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Generalized Metric Spaces over Lattice-Ordered Monoids

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Abstract. This paper introduces a generalized framework for metric spaces in which the traditional real-valued distance function is replaced by a function valued in a **lattice-ordered monoid (LOM)**-an algebraic structure that simultaneously supports a partial order and a commutative monoidal operation. Within this setting, we develop the foundational theory of **LOM-metric spaces**, defining open balls, topologies, and key topological properties based purely on the internal structure of the underlying monoid. We establish sufficient algebraic conditions such as **residual closure**, **finite meet property**, **dense ordering**, and **sub-additivity**, under which classical separation axioms (T_0, T_1 and T_2) hold. Further, we explore generalized notions of boundedness, Cauchy sequences, and completeness. The robustness of this framework is demonstrated by proving generalized versions of several fundamental results of analysis, including a **Generalised Sandwich Theorem**, **Cantor's Intersection Theorem**, and the **Baire Category Theorem**. This unified, algebraically driven perspective not only subsumes the classical theory but also enables new applications in fuzzy systems, logic, domain theory, and non-classical models of computation.

1 Introduction

Metric spaces form the foundation for much of classical topology and analysis, defined by a real-valued function satisfying symmetry, non-negativity, the triangle inequality and identity of indiscernible. These spaces enable rigorous notions of continuity, compactness and convergence. However, in many mathematical and computational contexts such as fuzzy set theory, lattice theory and domain theory, real-valued metrics may not adequately capture the intrinsic structure of the space or application.

To address this limitation, we propose a generalization of metric spaces by replacing the codomain $\mathbb{R}^+ \cup \{0\}$ with a lattice-ordered monoid (LOM)-a structure that simultaneously supports a partial order (lattice) and a binary operation (monoid). This leads to the definition of LOM-metric spaces, in which distance functions are valued in such ordered algebraic structures.

In this paper, we undertake a systematic development of the topological and analytical theory of these spaces. Our primary objectives are to :

- Define and explore the basic structure of LOM-metric spaces.
- Introduce topological concepts (open balls, open sets) purely through the algebraic properties of the LOM.
- Investigate the classical separation axioms (T_0, T_1, and T_2) and establish their dependence on structural features of the LOM.
- Develop a comprehensive theory of convergence, Cauchy sequences, and completeness.

- Demonstrate the framework's power by proving generalized versions of fundamental analytical results, including the Generalised Sandwich Theorem, Cantor's Intersection Theorem, and the Baire Category Theorem.

Our treatment is self-contained and algebraic, highlighting how core topological properties emerge naturally from lattice and monoid operations. This framework not only generalizes classical metric spaces but also supports new analytic approaches in non-traditional settings such as fuzzy logic, information systems, and generalized computation.

For basic study related to metric space and topology one may see [10] and for that of background in lattice theory one may see [3].

2 Related Works

Classical metric spaces, as introduced in early 20th-century analysis, have long served as the standard framework for formalizing notions of distance, convergence, and continuity in topology and analysis. Over time, various generalizations have been developed to accommodate more abstract settings and applications.

Among these, b-metric spaces (Czerwik, 1993) [4] and modular metric spaces (Das et al., 2021) [5] relax the triangle inequality, enabling broader fixed-point results and applications in functional analysis. In **fuzzy metric spaces**, introduced by George and Veeramani (1994) [6], the concept of distance incorporates degrees of uncertainty using t-norms, while **probabilistic metric spaces** (Menger, 1942) [11] model distances using distribution functions to handle randomness.

More recent generalizations include metrics valued in **ordered structures** or **posets**, which have become important in **domain theory**, **computational semantics**, and **theoretical computer science**. For instance, the work of Reich and Zaslavski (2022) [14] and Guran et al. (2020) [8] explores fixed point theorems in generalized and non-linear metric spaces. Petrov et al. (2024) [12] extend this further to semi-metric structures defined over graphs and triangle functions.

A particularly relevant line of research is that of **Ashale, Abebe, and Venkateswarlu (2024)** [1], who introduced *autometrized lattice-ordered monoids (AL-monoids)*, investigating their internal algebraic structure and metric ideals. However, their focus remains largely on algebraic properties rather than topological development.

Other notable generalization replace the standard real line with alternative algebraic structures, such as vector valued metric spaces ([13]) and hyperbolic-valued metric spaces ([9])

Despite these efforts, a **complete and unified topological theory** of metric spaces valued in **lattice-ordered monoids** including definitions of open and closed sets, convergence, boundedness, and separation axioms rooted in algebraic properties has not been fully developed. This paper fills that gap by formulating a self-contained and general framework that extends classical metric theory to an algebraically enriched setting.

Our work not only generalizes existing metric frameworks but also introduces new concepts such as **monoid-scaled boundedness**, **order-coherent convergence**, and a **generalised sandwich theorem**, thereby significantly broadening the theoretical landscape and opening new avenues for application.

3 Basic Definitions and Examples

In this section, we formally define the Lattice-Ordered Monoid (LOM) and the concept of a metric valued in it. We then provide several illustrative examples, showing how this framework not only recovers classical metric spaces but also extends to fundamentally different structures like Boolean algebras and power sets.

Definition 3.1 (Lattice-Ordered Monoid). A **lattice-ordered monoid (LOM)** is a triple $(M, \oplus, \preccurlyeq)$ such that

(i) $(M, \oplus)$ is a **commutative monoid** with identity element $O_M \in M$,

(ii) $(M, \preccurlyeq)$ is a **lattice**, i.e., for all $a, b \in M$ the **join** $a \vee b$ and the **meet** $a \wedge b$ exist in M,

(iii) The operation $\oplus$ is **monotone** with respect to $\preccurlyeq$:

$$\forall\, a,\ b,\ c,\ d \in M,\ a \preccurlyeq b,\ \ c \preccurlyeq d \Rightarrow a \oplus c \preccurlyeq b \oplus d.$$

Definition 3.2 (Metric over a Lattice-Ordered Monoid). Let $(M, \oplus, \preccurlyeq)$ be a lattice-ordered monoid, and let X be a non-empty set. A function $d : X \times X \to M$is called a **metric over M** if, for all $x, y, z \in X$ the following hold:

(i) (**Non-negativity and Identity**) $O_M \preccurlyeq d\,(x,\ y)$ and $d\,(x,\ y) = O_M \Longleftrightarrow x = y$.

(ii) (**Symmetry**)

$$d\,(x,\ y) = d\,(y,\ x).$$

(iii) (**Triangle Inequality**)

$$d\,(x,\ z) \preccurlyeq d\,(x,\ y) \oplus d\,(y,\ z)$$

A triple $(X,\ d,\ M)$ satisfying the above is called a **metric space over the lattice-ordered monoid$M$**.

Example 3.3. [Classical Metric Space] Let $M = [0, \infty),\ \oplus = +$ (the usual addition) and $\preccurlyeq = \leq$ (the usual order).

(i) $(M, +, \leq)$is a commutative monoid.

(ii) $(M, \leq)$ is a complete lattice.

(iii) Addition is monotone with respect to $\leq$.

Then a function $d : X \times X \to [0, \infty)$ satisfying the usual metric axioms is a **metric over the LOM** $([0, \infty), +, \preccurlyeq)$. So, classical metric spaces are special cases of LOM-metric spaces.

