s-\tau_2,x)| e^{-c_2\tau_2} + \mu_3 |u_3(s,x)| \right) ds \\
&\leq B + 3B^2 t \\
&\leq \widetilde{B}(1+t),
\end{aligned}$$

and

$$\begin{aligned}
|F_4u(t,x)| &\leq |\frac{\partial u_3}{\partial x}(t,0)| + |u_4(t,x)| \\
&\leq 2B \\
&\leq \widetilde{B}(1+t),
\end{aligned}$$

and

$$\begin{aligned}
|F_5u(t,x)| &\leq |\frac{\partial u_3}{\partial x}(t,1)| + |u_5(t,x)| \\
&\leq 2B \\
&\leq \widetilde{B}(1+t).
\end{aligned}$$

For $(t,x) \in [-\tau,0] \times [0,1]$, we have

$$\begin{aligned} |F_j u(t,x)| &= \phi_j(x) \\ &\leq B \\ &\leq \widetilde{B}(1+|t|), \quad j \in \{1,2,3\}, \end{aligned}$$

and

$$\begin{aligned} |F_4 u(t,x)| &\leq |\frac{\partial u_3}{\partial x}(0,0)| + |u_4(t,x)| \\ &\leq 2B \\ &\leq \widetilde{B}(1+|t|), \end{aligned}$$

and

$$\begin{aligned} |F_5 u(t,x)| &\leq |\frac{\partial u_3}{\partial x}(0,1)| + |u_5(t,x)| \\ &\leq 2B \\ &\leq \widetilde{B}(1+|t|). \end{aligned}$$

This completes the proof. □

For $u \in X$ and $(t,x) \in [-\tau,\infty) \times [0,1]$, define the operators

$$\begin{aligned} S_{1j}u(t,x) &= u_j(t,x) - F_j u(t,x), \quad j \in \{1,\dots,5\}, \\ S_1 u(t,x) &= (S_{11}u(t,x),\dots,S_{15}u(t,x)). \end{aligned}$$

Lemma 2.3. *Let $u \in X$ be such that $\|u\| \leq B$. Then*

$$|S_{1j}u(t,x)| \leq B_1(1+|t|), \quad (t,x) \in [-\tau,\infty) \times [0,1], \quad j \in \{1,\dots,5\},$$

where $B_1 = B + \widetilde{B}$.

Proof. By Lemma 2.2, we get

$$\begin{aligned} |S_{1j}u(t,x)| &= |u_j(t,x) - F_j u(t,x)| \\ &\leq |u_j(t,x)| + |F_j u(t,x)| \\ &\leq B + \widetilde{B}(1+|t|) \\ &\leq B_1(1+|t|), \quad (t,x) \in [-\tau,\infty) \times [0,1], \quad j \in \{1,\dots,5\}. \end{aligned}$$

This completes the proof. □

Let $g \in \mathcal{C}([-\tau,\infty),\mathbb{R}_+)$ be a function such that $g > 0$ on $(-\tau,\infty)$ and

$$2(1+|t|+\tau)^2 \int_{-\tau}^{t} g(t_1)dt_1 \leq A, \quad t \in [-\tau,\infty), \tag{2.1}$$

for some positive constant A. Such function g exists. Indeed, take

$$h(s) = \log \frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})},$$

$$l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)^3 h(s) < \infty,$$

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)^3 l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2)^3\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [26] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t) = Q(t+\tau), \quad t \in [-\tau, \infty).$$

Then there exists a constant $C > 0$ such that

$$2(1+|t|+\tau)^2 \int_{-\tau}^{t} g_1(\tau)d\tau \leq C, \quad (t,x) \in [-\tau,\infty).$$

Let

$$g(t) = \frac{A}{C}g_1(t), \quad t \in [-\tau, \infty).$$

Then

$$2(1+|t|+\tau)^2 \int_{-\tau}^{t} g(z)dz \leq A, \quad t \in [-\tau,\infty).$$

For $u \in X$ and $(t,x) \in [-\tau,\infty)\times[0,1]$, define the operators

$$S_{2j}u(t,x) = \int_{-\tau}^{t}\int_{0}^{x}(t-t_1)(x-s)^2 g(t_1)S_{1j}u(t_1,s)dsdt_1, \quad j \in \{1,\ldots,5\},$$

$$S_2u(t,x) = (S_{21}u(t,x),\ldots,S_{25}u(t,x)). \tag{2.2}$$

Lemma 2.4. *Suppose that the hypothesis* $(A1)$ *holds. If* $u \in X$ *and* $\|u\| \leq B$, *then*

$$\|S_2u\| \leq AB_1,$$

where A *is the constant from inequality (2.1) and* B_1 *is the constant defined in Lemma 2.3.*

Proof. Suppose that $(A1)$ holds, $u \in X$ such that $\|u\| \leq B$, $(t,x) \in [-\tau,\infty)\times[0,1]$ and $j \in \{1,\ldots,5\}$. Then, we have

$$\begin{aligned} |S_{2j}u(t,x)| &= \left|\int_{-\tau}^{t}\int_{0}^{x}(t-t_1)(x-s)^2 g(t_1)S_{1j}u(t_1,s)dsdt_1\right| \\ &\leq \int_{-\tau}^{t}\left|\int_{0}^{x}(t-t_1)(x-s)^2 g(t_1)|S_{1j}u(t_1,s)|ds\right|dt_1 \end{aligned}$$