Example 3.4 (Discrete Metric over a Boolean Algebra). Let $M = \{0, 1\}$, with $\oplus = \vee$(logical OR), and $\preccurlyeq$ is the usual order $0 \preceq 1$.

- $(M, \vee)$ is a commutative monoid with identity 0.
- $(M, \preccurlyeq)$ is a complete ordered lattice.
- $\vee$ is monotone with respect to $\preccurlyeq$.

Define:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Then (X, d, M) is a discrete metric space over a Boolean algebra.

Example 3.5 (Symmetric Difference Metric over the power set). Let X be any non-empty set. Define:

(i) $M = P(X)$, the power set of X,

(ii) $\oplus = \cup$ (set union),

(iii) $\preccurlyeq = \subseteq$ (set inclusion),

(iv) Identity element $O_M = \emptyset$.

Then:

(i) $(P(X),\ \cup)$ is a **commutative monoid** with identity $\emptyset$,

(ii) $(P(X),\ \subseteq)$ is a **complete lattice**,

(iii) Union is **monotone** with respect to inclusion:

$$A \subseteq B \Rightarrow A \cup C \subseteq B \cup C.$$

So $(P(X), \cup, \subseteq)$ is a **lattice-ordered monoid**.
Let us define the metric on $P(X)$ as:

$$d(A, B) = A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Then:

(i) $d(A, A) = A \triangle A = \emptyset$,

(ii) $d(A, B) = d(B, A)$,

(iii) $d(A, C) \subseteq d(A, B) \cup d(B, C)$, since symmetric difference satisfies a triangle-like inequality.

So $(P(X), d, P(X))$ is a metric space over the LOM $(P(X), \cup, \subseteq)$.

4 Open balls and open sets in the LOM metric space

With the foundational definitions of a LOM-metric space established, we now proceed to construct its intrinsic topology. Following the classical approach, our construction begins with the most fundamental topological building blocks: open and closed balls. In our generalised setting, these are defined not by real-valued radii, but by elements from the monoid M and the strict partial order $\prec$. This section will then use the concept of an open ball to define open sets and explore the crucial algebraic properties of the LOM, namely the **finite meet property** and **residual closure**, that ensure this collection of open sets forms a well-behaved topology. Furthermore, we will investigate various notions of boundedness that arise naturally from the structure of the monoid.

Definition 4.1 (**Closed Ball in a LOM-Metric Space**). Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ and let $\varepsilon \in M$ satisfy $O_M \prec \varepsilon$. Then, for any point $x \in X$, the **closed ball centred at** x **with radius** ε is defined as:

$$B[x, \varepsilon] := \{y \in X : d(x, y) \preccurlyeq \varepsilon\}.$$

Definition 4.2 (**Open Ball in a LOM-Metric Space**). Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$, and let $\varepsilon \in M$ satisfy $O_M \prec \varepsilon$. Then, for any point $x \in X$, the **open ball centred at** x **with radius** ε is defined as:
$B(x, \varepsilon) := \{y \in X : d(x, y) \prec \varepsilon\}$.

Remark 4.3. Since we don't have real-valued distances, we can't talk about "smaller than a number" - instead, we use the **strict order** $\prec$ induced by the partial order $\preccurlyeq$.

Example 4.4 (Open ball in a Symmetric Difference Metric over $P(S)$). Let us define a metric space where:

(i) $S = \{a, b, c\}$

(ii) $X = P(S)$, all subsets of S

(iii) $M = P(S)$

$\oplus = \cup, \preccurlyeq = \subseteq, O_M = \emptyset$
We define a metric $d : X \times X \to M$ as follows: if $A, B \subseteq S$, then $d(A, B) = A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Then (X, d, M) is a metric space over the LOM $(P(S), \cup, \subseteq)$.
Open Ball Example:
Take centre $x = \{a\}$, radius $\varepsilon = \{a, b\}$. Then:

$$B(x, \varepsilon) = \{y \subseteq S : d(x, y) \subsetneq \{a, b\}\}$$

Consists of all y such that symmetric difference with $\{a\}$ are proper subsets of $\{a, b\}$.

(i) $y = \{a\}$, (diff = $\emptyset$)

(ii) $y = \emptyset$, (diff = $\{a\}$)

(iii) $y = \{b\}$, (diff = $\{a, b\}$ — excluded)

(iv) $y = \{a, b\}$, (diff = $\{b\}$) — included

Therefore,

$$B(\{a\},\ \{a,\ b\}) \ =\ \{\emptyset,\ \{a\},\ \{a,\ b\}\}.$$

Definition 4.5 (**Open Set in a LOM-Metric Space**). Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A subset $U \subseteq X$ is said to be **open** if for every point $x \in U$, there exists some $\varepsilon \in M$ with $O_M \prec \varepsilon$ such that

$$B(x, \varepsilon) \subseteq U.$$

Proposition 4.6. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. Let $m_1, m_2 \in M$ such that $m_1 \preccurlyeq m_2$. Then for any $x \in X$, $B(x, m_1) \subseteq B(x, m_2)$.*

Proof. Let $y \in B(x, m_1)$. Then

$$d\,(x, y) \prec m_1 \tag{4.1}$$

and

$$m_1 \preccurlyeq m_2. \tag{4.2}$$

Since $(M, \oplus)$ is a lattice, so it is a partially ordered set and hence the relation $\preceq$ is transitive.

Hence from (4.1) and (4.2) we get

$$d\,(x,\ y) \prec m_2$$

$$\Rightarrow y\ \in B(x,\ m_2).$$

Therefore $B(x,\ m_1)\ \subseteq B(x,\ m_2)$. □

Definition 4.7 (**Finite Meet Property**). A lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ is said to have the *finite meet property* if, for any finite collection of elements $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\} \subseteq M$ with $O_M \prec \varepsilon_i$ for all $i = 1, 2, \ldots, k$, their meet satisfies $O_M \prec \bigwedge_{i=1}^{k} \varepsilon_i$.

Theorem 4.8 (**Open Sets Form a Topology**). *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ that satisfies the finite meet property. The collection of open sets in X, forms a topology. Specifically:*

(i) The empty set ϕ and the entire space X are open.

(ii) The union of any collection of open sets is open.

(iii) The intersection of finitely many open sets is open.

Proof. (i) **Empty Set and Entire Space**:

a. The empty set ϕ is open, as there are no points $x \in \phi$ requiring an open ball to be contained within it.

b. The entire space X is open, since for any $x \in X$ and any $\varepsilon \in M$ with $O_M \prec \varepsilon$, the open ball $B(x, \varepsilon) = \{y \in X \mid d(x, y) \prec \varepsilon\}$ satisfies $B(x, \varepsilon) \subseteq X$.

(ii) **Arbitrary Unions**: Let $\{U_i\}_{\mathrm{i}\in\mathrm{I}}$ be a collection of open sets and define $U = \bigcup_{\mathrm{i}\in\mathrm{I}} U_i$. For any $x \in U$, there exists some i∈I such that $x \in U_i$. Since U_i is open, there exists $\varepsilon \in M$ with $O_M \prec \varepsilon$ such that $B(x, \varepsilon) \subseteq U_i$. Since $U_i \subseteq U$, it follows that $B(x, \varepsilon) \subseteq U$. Thus, U is open.