$$\begin{aligned} &\leq B_1(1+|t|+\tau)\int_{-\tau}^{t}(1+|t_1|)g(t_1)dt_1 \\ &\leq B_1(1+|t|+\tau)^2\int_{-\tau}^{t}g(t_1)dt_1 \\ &\leq AB_1, \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\partial}{\partial t}S_{2j}u(t,x)\right| &= \left|\int_{-\tau}^{t}\int_{0}^{x}(x-s)^2g(t_1)S_{1j}u(t_1,s)dsdt_1\right| \\ &\leq \int_{-\tau}^{t}\left|\int_{0}^{x}(x-s)^2g(t_1)|S_{1j}u(t_1,s)|ds\right|dt_1 \\ &\leq B_1\int_{-\tau}^{t}(1+|t_1|)g(t_1)dt_1 \\ &\leq B_1(1+|t|+\tau)^2\int_{-\tau}^{t}g(t_1)dt_1 \\ &\leq AB_1, \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\partial}{\partial x}S_{2j}u(t,x)\right| &= \left|2\int_{-\tau}^{t}\int_{0}^{x}(t-t_1)(x-s)g(t_1)S_{1j}u(t_1,s)dsdt_1\right| \\ &\leq 2\int_{-\tau}^{t}\left|\int_{0}^{x}(t-t_1)(x-s)g(t_1)|S_{1j}u(t_1,s)|ds\right|dt_1 \\ &\leq 2B_1(1+|t|+\tau)\int_{-\tau}^{t}(1+|t_1|)g(t_1)dt_1 \\ &\leq 2B_1(1+|t|+\tau)^2\int_{-\tau}^{t}g(t_1)dt_1 \\ &\leq AB_1, \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\partial^2}{\partial x^2}S_{2j}u(t,x)\right| &= \left|2\int_{-\tau}^{t}\int_{0}^{x}(t-t_1)g(t_1)S_{1j}u(t_1,s)dsdt_1\right| \\ &\leq 2\int_{-\tau}^{t}\left|\int_{0}^{x}(t-t_1)g(t_1)|S_{1j}u(t_1,s)|ds\right|dt_1 \\ &\leq 2B_1(1+|t|+\tau)\int_{-\tau}^{t}(1+|t_1|)g(t_1)dt_1 \\ &\leq 2B_1(1+|t|+\tau)^2\int_{-\tau}^{t}g(t_1)dt_1 \\ &\leq AB_1. \end{aligned}$$

Thus, $\|S_2u\| \leq AB_1$. This completes the proof. □

Lemma 2.5. *Suppose that the assumption* $(A1)$ *holds. If* $u=(u_1,u_2,u_3,u_4,u_5)\in X$ *satisfies the equation*

$$S_2u(t,x)=D,\quad (t,x)\in[-\tau,\infty)\times[0,1], \tag{2.3}$$

for some constant vector D, *then* (u_1,u_2,u_3) *is a solution to the problem* (1.1)–(1.3).

Proof. Recall the definition of the j-th component of S_2: For $(t,x)\in[-\tau,\infty)\times[0,1]$,

$$S_{2j}u(t,x)=\int_{-\tau}^{t}\int_{0}^{x}(t-t_1)(x-s)^2g(t_1)S_{1j}u(t_1,s)dsdt_1,\quad j\in\{1,\dots,5\}.$$

Since equation (2.3) holds identically for all (t,x), we can differentiate both sides. Differentiating twice with respect to t and three times with respect to x yields:

$$\frac{\partial^5}{\partial x^3 \partial t^2} S_{2j}u(t,x) = 0, \quad (t,x) \in [-\tau,\infty)\times[0,1], \quad j \in \{1,\dots,5\}.$$

Let us compute this derivative step by step. For $(t,x) \in [-\tau,\infty)\times[0,1]$ and $j \in \{1,\dots,5\}$, we have

$$\begin{aligned}
\frac{\partial}{\partial t} S_{2j}u(t,x) &= \int_{-\tau}^{t}\int_{0}^{x} (x-s)^2 g(t_1) S_{1j}u(t_1,s)\,ds dt_1,\\
\frac{\partial^2}{\partial t^2} S_{2j}u(t,x) &= \int_{0}^{x} (x-s)^2 g(t) S_{1j}u(t,s)\,ds,\\
\frac{\partial^3}{\partial x \partial t^2} S_{2j}u(t,x) &= 2\int_{0}^{x} (x-s) g(t) S_{1j}u(t,s)\,ds,\\
\frac{\partial^4}{\partial x^2 \partial t^2} S_{2j}u(t,x) &= 2\int_{0}^{x} g(t) S_{1j}u(t,s)\,ds,
\end{aligned}$$

and

$$\frac{\partial^5}{\partial x^3 \partial t^2} S_{2j}u(t,x) = 2g(t) S_{1j}u(t,x).$$

Since

$$\frac{\partial^5}{\partial x^3 \partial t^2} S_{2j}u(t,x) = 0, \quad \text{for all } (t,x) \in [-\tau,\infty)\times[0,1],$$

then

$$g(t) S_{1j}u(t,x) = 0 \quad \text{for all } (t,x) \in [-\tau,\infty)\times[0,1], \quad j \in \{1,\dots,5\}.$$

Since $g(t) > 0$ on $(-\tau,\infty)$, it follows that $S_1u(t,x) = 0$ on $(-\tau,\infty)\times[0,1]$. By the continuity of S_1u, this equality extends to the domain $[-\tau,\infty)\times[0,1]$. Since each solution of the equation $S_1u = 0$ is a fixed point of the operator F, we deduce thanks to Lemma 2.1 that (u_1,u_2,u_3) is a solution to the problem (1.1)–(1.3). This completes the proof. □

3 Proof of Theorem 1.1

The following fixed point theorem, which extends the result in [20, Theorem 3.1.7], will be used to establish the existence of at least one nontrivial nonnegative solution to problem (1.1)-(1.3). A similar result can be found in [3, Theorem 2.7].

Let $\mathcal{P}$ be a cone in a Banach space E. For two numbers $0 < r < R$, we set

$$\mathcal{P}_r = \{x \in \mathcal{P} : \|x\| < r\}, \text{ and } \overline{\mathcal{P}}_{r,R} = \{x \in \mathcal{P} : r \le \|x\| \le R\}.$$

Theorem 3.1. *[21, Theorem 1] Let $\alpha,\beta > 0$, $\alpha \neq \beta$, and define $r = \min(\alpha,\beta)$ and $R = \max(\alpha,\beta)$. Assume that $T : \mathcal{P} \to E$ is an expansive map and $S : \overline{\mathcal{P}_R} \to E$ is a completely continuous map. Suppose that $S(\overline{\mathcal{P}_R}) \subset (I-T)(\mathcal{P})$, and there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that the following conditions hold:*

$$\begin{aligned}
Sx &\neq (I-T)(x-\lambda u_0) \text{ for all } \lambda > 0,\ x \in \partial\mathcal{P}_\alpha \cap (\mathcal{P}+\lambda u_0),\\
Sx &\neq (I-T)(\lambda x) \text{ for all } \lambda \ge 1+\epsilon,\ x \in \partial\mathcal{P}_\beta, \text{ for some } \epsilon > 0 \text{ sufficiently small.}
\end{aligned}$$

Then $T+S$ has a fixed point $x \in \overline{\mathcal{P}}_{r,R}$.

Let X be the space used in the previous section and r and R be positive constants that satisfy the following conditions

$$r < R \le B,$$

where B is the constant which appears in condition $(A1)$. Set

$$\widetilde{P} = \{u \in X : u \ge 0 \quad \text{on} \quad [-\tau,\infty)\times[0,1]\}.$$

With $\mathcal{P}$ we will denote the closure of the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$ and $(t,x) \in [-\tau,\infty) \times [0,1]$, define the operators T_1 and S_3 as follows

$$\begin{aligned} T_1v(t,x) &= (1+m\epsilon)v(t,x), \\ S_3v(t,x) &= -\epsilon S_2v(t,x) - \epsilon A\mathcal{B}_1 - m\epsilon v(t,x), \end{aligned}$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (2.2). Here, A is the constant which appears in formula (2.1) and $\mathcal{B}_1 = (B_1,\dots,B_1) \in \mathbb{R}^5$ with B_1 is the constant of Lemma 2.3. From Lemma 2.5, we deduce that if $(w_1,w_2,w_3,w_4,w_5) \in X$ is a fixed point of the operator $T_1 + S_3$, then (w_1,w_2,w_3) is a solution to the IBVP (1.1)–(1.3). Define

$$\mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \text{ and } \mathcal{P}_R = \{v \in \mathcal{P} : \|v\| < R\}.$$

(i) For $v_1, v_2 \in \mathcal{P}$, we have

$$\|T_1v_1 - T_1v_2\| = (1+m\epsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \mathcal{P} \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(ii) For $v \in \overline{\mathcal{P}_R}$, we get

$$\begin{aligned} \|S_3v\| &\le \epsilon\|S_2v\| + \epsilon AB_1 + m\epsilon\|v\| \\ &\le \epsilon\Big(2AB_1 + mR\Big). \end{aligned}$$

Therefore $S_3(\overline{\mathcal{P}_R})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_R} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_R})$ is equi-continuous. Consequently, $S_3 : \overline{\mathcal{P}_R} \to X$ is completely continuous.

(iii) Let $v_1 \in \overline{\mathcal{P}_R}$. Set

$$v_2 = v_1 + \frac{1}{m}S_2v_1 + \frac{1}{m}A\mathcal{B}_1.$$

We have $v_2 \ge 0$ on $[-\tau,\infty) \times [0,1]$. Therefore $v_2 \in \mathcal{P}$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2v_1 - \epsilon A\mathcal{B}_1$$

or

$$\begin{aligned} (I - T_1)v_2 &= -\epsilon m v_2 \\ &= S_3v_1. \end{aligned}$$

Consequently, $S_3(\overline{\mathcal{P}_R}) \subset (I - T_1)(\mathcal{P})$.

(iv) Assume that for any $v_0 \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}$, there exist $\lambda > 0$ and $v \in \partial\mathcal{P}_r \cap (\mathcal{P} + \lambda v_0)$ such that

$$S_3v = (I - T_1)(v - \lambda v_0).$$

Then

$$-\epsilon S_2v - \epsilon A\mathcal{B}_1 - m\epsilon v = -m\epsilon(v - \lambda v_0),$$

or

$$-S_2v = m\lambda v_0 + A\mathcal{B}_1.$$

Hence,

$$\|S_2v\| = \|m\lambda v_0 + A\mathcal{B}_1\| > AB_1.$$

This contradicts Lemma 2.4.

(v) Let $\epsilon_1 = \frac{3AB_1}{mR}$. Suppose that there exists a $v_1 \in \partial\mathcal{P}_R$ and $\lambda_1 \ge 1 + \epsilon_1$ such that

$$S_3v_1 = (I - T_1)(\lambda_1 v_1).$$

Then,

$$\epsilon S_2v_1 + \epsilon A\mathcal{B}_1 + m\epsilon v_1 = \lambda_1 m\epsilon v_1.$$

Hence,

$$\begin{aligned}\lambda_1 m\epsilon R &= \lambda_1 m\epsilon\|v_1\| \\ &\leq \epsilon\|S_2 v_1\| + \epsilon AB_1 + m\epsilon\|v_1\| \\ &\leq 2\epsilon AB_1 + m\epsilon R,\end{aligned}$$

whereupon

$$\begin{aligned}\lambda_1 &\leq 1 + \frac{2AB_1}{mR} \\ &< 1 + \epsilon_1.\end{aligned}$$

This is a contradiction.

Therefore all conditions of Theorem 3.1 hold. Hence, the problem (1.1)–(1.3) has at least one nonnegative solution.

Example 3.2. In the context of our generalized HBV model (1.1), we consider the function given by

$$f(u_1, u_2, u_3) = \frac{\beta u_1}{1 + \delta_1 u_1 + \delta_2 u_3 + \delta_2\alpha + \delta_2 |u_3 - \alpha|}$$

where $\beta, \delta_1, \delta_2, \alpha \geq 0$ are constants. The function $f(u_1, u_2, u_3)u_3 = \frac{\beta u_1 u_3}{1+\delta_1 u_1+\delta_2 u_3+\delta_2\alpha+\delta_2|u_3-\alpha|}$ is a novel extension of the classical Beddington-DeAngelis incidence function, which can be used to model the rate of new infections in hepatitis B [30, 32], where u_1 is healthy cells and u_3 is free virus. It incorporates standard mass action kinetics $\beta u_1 u_3$ and saturation effects $1 + \delta_1 u_1 + \delta_2 u_3$ but adds a key nonlinear regulatory term, $\delta_2 |u_3 - \alpha|$, which dependents on a critical viral concentration, α. This creates a "phase transition" in the infection's dynamics, causing the system's behavior to change significantly once the viral load crosses this threshold, which could correspond to triggering a strong immune response [2, 13] or a shift from acute to chronic infection [1]. Moreover, the product $f(u_1, u_2, u_3)u_3$ is continuous and strictly increasing for $u_1,\ u_3 > 0$ ensuring that higher concentrations of healthy cells or virus particles lead to increased infection rates. This behavior is consistent with biological principles, although saturation prevents unlimited growth, thus strengthening its biological plausibility.