(iii) **Finite Intersections**: Let $U_1, U_2, \ldots, U_k$ be a finite collection of open sets, and let $U = \bigcap_{i=1}^{k} U_i$. For any $x \in U$, since each U_i is open, there exists $\varepsilon_i \in M$ with $O_M \prec \varepsilon_i$ such that $B(x, \varepsilon_i) \subseteq U_i$ for each $i = 1, 2, \ldots, k$. Since $(M, \oplus, \preccurlyeq)$ is a lattice, the meet $\varepsilon = \bigwedge_{i=1}^{k} \varepsilon_i$ exists in M. By the finite meet property, since $O_M \prec \varepsilon_i$, for each $i = 1, 2, \ldots, k$ it follows that $O_M \prec \varepsilon = \bigwedge_{i=1}^{k} \varepsilon_i$. By Proposition 4.1, for any $m_1, m_2 \in M$ with $m_1 \preccurlyeq m_2$, we have $B(x, m_1) \subseteq B(x, m_2)$. Since $\varepsilon \preccurlyeq \varepsilon_i$ for each i, it follows that $B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq U_i$ for each $i = 1, 2, \ldots, k$. Thus, $B(x, \varepsilon) \subseteq \bigcap_{i=1}^{k} U_i = U$. Since $O_M \prec \varepsilon$, the open ball $B(x, \varepsilon)$ satisfies the condition for U to be open at x. Therefore, U is open.

□

Definition 4.9. A LOM $(M, \oplus, \preccurlyeq)$ is said to be residually closed if for any $a, b \in M$ with $a \prec b$, there exists $\delta \in M$ such that $O_M \prec \delta$ and $a \oplus \delta = b$.

Example 4.10. (i) $(\mathbb{R}, +, \leq)$, $([0, \infty), +, \leq)$ are residually closed LOM.

(ii) Let X be any non-empty set. Then the LOM (P(X), ∪, ⊆) is residually closed.

Theorem 4.11. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If $(M, \oplus, \preccurlyeq)$ is residually closed then every open ball in (X, d, M) is an open set in (X, d, M).*

Proof. Let $B(x, m)$ be an open ball centred at x and radius m, where $O_M \prec m$.
Now let $y \in B(x, m)$. Then either $y = x$ or $y \neq x$.
If $y = x$ then in this case $B(x, m) \subseteq B(x, m)$.
If $y \neq x$ then $O_M \prec d(x, y) \prec m$.
Since $d(x, y) \prec m$ and the LOM $(M, \oplus, \preccurlyeq)$ is residually closed, there exists $\delta \in M$ such that $O_M \prec \delta$ and $d(x, y) \oplus \delta = m$.
Now consider the open ball $B(y, \delta) := \{z \in X \mid d(y, z) \prec \delta\}$.
Now let $z \in B(y, \delta)$ be any point. Then by triangle inequality we have
$d(x, z) \preccurlyeq d(x, y) \oplus d(y, z) \prec d(x, y) \oplus \delta = m$, since $\oplus$ is **monotone** with respect to $\preccurlyeq$.
This implies that $d(x, z) \prec m$.
Therefore $z \in B(x, m)$. So $B(y, \delta) \subseteq B(x, m)$. Hence for every $y \in B(x, m)$, there exists an open ball $B(y, \delta)$ such that $y \in B(y, \delta) \subseteq B(x, m)$ and therefore, every open ball is an open set. □

Corollary 4.12. *Since $(\mathbb{R}, +, \leq)$ is residually closed, so in any classical metric space over $\mathbb{R}$, every open ball is automatically an open set.*

Example 4.13. The converse of the above theorem may not be true as we can see from the following example: Let $M = \{0, a, b\}$ be equipped with the order $0 \prec a \prec b$. Define the binary operation $\oplus$ on M as:

$$x \oplus y = \mathbf{min}(x, y).$$

That is, the operation returns the meet of the two elements in the lattice. Then:

(i) $(M, \oplus)$ is a commutative monoid with identity element b,

(ii) $(M, \preccurlyeq)$ is a lattice,

(iii) The operation $\oplus = min$ is monotone with respect to $\preccurlyeq$.

Thus, $(M, \oplus, \preccurlyeq)$ is a valid lattice-ordered monoid (LOM). However, this LOM is **not residually closed**.
To see this, take $a \prec b$. According to the definition of residual closure, there must exist some $\delta \in M$ with $0 \prec \delta$ such that $a \oplus \delta = b$.
But since $\oplus = min$, we have $a \oplus \delta = \min(a, \delta) \preccurlyeq a \prec b$. So $a \oplus \delta \neq b$ for any $\delta \in M$.
Hence, no such δ exists, and residual closure fails.
Now consider the set $X = \{x_1, x_2, x_3\}$, and define a distance function $d : X \times X \to M$ by:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ a, & \text{if } x \neq y. \end{cases}$$

Then it is easy to verify that d satisfies the axioms of a metric over M.
Now we examine the open balls.
Since $0 \prec a \prec b$, the radius of every open ball is either a or b.
Open Ball $B(x, b)$:
Let $x \in X$ and radius $\varepsilon = b$. Then:

$$B(x, b) = \{y \in X \mid d(x, y) \prec b\}.$$

Since $d(x, y) \in \{0, a\}$, and both $0, a \prec b$, it follows that:

$$B(x, b) = X$$

which is trivially open.
Open Ball $B(x, a)$:
Now let ε=a. Then $B(x, a) = \{y \in X : d(x, y) \prec a\}$.
Here $d(x, x) = 0 \prec a,\ d(x, y) = a\ for\ y \neq x$.
So $B(x, a) = \{x\}$.
This singleton set is open because for every point $x \in B(x, a)$, there exists ε=a$\succ$0 such that $B(x, \varepsilon) \subseteq \{x\}$.
In this example, every open ball is an open set, even though the underlying LOM $(M, \oplus, \preccurlyeq)$ is not residually closed. Therefore, the converse of the theorem does not hold in general.

Definition 4.14. Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A subset B of X is said to be bounded if there exist $\mu \in M$ with $O_M \prec \mu$ such that $d(x, y) \preccurlyeq \mu$ for all $x, y \in B$.

Theorem 4.15. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If a subset B of X is bounded, that is, there exists $\mu \in M$ with $O_M \prec \mu$ such that $d(x, y) \preccurlyeq \mu$ for all $x, y \in B$ and*

(i) if μ is a maximal element of M, then B is contained in a closed ball in X.

(ii) if μ is not a maximal element of M, then B is contained in an open ball in X.