On the other hand, the function f satisfies the condition (A1). Indeed, f is continuous on $\mathbb{R}^3$ and satisfies

$$|f(u_1, u_2, u_3)| \leq \beta |u_1|, \ (u_1, u_2, u_3) \in \mathbb{R}^3,$$

which corresponds to

$$k_1 = k_2 = k_3 = 1,$$

$$d_0 = 0,\ d_{1k_1} = \beta,\ d_{1k_2} = 0,\ d_{1k_3} = 0,$$

and

$$p_{1k_1} = 1,\ p_{1k_2} = 0,\ p_{1k_3} = 0.$$

Let $\phi_1,\ \phi_2,\ \phi_3$ be any nonnegative functions in $C^2([0, 1])$ and

$$M = \max\{\max_{x\in[0,1]} \phi_j(x), \max_{x\in[0,1]} \left|\frac{\partial\phi_j(x)}{\partial x}\right|, \max_{x\in[0,1]} \left|\frac{\partial^2\phi_j(x)}{\partial x^2}\right|, j = 1, 2, 3\}.$$

We put

$$B = \max\{1, \beta, \tau_1, \tau_2,\ b,\ \mu_1,\ c_1,\ \mu_2,\ a,\ \gamma,\ c_2,\ \mu_3,\ M\}.$$

where $\tau_1, \tau_2, b, \mu_1, c_1, \mu_2, a, \gamma, c_2, \mu_3$ are the nonnegative constants of the system (1.1). Consequently, Theorem 1.1 applies, ensuring the existence of at least one bounded nonnegative solution to the initial-boundary value problem (1.1)–(1.3) for this biologically motivated and non-smooth incidence function.

4 Proof of Theorem 1.2

We first recall the precise definition of Ulam-Hyers stability for our problem. For more details on this theory, readers are referred to [19], [23] and [28] and references therein.

Definition 4.1. The initial-boundary value problem (1.1)-(1.3) is said to have Ulam-Hyers stability in $[-\tau, T] \times [0,1]$, $T > 0$, if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for any functions $v_1, v_2, v_3 \in C^1([-\tau, T], C^2([0,1]))$ satisfying

$$\left|\frac{\partial v_1}{\partial t}(t,x) - b + f(v_1(t,x), v_2(t,x), v_3(t,x))v_3(t,x) + \mu_1 v_1(t,x)\right| \leq \epsilon$$

$$\left|\frac{\partial v_2}{\partial t}(t,x) - f(v_1(t-\tau_1,x), v_2(t-\tau_1,x), v_3(t-\tau_1,x))v_3(t-\tau_1,x)e^{-c_1\tau_1} + \mu_2 v_2(t,x)\right| \leq \epsilon$$

$$\left|\frac{\partial v_3}{\partial t}(t,x) - a\frac{\partial^2 v_3}{\partial x^2}(t,x) - \gamma v_2(t-\tau_2,x)e^{-c_2\tau_2} + \mu_3 v_3(t,x)\right| \leq \epsilon, \quad (t,x) \in (0,T] \times [0,1], \tag{4.1}$$

$$\begin{aligned} &|v_1(t,x) - \phi_1(x)| < \epsilon \\ &|v_2(t,x) - \phi_2(x)| < \epsilon \\ &|v_3(t,x) - \phi_3(x)| < \epsilon, \quad (t,x) \in [-\tau, 0] \times [0,1], \end{aligned} \tag{4.2}$$

and

$$\frac{\partial v_3}{\partial x}(t,0) = \frac{\partial v_3}{\partial x}(t,1) = 0, \quad t \in (0,T], \tag{4.3}$$

there exists a solution (u_1, u_2, u_3) of (1.1)-(1.3) such that:

$$\begin{aligned} |u_1(t,x) - v_1(t,x)| &\leq c\epsilon, \\ |u_2(t,x) - v_2(t,x)| &\leq c\epsilon, \\ |u_3(t,x) - v_3(t,x)| &\leq c\epsilon, \quad (t,x) \in [-\tau, T] \times [0,1]. \end{aligned}$$

Let $\epsilon > 0$ be arbitrarily and $T > 0$ fixed. Suppose (v_1, v_2, v_3) satisfies the approximate system in the above definition. The first two inequalities of (4.1) yields:

$$\left|v_1(t,x) - v_1(0,x) - bt + \int_0^t (f(v_1(s,x), v_2(s,x), v_3(s,x))v_3(s,x)ds + \mu_1 \int_0^t v_1(s,x))ds\right| \leq \epsilon t,$$

$$\left|v_2(t,x) - v_2(0,x) - \int_0^t (f(v_1(s-\tau_1,x), v_2(s-\tau_1,x), v_3(s-\tau_1,x))v_3(s-\tau_1,x)e^{-c_1\tau_1} - \mu_2 v_2(s,x))ds\right| \leq \epsilon t,$$

$(t,x) \in (0,T] \times [0,1]$. Let

$$q(t,x) = \frac{\partial v_3}{\partial t}(t,x) - a\frac{\partial^2 v_3}{\partial x^2}(t,x) - \gamma v_2(t-\tau_2,x)e^{-c_2\tau_2} + \mu_3 v_3(t,x).$$

So,

$$|q(t,x)| \leq \epsilon, \text{ for } (t,x) \in (0,T] \times [0,1].$$

Let (u_1, u_2, u_3) be a solution of (1.1)-(1.3) given by Theorem 1.1 with:

$$u_j(t,x) = \phi_j(x), \text{ for } (t,x) \in [-\tau, 0] \times [0,1], \ j = 1, 2, 3.$$