Proof. Let B be bounded. Then there exists $\mu \in M$ with $O_M \prec \mu$ such that $d(x, y) \preccurlyeq \mu$ for all $x, y \in B$.
Case 1: Let μ be a maximal element in M.
For fixed $x_0 \in B$ and for any $y \in B$, $d(x_0, y) \preccurlyeq \mu$.
Consider the closed ball $B[x_0, \mu] = \{z \in X : d(x_0, z) \preccurlyeq \mu\}$. Then since $d(x_0, y) \preccurlyeq \mu$, $y \in B[x_0, \mu]$. Since $y \in B$ is arbitrary, $B \subseteq B[x_0, \mu]$. Therefore, B is contained in a closed ball in X.
Case 2: Let μ be not a maximal element in M.
Then there exists at least one element μ' in M such that $\mu \prec \mu'$ otherwise μ will be a maximal element of M. For fixed $x_0 \in B$ and for any $y \in B$, $d(x_0, y) \preccurlyeq \mu$. Consider the open ball $B(x_0, \mu') = \{z \in X : d(x_0, z) \prec \mu'\}$. Since $d(x_0, y) \preccurlyeq \mu \prec \mu'$, $y \in B(x_0, \mu')$.
Since $y \in B$ is arbitrary, $B \subseteq B(x_0, \mu')$. Therefore, B is contained in an open ball in X. □

Definition 4.16 (**Monoid-Scaled Boundedness**). Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A subset $B \subseteq X$ is monoid-scaled bounded if there exists an element $\mu \in M$ with $O_M \prec \mu$ such that for every $x, y \in B$, there exists a positive integer k (depending on x, y) satisfying: $d(x, y) \preccurlyeq \mu \oplus \mu \oplus ... \oplus \mu$ (k times).
Here, μ is a scaling element and k is the scaling index for the pair (x, y).

Note: It is clear from the above definition that if a set is bounded, then it is scaled bounded, but a scaled bounded set may not be bounded. For example, consider the set $\{1,\ 1+\frac{1}{2},\ ,\ 1+\frac{1}{2}+\frac{1}{3}, \dots\}$. This set is not bounded in a classical metric space but is a scaled bounded with a scaling element 1.

Theorem 4.17. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ that is residually closed and satisfies the finite meet property. If a subset $B \subseteq X$ is monoid-scaled bounded with scaling element $\mu \in M$ and if the scaling index for the pair $x, y \in B$ is finite then:*

(i) If μ is a maximal element in M then B is contained in a closed ball.

(ii) If μ is not a maximal element in M and M is densely ordered then B is contained in an open ball.

Proof. Let $B \subseteq X$ be monoid-scaled bounded with scaling element $\mu \in M$. Then for all $x, y \in B$, there exists a $k_{(x,y)} \in \mathbb{N}$ such that: $d(x,\ y) \preccurlyeq \mu \oplus \mu \oplus \ ... \ \oplus \mu$ ($k_{(x,y)}$ times).
Let $K = \max\{k_{(x,y)} : x, y \in B\}$. Then K is finite since the scaling index for the pair $x,\ y \in B$ is finite. Now let $\nu = \mu \oplus \mu \oplus \ ... \ \oplus \mu$ (K times). Then since $\oplus$ is monotone and $O_M \prec \mu$, $O_M \prec \mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu$ (K times)$= \nu$.
Case 1: Let μ be a maximal element in M.
Since $\oplus$ is monotone, $\mu \preccurlyeq \mu \oplus \mu \oplus \ ... \ \oplus \mu$ (K times) $= \nu$. And since μ is a maximal element in M, $\mu = \nu$. For fixed $x_0 \in B$ and for any $y \in B$, $d(x_0, y) \preceq \mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu$ ($k_{(x_0,y)}$ times). Again, $\mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu$ ($k_{(x_0,y)}$) $\preccurlyeq \ \mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu$ (K times), since $\oplus$ is monotone and $k_{(x_0,y)} \leq K$. Therefore, $\mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu$ ($k_{(x_0,y)}$ times) $\ \preceq \nu = \mu$. Since the partial order $\preccurlyeq$ is transitive, $d(x_0, y) \preceq \mu$. Consider the closed ball $B\,[x_0, \mu] = \{z \in X :\ d(x_0, z) \preceq \mu\}$. Then since $d(x_0, y) \preceq \mu$, $y \in B\,[x_0, \mu]$. Since $y \in B$ is arbitrary, $B \subseteq B\,[x_0, \mu]$. Hence, B is contained in a closed ball in X.
Case 2: Let μ be not a maximal element in M.
Then there exists at least one element μ' in M such that $\mu \prec \mu'$ otherwise μ will be a maximal element of M. For fixed $x_0 \in B$ and for any $y \in B$, $d(x_0, y) \preceq \mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu$ ($k_{(x_0,y)}$ times). Again, $\mu \oplus \mu \oplus ... \oplus \mu$ ($k_{(x_0,y)}$ times) $\prec \ \mu' \oplus \mu' \oplus ... \oplus \mu'$ ($k_{(x_0,y)}$ times), since $\oplus$ is monotone. So, $\mu \oplus \mu \oplus ... \oplus \mu$ ($k_{(x_0,y)}$ times)$\prec \mu' \oplus \mu' \oplus ... \oplus \mu' = \upsilon'$ (K times), for some $\upsilon' \in M$. Since $\oplus$ is monotone and $k_{(x_0,y)} \leq K$. Therefore, $\mu \ \oplus \ \mu \ \oplus \ ... \ \oplus \ \mu\ (k_{(x_0,y)}) \prec \upsilon'$. Since the partial order $\preccurlyeq$ is transitive, $d(x_0, y) \prec \upsilon'$. Consider the open ball $B(x_0, \upsilon') = \{z \in X :\ d(x_0, z) \prec \upsilon'\}$. Then $d(x_0, y) \prec \upsilon'$, $y \in B(x_0, \upsilon')$. Since $y \in B$ is arbitrary, $B \subseteq B(x_0, \upsilon')$. Therefore, B is contained in an open ball in X. □

5 Neighbourhood, Limit Point, Interior and Closure

Building upon the topology of open sets established in Section 4, we now formally define a collection of essential topological concepts that allow for a deeper analysis of the space's structure. This section introduces the notions of a **neighbourhood** of a point, which provides a local perspective, and the **limit point** of a set, which is crucial for understanding convergence and accumulation. These concepts lead directly to the definitions of **closed sets**, the **closure** of a set, and **dense sets**. Our focus remains on demonstrating how these familiar topological ideas are characterized within the LOM-metric framework and how their properties are intimately linked to the algebraic features of the underlying monoid.

Definition 5.1 (**Neighbourhood of a Point**). Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A subset U of (X, d, M) is said to be a neighbourhood of a point $x \in X$ if there exists an r with $O_M \prec r$ such that $B(x, r) \subseteq U$.

Example 5.2. In a metric space (X, d, M) over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$, every open ball $B(x, r)$ is a neighbourhood of each of its points.

Theorem 5.3. *A set U in a metric space (X, d, M) over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ is open iff U is a neighbourhood of each point of it.*

Proof. It directly follows from the definitions of open sets and neighbourhood. □

Theorem 5.4. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ and $x \in X$. Then*

(i) x has at least one neighbourhood.

(ii) If U is a neighbourhood of x and $U \subseteq V$, then V is a neighbourhood of x.

(iii) If the lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ has finite meet property and U, V be neighbourhoods of x then $U \cap V$ is a neighbourhood of x i.e. intersection of two (and hence finitely many) neighbourhoods of x is a neighbourhood of x.

Proof. (i) Clearly X is a neighbourhoods of x.

(ii) Since U is a neighbourhood of x, there exist $O_M \prec r$ such that $x \in B(x,r) \subseteq U$. Then $x \in B(x,r) \subseteq U \subseteq V$ and hence V is a neighbourhood of x.

(iii) Since U and V are neighbourhoods of x, there exist $O_M \prec r_1$ and $O_M \prec r_2$ such that $x \in B(x,r_1) \subseteq U$ and $x \in B(x,r_2) \subseteq V$. Since the lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ has finite meet property, $O_M \prec r_1$ and $O_M \prec r_2$ implies $O_M \prec r_1 \wedge r_2$. Let $r_1 \wedge r_2 = r$. Then $r \preccurlyeq r_1$ and $r \preccurlyeq r_2$, and by Proposition 1, $B(x,\ r) \subseteq B(x,\ r_1)$ and $B(x,\ r) \subseteq B(x,\ r_2)$. Therefore, $x \in B(x,\ r) \subseteq B(x,\ r_1) \subseteq U$ and $x \in B(x,\ r) \subseteq B(x,\ r_2) \subseteq V$. So, $x \in B(x,\ r) \subseteq U \cap V$ and hence $U \cap V$ is a neighbourhood of x.

□

Definition 5.5 (**Limit Point, Isolated Point and Derived Set**). Let (X,d,M) be a metric space over a lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $x \in X$. Let A be any subset of X.

(i) The point x is said to be limit point of A if every neighbourhood of x contains a point $y \neq x$ such that $y \in A$.

(ii) The point x is called isolated point of A if $x \in A$ and x is not a limit point of A.

(iii) The set of all limit points (if any) of A is called the derived set of A and is denoted by A'.

Definition 5.6 (**Closed set**). Let (X,d,M) be a metric space over a lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $A \subseteq X$. The set A is said to be closed if its complement $X \setminus A$ is open.

Theorem 5.7. *Let $(X,\ d,\ M)$ be a metric space over a lattice-ordered monoid $(M,\ \oplus,\ \preccurlyeq)$. Then:*

(i) Arbitrary intersection of closed sets in X is closed.

(ii) Finite union of closed sets in X is closed if the lattice-ordered monoid $(M,\ \oplus,\ \preccurlyeq)$ has finite meet property.

Proof.

(i) Let $\{F_\alpha\}_{\alpha\in\wedge}$ be an arbitrary family of closed sets in X. Let $F = \bigcap_{\alpha\in\wedge} F_\alpha$.

Now, $X\diagdown F = X\diagdown\bigcap_{\alpha\in\wedge} F_\alpha = \bigcup_{\alpha\in\wedge}(X\diagdown F_\alpha)$. Since F_α is closed for each $\alpha \in \wedge$, $X\diagdown F_\alpha$ is open for each $\alpha \in \wedge$. We know arbitrary union of open sets in X is open, so $X\diagdown F$ is open. Hence F is closed in X.

(ii) Let $\{F_1, F_2, \cdots, F_n\}$ be a finite family of closed sets. Then $X\diagdown F_i$ is open for $i = 1,2,\ldots,n$. Let $G_i = X\diagdown F_i\ ,\ i = 1,2,\ldots\ldots,n$. Then $\bigcup_{i=1}^n F_i = \bigcup_{i=1}^n (X\diagdown G_i) = X\diagdown\bigcap_{i=1}^n G_i$. Since the lattice-ordered monoid $(M,\ \oplus,\ \preccurlyeq)$ has finite meet property, $\bigcap_{i=1}^n G_i$ is open in X. Hence, $X\diagdown\bigcap_{i=1}^n G_i$ is closed in X i.e., $\bigcup_{i=1}^n F_i$ is closed.

□

Theorem 5.8. *Let (X,d,M) be a metric space over a lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $A \subseteq X$. Then A is closed if and only if A contains all of its limit points i.e., $A' \subseteq A$.*

Proof. Let A be closed. We shall prove that $A' \subseteq A$. Let $x \in A'$.
If $x \in A$, then there is nothing to prove. If $x \notin A$, then $x \in X\diagdown A$. Since A is closed $X\diagdown A$ is open. Therefore, $X\diagdown A$ is a neighbourhood of x which contains no points of A other than x which is a contradiction to the fact that $x \in A'$. So, $x \in A$, thus $A' \subseteq A$.
Conversely, assume that $A' \subseteq A$. We shall prove that A is closed. Let $x \in X\diagdown A$. Then $x \notin A,\ x \notin A'$. So, there exists a neighbourhood N of x which does not contain any elements of A i.e., $x \in N \subseteq X\diagdown A$. Therefore, $X\diagdown A$ is open and hence A is closed. □

Definition 5.9. A monoid $(M,\oplus)$ is said to satisfy cancellation law if for all a,b,c in M $a\oplus b = a\oplus c \Longrightarrow b = c$ and $b\oplus a = c\oplus a \Longrightarrow b = c$.

Theorem 5.10. *Let $(X,\ d,\ M)$ be a metric space over a lattice-ordered monoid $(M,\ \oplus,\ \preccurlyeq)$. If $(M,\ \oplus,\ \preccurlyeq)$ is residually closed and satisfies cancellation law then each closed ball in X is a closed set.*

Proof. Let $B[x,\varepsilon]$ be a closed ball in $(X,\ d,\ M)$ with centre at $x \in X$ and radius $\varepsilon \in M$ with $O_M \prec \varepsilon$. We have to prove $B[x,\ \varepsilon]$ is a closed set. It is sufficient to prove that $X \diagdown B[x,\varepsilon]$ is open. Let $y \in X \diagdown B[x,\varepsilon]$. Then $y \notin B[x,\varepsilon]$ and therefore $d(x,y) \succ \epsilon$. Since $(M,\oplus,\preccurlyeq)$ is residually closed, there exists $\delta \in M$ such that $O_M \prec \delta$ and $d(x,y) \oplus \delta = \epsilon$. We now consider the open ball $B(y,\delta)$. Let $z \in B(y,\delta)$, then $d(y,z) \prec \delta$. Then by triangle inequality we have $d(x,z) \oplus d(z,y) \succcurlyeq d(x,y)$ i.e., $d(x,z) \oplus \delta \succcurlyeq d(x,y) \succ \epsilon = d(x,y) \oplus \delta \Longrightarrow d(x,z) \succ d(x,y)$ [by cancellation law]. Therefore, $d(x,z) \succ \epsilon$ [because $d(x,y) \succ \epsilon$].
$\therefore z \notin B[x,\epsilon]$ i.e., $z \in X \diagdown B[x,\epsilon]$. So, $B(y,\delta) \subseteq X \diagdown B[x,\epsilon]$. Also, $y \in B(y,\delta) \subseteq X \diagdown B[x,\epsilon]$. This implies that $X \diagdown B[x,\epsilon]$ is open and hence $B[x,\epsilon]$ is closed. □

Definition 5.11 (**Closure of a Set**). Let (X,d,M) be a metric space over a lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $A \subseteq X$. The closure of A, denoted by $\overline{A}$, is the union of A and the set of all limit points of A i.e., $\overline{A} = A \cup A'$.