The functions u_1 and u_2 can be expressed in integral form:

$$u_1(t,x) = \phi_1(x) + bt - \int_0^t f(u_1(s,x), u_2(s,x), u_3(s,x))u_3(s,x)ds - \mu_1 \int_0^t u_1(s,x)ds,$$

$$u_2(t,x) = \phi_2(x) + \int_0^t f(u_1(s-\tau_1,x),u_2(s-\tau_1,x),u_3(s-\tau_1,x))u_3(s-\tau_1,x)e^{-c\tau_1}ds$$
$$-\mu_2\int_0^t u_2(s,x)ds.$$

Let us consider the periodic extensions of the functions $u_j, v_j,\ j=2,3$ to $[-\tau,T]\times\mathbb{R}$ and of the functions $\phi_j,\ j=2,3$ to $\mathbb{R}$ defined by:

$$\widetilde{u}_j(t,x) = \begin{cases} u_j(t,x-2k) & \text{if } x\in[2k,2k+1] \\ u_j(t,2k+2-x) & \text{if } x\in[2k+1,2k+2], \quad k\in\mathbb{Z}, \quad j=2,3 \end{cases}$$

$$\widetilde{v}_j(t,x) = \begin{cases} v_j(t,x-2k) & \text{if } x\in[2k,2k+1] \\ v_j(t,2k+2-x) & \text{if } x\in[2k+1,2k+2], \quad k\in\mathbb{Z}, \quad j=2,3 \end{cases}$$

$$\widetilde{\phi}_j(x) = \begin{cases} \phi_j(x-2k) & \text{if } x\in[2k,2k+1] \\ \phi_j(2k+2-x) & \text{if } x\in[2k+1,2k+2], \quad k\in\mathbb{Z}, \quad j=2,3. \end{cases}$$

Let

$$\widetilde{q}(t,x) = \frac{\partial\widetilde{v}_3}{\partial t}(t,x) - a\frac{\partial^2\widetilde{v}_3}{\partial x^2}(t,x) - \gamma\widetilde{v}_2(t-\tau_2,x)e^{-c_2\tau_2} + \mu_3\widetilde{v}_3(t,x),\ (t,x)\in(0,T]\times\mathbb{R}. \quad (4.4)$$

Remark 4.2. (i) For each $j=2,3$ $\widetilde{u}_j$, $\widetilde{v}_j$ inherit on $[-\tau,T]\times\mathbb{R}$, the smoothness properties of u_j, v_j on $[-\tau,T]\times[0,1]$. More precisely, $\widetilde{u}_j,\ \widetilde{v}_j\in C^1([-\tau,T],C^2(\mathbb{R})),\ j=2,3$.

(ii) For all $x\in\mathbb{R}$, there exists $y\in[0,1]$ such that

$$\widetilde{u}_j(t,x) = u_j(t,y),\ \widetilde{v}_j(t,x) = v_j(t,y),\ \widetilde{\phi}_j(x) = \phi_j(y),\ j=2,3,$$

and

$$\frac{\partial^2\widetilde{v}_3}{\partial x^2}(t,x) = \frac{\partial^2 v_3}{\partial y^2}(t,y) \text{ and } \frac{\partial^2\widetilde{u}_3}{\partial x^2}(t,x) = \frac{\partial^2 u_3}{\partial y^2}(t,y),\ t\in[-\tau,T].$$

Then,

$$\max_{y\in[0,1]}|u_j(t,y)-v_j(t,y)| = \max_{x\in\mathbb{R}}|\widetilde{u}_j(t,x)-\widetilde{v}_j(t,x)|,\ j=2,3,\ t\in[-\tau,T],$$

and

$$\widetilde{q}(t,x) = q(t,y), t\in(0,T].$$

Then we deduce that

$$|\widetilde{q}(t,x)|\le\epsilon \quad \text{for } (t,x)\in(0,T]\times\mathbb{R}.$$

Solving the equation (4.4), we obtain

$$\widetilde{v}_3(t,x) = \frac{1}{2\sqrt{a\pi t}}\int_{-\infty}^{\infty}\widetilde{v}_3(0,\xi)e^{-\frac{(x-\xi)^2}{4at}-\mu_3 t}\,d\xi$$
$$+\int_0^t\frac{1}{2\sqrt{a\pi(t-s)}}\int_{-\infty}^{\infty}\left(\gamma\widetilde{v}_2(s-\tau_2,y)e^{-c_2\tau_2}+\widetilde{q}(s,y)\right)e^{-\frac{(x-y)^2}{4a(t-s)}-\mu_3(t-s)}\,dy\,ds,$$

$(t,x)\in(0,T]\times\mathbb{R}$. Therefore

$$\widetilde{u}_3(t,x) = \frac{1}{2\sqrt{a\pi t}}\int_{-\infty}^{\infty}\widetilde{\phi}_3(\xi)e^{-\frac{(x-\xi)^2}{4at}-\mu_3 t}d\xi$$
$$+\int_0^t\frac{1}{2\sqrt{a\pi(t-s)}}\int_{-\infty}^{\infty}\left(\gamma\widetilde{u}_2(s-\tau_2,y)e^{-c_2\tau_2}\right)e^{-\frac{(x-y)^2}{4a(t-s)}-\mu_3(t-s)}dyds,$$
$$(t,x)\in(0,T]\times\mathbb{R}.$$

Hence,

$$
\begin{aligned}
&|\widetilde{u}_3(t,x)-\widetilde{v}_3(t,x)|\\
=\;&\left|\frac{1}{2\sqrt{a\pi t}}\int_{-\infty}^{\infty}(\widetilde{\phi}_3(\xi)-\widetilde{v}(0,\xi))e^{-\frac{(x-\xi)^2}{4at}-\mu_3 t}d\xi\right.\\
&+\int_0^t\frac{1}{2\sqrt{a\pi(t-s)}}\int_{-\infty}^{\infty}(\gamma(\widetilde{u}_2(s-\tau_2,y)-\widetilde{v}_2(s-\tau_2,y))e^{-c_2\tau_2}\\
&\left.-\widetilde{q}(s,y))e^{-\frac{(x-y)^2}{4a(t-s)}-\mu_3(t-s)}dyds\right|\\
\leq\;&\frac{1}{2\sqrt{a\pi t}}\int_{-\infty}^{\infty}\left|\widetilde{\phi}_3(\xi)-\widetilde{v}(0,\xi)\right|e^{-\frac{(x-\xi)^2}{4at}-\mu_3 t}d\xi\\
&+\int_0^t\frac{1}{2\sqrt{a\pi(t-s)}}\int_{-\infty}^{\infty}(\gamma\left|\widetilde{u}_2(s-\tau_2,y)-\widetilde{v}_2(s-\tau_2,y)\right|e^{-c_2\tau_2}\\
&+|\widetilde{q}(s,y)|)e^{-\frac{(x-y)^2}{4a(t-s)}-\mu_3(t-s)}dyds\\
\leq\;&\epsilon+\epsilon t+\gamma e^{-c_2\tau_2}\int_0^t\frac{1}{2\sqrt{a\pi(t-s)}}\int_{-\infty}^{\infty}\left|\widetilde{u}_2(s-\tau_2,y)-\widetilde{v}_2(s-\tau_2,y)\right|e^{-\frac{(x-y)^2}{4a(t-s)}-\mu_3(t-s)}dyds\\
\leq\;&\epsilon+\epsilon t+\gamma e^{-c_2\tau_2}\int_0^t\frac{\max\limits_{y\in\mathbb{R}}\left|\widetilde{u}_2(s-\tau_2,y)-\widetilde{v}_2(s-\tau_2,y)\right|}{2\sqrt{a\pi(t-s)}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4a(t-s)}-\mu_3(t-s)}dyds\\
\leq\;&\epsilon+\epsilon t+\gamma e^{-c_2\tau_2}\int_0^t\max_{y\in[0,1]}\left|u_2(s-\tau_2,y)-v_2(s-\tau_2,y)\right|ds\\
\leq\;&\epsilon+\epsilon t+\gamma e^{-c_2\tau_2}\int_0^{\tau_2}\max_{y\in[0,1]}\left|\phi_2(y)-v_2(s-\tau_2,y)\right|ds\\
&+\gamma e^{-c_2\tau_2}\int_{\tau_2}^t\max_{y\in[0,1]}\left|u_2(s-\tau_2,y)-v_2(s-\tau_2,y)\right|ds\\
\leq\;&\epsilon+\epsilon t+\epsilon\gamma\tau_2 e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}\int_{\tau_2}^t\max_{y\in[0,1]}\left|u_2(s-\tau_2,y)-v_2(s-\tau_2,y)\right|ds\\
\leq\;&\epsilon+\epsilon t+\epsilon\gamma\tau_2 e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}\int_0^{t-\tau_2}\max_{y\in[0,1]}\left|u_2(s,y)-v_2(s,y)\right|ds\\
\leq\;&\epsilon+\epsilon T+\epsilon\gamma\tau_2 e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}\int_0^t\max_{y\in[0,1]}\left|u_2(s,y)-v_2(s,y)\right|ds,\quad (t,x)\in(0,T]\times\mathbb{R}.
\end{aligned}
$$

Finally, we have

$$
|\widetilde{u}_3(t,x)-\widetilde{v}_3(t,x)|\leq\epsilon+\epsilon T+\epsilon\gamma\tau_2 e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}\int_0^t\max_{y\in[0,1]}\left|u_2(s,y)-v_2(s,y)\right|ds,
$$

for all $(t,x) \in (0,T] \times \mathbb{R}$. Then

$$|u_3(t,x) - v_3(t,x)| \le \epsilon + \epsilon T + \epsilon\gamma\tau_2 e^{-c_2\tau_2} + \gamma e^{-c_2\tau_2} \int_0^t \max_{y\in[0,1]} |u_2(s,y) - v_2(s,y)|\, ds,$$

for all $(t,x) \in (0,T] \times [0,1]$.