Theorem 5.12. *Let (X,d,M) be a metric space over a lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $A \subseteq X$. Then:*

(i) *$\overline{A}$ is a closed set.*

(ii) *A is closed if and only if $A = \overline{A}$.*

(iii) *$\overline{A}$ is the smallest closed subset of X containing A.*

(iv) *$\overline{A}$ is the intersection of all closed subsets of X containing A.*

Proof.

(i) To show $\overline{A}$ is a closed, we show that $X \diagdown \overline{A}$ is open. Let $x \in X \diagdown \overline{A}$. Then $x \notin \overline{A}$. So, $x \notin A$ and $x \notin A'$. Since x is not a limit point of A and $x \notin A$, there exists a neighbourhood N of x such that $N \cap A = \phi$.

If $N \cap A' \neq \phi$, let $z \in N \cap A'$. N is a neighbourhood of z and z is a limit point of A. Then there exists $y \in A$ such that $y \in N$ with $y \neq z$ which is a contradiction to the fact that $N \cap A = \phi$. Therefore, $N \cap A' = \phi$. This implies that $N \cap (A \cup A') = \phi$ i.e., $N \cap \overline{A} = \phi$. Therefore $x \in N \subseteq X \diagdown \overline{A}$ and hence $X \diagdown \overline{A}$ is open and so $\overline{A}$ is closed.

(ii) Let A be closed. Then $A' \subseteq A$. Also, $\overline{A} = A \cup A' \subseteq A$ and hence $A = \overline{A}$.

Conversely, let $A = \overline{A}$. By (1) $\overline{A}$ is a closed and so A is closed.

(iii) Let B be any closed subset of X with $A \subseteq B$. Let $x \in \overline{A}$. If $x \in A$, then $x \in B$.

If $x \notin A$, it is a limit point of A. Then any neighbourhood N of x contains a point $y \in A$ such that $y \neq x$. But $y \in B$ and therefore x is a limit point of B. Since B is closed so $x \in B$ and thus $\overline{A} \subseteq B$. By (1), $\overline{A}$ is closed. Thus $\overline{A}$ is the smallest closed subset of X containing A.

(iv) Let $S = \bigcap \{B \subseteq X : B \text{ is closed and } B \supseteq A\}$. Since arbitrary intersection of closed sets is closed, S is closed. Clearly S is the smallest closed set containing A. By (3) $\overline{A} = S$.

□

Definition 5.13 (**Dense Set**). Let (X,d,M) be a metric space over a lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $A \subseteq X$. A is said to be dense in X if $\overline{A} = X$. The set A is called dense in itself if $A \subseteq A'$.

Theorem 5.14. *Let (X,d,M) be a metric space over a residually closed lattice-ordered monoid $(M,\oplus,\preccurlyeq)$ and $A \subseteq X$. Then the following are equivalent:*

(i) *A is dense in X.*

(ii) *F is closed in X and $A \subseteq F \Longrightarrow F = X$.*

(iii) *Every non empty open set in X intersects A.*

Proof. (1) $\Longrightarrow$ (2)
$A \subseteq F \Longrightarrow \overline{A} \subseteq \overline{F}$. Since $\overline{A} = X$ and $\overline{F} = F$, this implies that $X \subseteq F$. Therefore, $F = X$.
(2) $\Longrightarrow$ (3)

If G is a nonempty open set such that $G \cap A = \phi$, then $A \subseteq X \setminus G$. Since G is open, then $X \setminus G$ is a closed set. Then by (2) $X \setminus G = X$ i.e., $G = \phi$, a contradiction.
$(3) \Longrightarrow (1)$
Let $x \in X$ and N be any neighbourhood of x. Then there exists $r \in M$ with $O_M \prec r$ such that $x \in B(x, r) \subseteq N$. Since $(M, \oplus, \preccurlyeq)$ is residually closed, $B(x, r)$ is open in (X, d, M). Then by (3) $B(x, r)$ intersect A and so N intersect A i.e., every neighbourhood of x intersect A. Hence, x is adherent point of A i.e., $x \in \overline{A}$. This implies that $\overline{A} = X$ and hence A is dense in X. □

6 Separation Axioms

Having established the foundational topology of LOM-metric spaces, we now turn our attention to the **separation axioms**, which provide a crucial classification of topological spaces based on how effectively distinct points can be separated by open sets. This section examines the classical Kolmogorov (T_0), Fréchet (T_1) and Hausdorff (T_2) properties. A central thesis of this paper is that many familiar topological separation axioms can be explained not by the specific form of the metric d, but by the algebraic properties of the monoid M that lies behind it. We will establish a clear correspondence, showing that properties such as **dense ordering**, **residual closure**, and **sub-additivity** are sufficient algebraic conditions for a LOM-metric space to be T_0, T_1, and T_2 respectively.

Definition 6.1. A LOM $(M, \oplus, \preccurlyeq)$ is said to be densely ordered if for any $a, b \in M$ with $a \prec b$, there exists $\delta \in M$ such that $a \prec \delta \prec b$.

Theorem 6.2 (**T_0 or Kolmogorov**). *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If $(M, \oplus, \preccurlyeq)$ is densely ordered, then for every pair of distinct points x, y in X, there exists an open ball containing one but not the other point.*

Proof. Let $x, y \in X$ with $x \neq y$. Then by the definition of metric $O_M \prec d(x, y)$.
Since the LOM $(M, \oplus, \preccurlyeq)$ is densely ordered and $d(x, y) \in M$ such that $O_M \prec d(x, y)$, there exists $\varepsilon \in M$ such that $O_M \prec \varepsilon \prec d(x, y)$. Now consider the open ball $B(x, \varepsilon) = \{z \in X \, d(x, z) \prec \varepsilon\}$. Clearly $x \in B(x, \varepsilon)$ and $\varepsilon \prec d(x, y)$ implies $y \notin B(x, \varepsilon)$. Therefore, there exists an open ball $B(x, \varepsilon)$ that contains x but not y. □

Corollary 6.3. *Since $(\mathbb{R}, +, \leq)$ is densely ordered, every classical metric space over $\mathbb{R}$, satisfies the T_0 (Kolmogorov) property.*

The converse of the above theorem may not be true. Consider the following **counterexample:**

Example 6.4. Let $M = \{0, 1\}$, with order $0 \prec 1$ and monoid operation $\oplus = \vee$ (logical OR). Then:

(i) $(M, \oplus)$ is a commutative monoid with identity 0,

(ii) $(M, \preccurlyeq)$ is a complete ordered lattice,

(iii) $\oplus$ is monotone.

However, M is **not densely ordered**, as there exists no $m \in M$ such that $0 \prec m \prec 1$.
Define the metric over M as follows.

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Notice that this is the discrete metric over M. Then for any two distinct points $x \neq y$ the open ball $B(x, 1) = \{x\}$ contains x but not y, implying the space is T_0. Thus, we have a T_0 space without dense ordering, so the converse of the theorem does not hold.

Theorem 6.5 (T_1 **or Frechet**). *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If $(M, \oplus, \preccurlyeq)$ is residually closed and densely ordered, then for every pair of distinct points x, y in X, there exist open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Proof. Let $x, y \in X$ with $x \neq y$. Then by metric axiom $O_M \prec d(x, y)$.
Since M is densely ordered, there exists an element $\varepsilon \in M$ such that $O_M \prec \varepsilon \prec d(x, y)$.
Now consider the open ball

$$B(x, \varepsilon) = \{z \in X \mid d(x, z) \prec \varepsilon\}.$$

Then:

- $x \in B(x, \varepsilon)$, since $d(x, x) = O_M \prec \varepsilon$,
- $y \notin B(x, \varepsilon)$, since $\varepsilon \prec d(x, y)$ by construction.

Similarly, by symmetry of the metric:

- $d(y, x) = d(x, y) \succ O_M$, so, we can also find $\varepsilon' \in M$ with $O_M \prec \varepsilon' \prec d(y, x)$,
- So $y \in B(y, \varepsilon')$ but $x \notin B(y, \varepsilon')$.

Since M is residually closed, every open ball in (X, d, M) is an open set. Now $U = B(x, \varepsilon)$ and $V = B(y, \varepsilon')$ are as required. □

Corollary 6.6. *Every classical metric space over $(\mathbb{R}, +, \leq)$ is T_1, since $\mathbb{R}$ is residually closed and densely ordered.*

Definition 6.7. A LOM $(M, \oplus, \preccurlyeq)$ is said to have the sub-additive property if for any $a \in M$ with $O_M \prec a$, there exists $\varepsilon \in M$ such that $O_M \prec \varepsilon$ and $\varepsilon \oplus \varepsilon = a$.

Example 6.8.

(i) The LOM $(\mathbb{R}, +, \leq)$, $([0, \infty), +, \leq)$ have the sub-additive property.

(ii) Let X be any non-empty set. Then the LOM $(P(X), \cup, \subseteq)$ have the sub-additive property.

Theorem 6.9 (T_2 **or Hausdorff Property**). *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If $(M, \oplus, \preccurlyeq)$ is residually closed and if $(M, \oplus, \preccurlyeq)$ satisfies the sub additive property, then for every pair of distinct points x, y in X, there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.*

Proof. Let $x, y \in X$, with $x \neq y$. Then, by the metric axioms $O_M \prec d(x, y)$.
By the **sub-additive property**, there exists ϵ such that $O_M \prec \varepsilon$ and $\varepsilon \oplus \varepsilon = d(x, y)$.
Now consider the open balls:

$$U = B(x, \varepsilon) = \{z \in X \mid d(x, z) \prec \varepsilon\}$$
$$V = B(y, \varepsilon) = \{z \in X \mid d(y, z) \prec \varepsilon\}.$$

Then clearly $x \in U$ and $y \in V$. We claim that U and V are **disjoint**, i.e., $U \cap V = \phi$.
Suppose for contradiction that there exists $z \in U \cap V$. Then $d(x, z) \prec \varepsilon$, since $z \in U$ and $d(y, z) \prec \varepsilon$, since $z \in V$. By the triangle inequality $d(x, y) \preccurlyeq d(x, z) \oplus d(z, y)$. Since $d(z, y) = d(y, z)$ by symmetry, we have $d(x, y) \preccurlyeq d(x, z) \oplus d(y, z)$. Since $d(x, z) \prec \varepsilon$ and $d(y, z) \prec \varepsilon$ and $\oplus$ is monotone with respect to $\preccurlyeq$, we get $d(x, z) \oplus d(y, z) \prec \varepsilon \oplus \varepsilon = d(x, y)$.
Thus, $d(x, y) \prec d(x, y)$ which is not possible because the relation $\preccurlyeq$ is reflexive. Therefore, no such z exists and $U \cap V = \emptyset$. Finally, since by previous theorems, open balls are open sets (due to residual closure), U and V are open sets. Thus, (X, d, M) satisfies the Hausdorff (T_2) property. □

Remark 6.10. If a LOM have sub-additive property, it may not imply that the LOM is densely ordered. For example the LOM $(P(X), \cup, \subseteq)$ have the sub-additive property but is not densely ordered, where X is any non-empty set.

But still if a metric space (X, d, M) satisfies the Hausdorff (T_2) property then it satisfies the T_1 property. Hence, we have the following proposition.

Proposition 6.11. *In a LOM-metric space, the Hausdorff (T_2) property implies the Frechet (T_1) property.*

Proof. Since in a Hausdorff space, for any two distinct points x and y, there exist disjoint open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$, it immediately follows that $y \notin U$ and $x \notin V$, otherwise it will imply $U \cap V \neq \phi$. Thus, the T_1 property holds. □

Note: In the above three theorems we have proved that the T_0, T_1 or T_2 properties of a metric space over a LOM do not depend on the metric but depend on the structures of the LOM.

7 Convergence of sequences and completeness of spaces

Having explored the static topological properties of LOM-metric spaces, we now delve into their dynamic and analytical aspects by developing the theory of **convergence and completeness**. This section provides rigorous definitions for convergent sequences, Cauchy sequences, and what it means for a space to be complete, all framed by the algebraic properties of the underlying monoid. We first establish foundational results that generalize classical tools of analysis, proving criteria for the uniqueness of limits, a **Generalised Sandwich Theorem**, and the conditions under which convergent sequences are Cauchy. The section then culminates in demonstrating the profound structural consequences of completeness by proving generalized versions of two cornerstone theorems of analysis, the **Cantor Intersection Theorem** and the **Baire Category Theorem**.

Definition 7.1. Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A sequence (x_n) in X is said to converge to a point $x \in X$ if for every $\varepsilon \in M$ with $O_M \prec \varepsilon$, there exists $k \in \mathbb{N}$ such that $\forall n \geq k$, $d(x_n, x) \prec \varepsilon$. We write: $\lim_{n\to\infty} x_n = x$ or simply, $x_n \to x$.

Definition 7.2. Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A sequence (x_n) in X is said to be a Cauchy sequence if for every $\varepsilon \in M$ with $O_M \prec \varepsilon$, there exists $k \in \mathbb{N}$ such that $\forall m, n \geq k$, $d(x_n, x_m) \prec \varepsilon$. The space (X, d, M) is said to be complete if every Cauchy sequence converges to a point in X.

Definition 7.3. Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. A sequence (x_n) in X is said to be bounded if there exist a point $x_0 \in X$ and an element $\mu \in M$ with $O_M \preccurlyeq \mu$ such that $d(x_n, x_0) \preccurlyeq \mu$.