On the other hand we have

$$|u_2(t,x) - v_2(t,x)|$$

$$\begin{aligned}
&= \left| \phi_2(x) + \int_0^t f(u_1(s-\tau_1,x), u_2(s-\tau_1,x), u_3(s-\tau_1,x)) u_3(s-\tau_1,x) e^{-c\tau_1} ds \right. \\
&\quad \left. -\mu_2 \int_0^t u_2(s,x) ds - v_2(t,x) \right| \\
&\le \left| v_2(t,x) - v_2(0,x) - \int_0^t (f(v_1(s-\tau_1,x), v_2(s-\tau_1,x), v_3(s-\tau_1,x)) v_3(s-\tau_1,x) e^{-c_1\tau_1} ds \right. \\
&\quad \left. -\mu_2 \int_0^t v_2(s,x) ds \right| + |\phi_2(x) - v_2(0,x)| \\
&\quad + \left| \int_0^t f(u_1(s-\tau_1,x), u_2(s-\tau_1,x), u_3(s-\tau_1,x)) u_3(s-\tau_1,x) e^{-c\tau_1} ds \right. \\
&\quad \left. - \int_0^t (f(v_1(s-\tau_1,x), v_2(s-\tau_1,x), v_3(s-\tau_1,x)) v_3(s-\tau_1,x) e^{-c_1\tau_1} ds \right| \\
&\quad + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)|\, ds \\
&\le \epsilon t + \epsilon + W e^{-c_1\tau_1} \int_0^t |u_3(s-\tau_1,x) - v_3(s-\tau_1,x)| ds + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)| ds \\
&\le \epsilon t + \epsilon + W e^{-c_1\tau_1} \int_0^{\tau_1} |\phi_3(x) - v_3(s-\tau_1,x)| ds \\
&\quad + W e^{-c_1\tau_1} \int_{\tau_1}^t |u_3(s-\tau_1,x) - v_3(s-\tau_1,x)| ds + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)| ds \\
&\le \epsilon t + \epsilon + \epsilon\tau_1 W e^{-c_1\tau_1} + W e^{-c_1\tau_1} \int_0^{t-\tau_1} |u_3(s,x) - v_3(s,x)| ds + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)|\, ds \\
&\le \epsilon t + \epsilon + \epsilon\tau_1 W e^{-c_1\tau_1} + W e^{-c_1\tau_1} \int_0^t |u_3(s,x) - v_3(s,x)| ds + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon t + \epsilon + \epsilon\tau_1 W e^{-c_1\tau_1} + W e^{-c_1\tau_1} \int_0^t (\epsilon + \epsilon T + \epsilon\gamma\tau_2 e^{-c_2\tau_2} \\
&\quad + \gamma e^{-c_2\tau_2} \int_0^s \max_{y\in[0,1]} |u_2(\zeta, y) - v_2(\zeta, y)|\, d\zeta) ds \\
&\quad + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)|\, ds \\
&\leq \epsilon t + \epsilon + \epsilon\tau_1 W e^{-c_1\tau_1} + W e^{-c_1\tau_1} (\epsilon t + \epsilon T t + \epsilon\gamma\tau_2 e^{-c_2\tau_2} t \\
&\quad + \gamma e^{-c_2\tau_2} t \int_0^t \max_{y\in[0,1]} |u_2(s,y) - v_2(s,y)|\, ds) \\
&\quad + \mu_2 \int_0^t |u_2(s,x) - v_2(s,x)|\, ds \\
&\leq \epsilon t + \epsilon + \epsilon W e^{-c_1\tau_1} (\tau_1 + t + Tt) + W e^{-c_1\tau_1}\gamma\tau_2 e^{-c_2\tau_2} t \int_0^t \max_{y\in[0,1]} |u_2(s,y) - v_2(s,y)|\, ds \\
&\quad + \mu_2 \int_0^t \max_{x\in[0,1]} |u_2(s,x) - v_2(s,x)|\, ds \\
&\leq \epsilon T + \epsilon + \epsilon W e^{-c_1\tau_1} (\tau_1 + T + T^2) \\
&\quad + (W e^{-c_1\tau_1}\gamma\tau_2 e^{-c_2\tau_2} T + \mu_2) \int_0^t \max_{x\in[0,1]} |u_2(s,x) - v_2(s,x)|\,,
\end{aligned}
$$

for all $(t,x) \in (0,T] \times [0,1]$.

Finally, we have

$$
\begin{aligned}
|u_2(t,x) - v_2(t,x)| \quad &\leq \quad \epsilon T + \epsilon + \epsilon(\tau_1 + T + T^2) W e^{-c_1\tau_1} \\
&\quad + (W e^{-c_1\tau_1}\gamma\tau_2 e^{-c_2\tau_2} T + \mu_2) \int_0^t \max_{x\in[0,1]} |u_2(s,x) - v_2(s,x)|\, ds,
\end{aligned}
$$

for all $(t,x) \in (0,T] \times [0,1]$. Then

$$
\begin{aligned}
\max_{x\in[0,1]} |u_2(s,x) - v_2(s,x)| \quad &\leq \quad \epsilon T + \epsilon + \epsilon(\tau_1 + T + T^2) W e^{-c_1\tau_1} \\
&\quad + (W e^{-c_1\tau_1}\gamma\tau_2 e^{-c_2\tau_2} T + \mu_2) \int_0^t \max_{x\in[0,1]} |u_2(s,x) - v_2(s,x)|\, ds,
\end{aligned}
$$

for all $t \in (0,T]$. We now establish a crucial estimate for the difference between u_2 and v_2. Let us define

$$\phi(t) = \max_{x\in[0,1]} |u_2(t,x) - v_2(t,x)|.$$

This function $\phi \in C((0,T], \mathbb{R}_+)$ and satisfies the following integral inequality:

$$\phi(t) \leq \epsilon T + \epsilon + \epsilon(\tau_1 + T + T^2) W e^{-c_1\tau_1} + (W e^{-c_1\tau_1}\gamma\tau_2 e^{-c_2\tau_2} T + \mu_2) \int_0^t \phi(s) ds.$$

Applying the Gronwall inequality, we get

$$\begin{aligned}\phi(t) &\leq (\epsilon T+\epsilon+\epsilon(\tau_1+T+T^2)We^{-c_1\tau_1})e^{(We^{-c_1\tau_1}\gamma\tau_2e^{-c_2\tau_2}T+\mu_2)\int_0^t ds}\\ &\leq (\epsilon T+\epsilon+\epsilon(\tau_1+T+T^2)We^{-c_1\tau_1})e^{(We^{-c_1\tau_1}\gamma\tau_2e^{-c_2\tau_2}T+\mu_2)t}\\ &\leq \epsilon(T+1+(\tau_1+T+T^2)We^{-c_1\tau_1})e^{(We^{-c_1\tau_1}\gamma\tau_2e^{-c_2\tau_2}T+\mu_2)T}\\ &\leq k_1\epsilon \quad \text{for all } t\in(0,T],\end{aligned}$$

where the non negative constant k_1 is given by

$$k_1=(T+1+(\tau_1+T+T^2)We^{-c_1\tau_1})e^{(We^{-c_1\tau_1}\gamma\tau_2e^{-c_2\tau_2}T+\mu_2)T}.$$

This establishes our desired uniform bound

$$\max_{x\in[0,1]}|u_2(t,x)-v_2(t,x)|\leq k_1\epsilon,\quad \text{for all } t\in(0,T],$$

and consequently

$$|u_2(t,x)-v_2(t,x)|\leq k_1\epsilon,\quad \text{for all } (t,x)\in(0,T]\times[0,1].$$

Returning to our earlier inequality for $|u_3-v_3|$ and incorporating the bound we just established for $|u_2-v_2|$, we obtain

$$\begin{aligned}|u_3(t,x)-v_3(t,x)| &\leq \epsilon+\epsilon T+\epsilon\gamma\tau_2e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}\int_0^t\max_{y\in[0,1]}|u_2(s,y)-v_2(s,y)|\,ds\\ &\leq \epsilon+\epsilon T+\epsilon\gamma\tau_2e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}\int_0^t(\epsilon k_1)ds\\ &\leq \epsilon+\epsilon T+\epsilon\gamma\tau_2e^{-c_2\tau_2}+\epsilon\gamma e^{-c_2\tau_2}k_1t\\ &\leq \epsilon(1+T+\gamma\tau_2e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}k_1T)\\ &\leq \epsilon k_2,\quad \text{for all } (t,x)\in(0,T]\times[0,1],\end{aligned}$$

where the non negative constant k_2 is given by $k_2=(1+T+\gamma\tau_2e^{-c_2\tau_2}+\gamma e^{-c_2\tau_2}k_1T)$. Finally, we establish the corresponding bound for the first component:

$$|u_1(t,x)-v_1(t,x)|$$

$$\begin{aligned}&= \left|\phi_1(x)+bt-\int_0^t f(u_1(s,x),u_2(s,x),u_3(s,x))u_3(s,x)ds-\mu_1\int_0^t u_1(s,x)ds-v_1(t,x)\right|\\ &\leq \left|v_1(t,x)-v_1(0,x)-bt+\int_0^t(f(v_1(s,x),v_2(s,x),v_3(s,x))v_3(s,x)ds+\mu_1\int_0^t v_1(s,x))ds\right|\\ &\quad+\int_0^t|f(u_1(s,x),u_2(s,x),u_3(s,x))u_3(s,x)-f(v_1(s,x),v_2(s,x),v_3(s,x))v_3(s,x)|ds\\ &\quad+|\phi_1(x)-v_1(0,x)|+\mu_1\int_0^t|u_1(s,x)-v_1(s,x)|ds\\ &\leq \epsilon t+\epsilon+W\int_0^t|u_3(s,x)-v_3(s,x)|ds+\mu_1\int_0^t|u_1(s,x)-v_1(s,x)|ds\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon t + \epsilon + W \int_0^t k_2 \epsilon ds + \mu_1 \int_0^t |u_1(s,x) - v_1(s,x)| ds \\
&\leq \epsilon t + \epsilon + \epsilon W k_2 t + \mu_1 \int_0^t |u_1(s,x) - v_1(s,x)| ds \\
&\leq \epsilon T + \epsilon + \epsilon W k_2 T + \mu_1 \int_0^t |u_1(s,x) - v_1(s,x)| ds, \quad (t,x) \in (0,T] \times [0,1].
\end{aligned}$$

Applying the Gronwall inequality, we get

$$\begin{aligned}
|u_1(t,x) - v_1(t,x)| &\leq (\epsilon T + \epsilon + \epsilon W k_2 T) e^{\mu_1 \int_0^t ds} \\
&\leq (\epsilon T + \epsilon + \epsilon W k_2 T) e^{\mu_1 t} \\
&\leq \epsilon (T + 1 + W k_2 T) e^{\mu_1 T} \\
&\leq \epsilon k_3, \quad (t,x) \in (0,T] \times [0,1],
\end{aligned}$$

where the non negative constant k_3 is given by

$$k_3 = (T + 1 + W k_2 T) e^{\mu_1 T}.$$

Set

$$c = \max\{1, k_1, k_2, k_3\}.$$

Then we have

$$\begin{aligned}
|u_1(t,x) - v_1(t,x)| &\leq c\epsilon, \\
|u_2(t,x) - v_2(t,x)| &\leq c\epsilon, \\
|u_3(t,x) - v_3(t,x)| &\leq c\epsilon,
\end{aligned}$$

for all $(t,x) \in [-\tau, T] \times [0,1]$. This establishes that the system (1.1)-(1.3) is Ulam-Hyers stable according to Definition 4.1. This completes the proof of Theorem 1.2.

Example 4.3. Let us consider $(u_1, u_2, u_3) \mapsto \dfrac{\beta u_1}{1 + \delta_1 u_1 + \delta_2 u_3 + \delta_2 \alpha + \delta_2 |u_3 - \alpha|}$ the previously analyzed function in Example 3.2 and define

$$f(u_1, u_2, u_3) = \begin{cases} \dfrac{\beta u_1}{1 + \delta_1 u_1 + \delta_2 u_3 + \delta_2 \alpha + \delta_2 |u_3 - \alpha|} & \text{if } u_1, u_2, u_3 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\beta, \delta_1, \delta_2, \alpha \geq 0$ are constants. Recall that the function $(u_1, u_2, u_3) \mapsto f(u_1, u_2, u_3) u_3$ describes a novel extension of the classical Beddington-DeAngelis incidence function, which can be used to model the rate of new infections in hepatitis B. This function is Lipschitz continuous on $\mathbb{R}^3$, and if we assume that $|u_1 - v_1| \leq |u_3 - v_3|$, which means that the magnitude of change in the viral population u_3 is greater or equal to the magnitude change in the healthy cell population u_1, one can easily show that there exists a constant $W > 0$ such that

$$|f(u_1, u_2, u_3) u_3 - f(v_1, v_2, v_3) v_3| \leq W |u_3 - v_3|.$$

Since f satisfies the condition $(A1)$ and the Lipschitz type condition (1.4), Theorem 1.2 ensures that the initial-boundary value problem (1.1)–(1.3) is Ulam-Hyers stable on $[-\tau, T] \times [0,1]$ for any $T > 0$.

Conclusion

In this work, we investigate the existence of nonnegative bounded solutions for an initial-boundary value problem arising from a generalized hepatitis B virus (HBV) model. Unlike

previous studies requiring continuously differentiable incidence functions, we extend the analysis to the more general case of merely continuous functions. Our approach is based on the use of fixed-point theory for sums of operators. The key steps in proving the main result are as follows: first, we derive an integral representation of the solutions for the considered problem. Then, two suitable operators are constructed such that the fixed points of their sum correspond to solutions of the problem.

Furthermore, we establish Ulam-Hyers stability under sufficient conditions. This property is critical for real-world applicability, as it ensures that numerical approximations or parameter estimations remain reliable despite inherent uncertainties in clinical or epidemiological data. Such robustness bridges the gap between theoretical models and their practical use in public health decision-making.

Finally, our framework offers an adaptable tool for studying other complex epidemiological systems with non-smooth dynamics, opening new ways for mathematically rigorous yet biologically relevant modeling.

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