Theorem 7.4. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If $(M, \oplus, \preccurlyeq)$ is sub-additive then the limit of any convergent sequence in X is unique.*

Proof. Suppose a sequence (x_n) in X converges to two distinct points x, y in X.
Since $x \neq y$, $O_M \prec d(x, y)$. By the sub-additive property, there exists $\epsilon \in M$ with $O_M \prec \epsilon$ such that $\epsilon \oplus \epsilon = d(x, y)$. Since $x_n \to x$ and $O_M \prec \epsilon$ there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $d(x_n, x) \prec \varepsilon$. Similarly, since $x_n \to y$ and $O_M \prec \epsilon$ there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $d(x_n, y) \prec \epsilon$.
Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have $d(x_n, x) \prec \varepsilon$ and $d(x_n, y) \prec \varepsilon$.
Since $\oplus$ is monotone, $d(x_n, x) \oplus d(x_n, y) \prec \varepsilon \oplus \varepsilon = d(x, y)$. Again, by triangle inequality, $d(x, y) \preccurlyeq d(x, x_n) \oplus d(x_n, y) = d(x_n, x) \oplus d(x_n, y) \prec \varepsilon \oplus \varepsilon = d(x, y)$. This implies $d(x, y) \prec d(x, y)$, which contradicts the reflexivity of the partial order '$\preccurlyeq$'. Hence our assumption that $x \neq y$ is wrong. Therefore, the sequence (x_n) in X converges to unique limit. □

Theorem 7.5. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. Then every Cauchy sequence in (X, d, M) is a bounded sequence.*

Proof. Let (x_n) be a Cauchy sequence in X. Then for every $\varepsilon \in M$ with $O_M \prec \varepsilon$, there exists $k \in \mathbb{N}$ such that $\forall m, n \geq k$, $d(x_n, x_m) \prec \varepsilon$. Fix $x_k \in X$. Now for any $n \in \mathbb{N}$,
Case 1: If $n \geq k$, then $d(x_n, x_k) \prec \varepsilon$.
Case 2: If $n < k$, that is if $n = 1, 2, \ldots k-1$, then by triangle inequality:

$$d(x_n, x_k) \preccurlyeq d(x_n, x_1) \oplus d(x_1, x_2) \oplus \ \ldots \ \oplus d(x_{k-1}, x_k).$$

Let $\mu_0 = \bigvee_{i=1}^{k-1} d(x_i, x_{i+1}) \vee d(x_n, x_1)$, which exists in M since $(M, \preccurlyeq)$ is a lattice. Then $d(x_n, x_1) \preccurlyeq \mu_0$ and $d(x_i, x_{i+1}) \preccurlyeq \mu_0$ for $i = 1, 2, \ldots k-1$. Since $\oplus$ is monotone, $d(x_n, x_1) \oplus d(x_1, x_2) \oplus \ \ldots \ \oplus d(x_{k-1}, x_k) \preccurlyeq \mu_0 \oplus \mu_0 \oplus \cdots \oplus \mu_0$ (k times). Define $\mu = \varepsilon \vee (\mu_0 \oplus \mu_0 \oplus \cdots \oplus \mu_0)$ (k times). Then clearly $\mu \in M$ and $O_M \prec \mu$. Now if $n \geq k$, then $d(x_n, x_k) \prec \varepsilon \preccurlyeq \mu$ and if $n < k$, then $d(x_n, x_k) \preccurlyeq \mu_0 \oplus \mu_0 \oplus \cdots \oplus \mu_0$ (k times)$\preccurlyeq \mu$. Therefore, $d(x_n, x_k) \preccurlyeq \mu$ for all $n \in \mathbb{N}$, and hence the sequence (x_n) is bounded. □

Theorem 7.6. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$. If $(M, \oplus, \preccurlyeq)$ have the sub-additive property then every convergent sequence in (X, d, M) is a Cauchy sequence.*

Proof. Let (x_n) be a convergent sequence in (X, d, M) and suppose $x_n \to x$. Now let $\varepsilon \in M$ with $O_M \prec \varepsilon$. Since $O_M \prec \varepsilon$ and $(M, \oplus, \preccurlyeq)$ have the sub-additive property, there exists $\delta \in M$ with $O_M \prec \delta$ such that $\delta \oplus \delta = \varepsilon$. Now since $\delta \in M$ with $O_M \prec \delta$ and $x_n \to x$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_n, x) \prec \delta$. Now, take any $m, n \geq N$. By the triangle inequality:

$$d(x_m, x_n) \preccurlyeq d(x_m, x) \oplus d(x, x_n). \tag{7.1}$$

Since $m, n \geq N$, $d(x_m, x) \prec \delta$ and $d(x_n, x) \prec \delta$. Now since $d(x_m, x) \prec \delta$, $d(x_n, x) \prec \delta$ and $\oplus$ is monotone, we have

$$d(x_m, x) \oplus d(x, x_n) \prec \delta \oplus \delta = \varepsilon. \tag{7.2}$$

Since the partial order relation " $\preccurlyeq$ " is transitive, from (7.1) and (7.2) we have

$$d(x_m, x_n) \prec \varepsilon.$$

Thus $d(x_m, x_n) \prec \varepsilon$ for all $m, n \geq N$. Hence the sequence (x_n) is a Cauchy sequence. □

Remark 7.7. It is clear from the above theorem that in a classical metric space, every convergent sequence is a Cauchy sequence but in a metric space over a LOM this may not be the case unless the LOM have the sub-additive property.

Theorem 7.8. *Let (X, d, M) be a metric space over the LOM $M = \{0, 1\}$, with $\oplus = \vee$, with the order $0 \prec 1$, and $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ otherwise (the discrete metric). Then (X, d, M) is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. Now let $\varepsilon \in M$ with $O_M \prec \varepsilon$. Since $O_M = 0 \prec 1$, we must have $\varepsilon = 1$. Since (x_n) is Cauchy sequence in X and $\varepsilon \in M$ with $O_M \prec \varepsilon$, there exists $k \in \mathbb{N}$ such that $\forall m, n \geq k$, $d(x_n, x_m) \prec \varepsilon = 1$. So, for $d(x_n, x_m) \prec 1$ in M, we must have $d(x_n, x_m) = 0 = O_M$. Therefore $\forall m, n \geq k$, $d(x_n, x_m) = O_M$. This implies $x_n = x_m$ for all $m, n \geq k$. Thus, the sequence (x_n) is eventually constant, say $x_n = x$ for all $n \geq k$. So, $d(x_n, x) = O_M \prec \varepsilon$ for all $n \geq k$, implying that $x_n \to x$. Therefore, the sequence (x_n) is convergent. Hence (X, d, M) is complete. □

Theorem 7.9. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ that is residually closed and satisfies the finite meet property. If (X, d, M) is complete, then every Cauchy sequence in X converges to a point in the topology induced by the collection of open sets defined by open balls. Also, if $(M, \oplus, \preccurlyeq)$ is densely ordered then this limit is unique.*

Proof. Let (x_n) be a Cauchy sequence in (X, d, M). Since (X, d, M) is complete, there exists a point $x \in X$ such that for every $\varepsilon \in M$ with $O_M \prec \varepsilon$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec \varepsilon$. We must show that (x_n) converges to x in the topology induced by open balls, i.e., for every open set U containing x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. By the finite meet property, the collection of open sets $U \subseteq X$ such that for every $y \in U$, there exists $\varepsilon \in M$ with $O_M \prec \varepsilon$ and $B(y, \varepsilon) \subseteq U$ forms a topology (Theorem 4.8). Since $(M, \oplus, \preccurlyeq)$ is residually closed, every open ball $B(y, \varepsilon) = \{z \in X : d(y, z) \prec \varepsilon\}$ is an open set. Let U be an open set containing x. By the definition of open set, there exists $\varepsilon \in M$ with $O_M \prec \varepsilon$ such that $B(x, \varepsilon) \subseteq U$. By completeness, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec \varepsilon$. Thus, $x_n \in B(x, \varepsilon)$, and since $B(x, \varepsilon) \subseteq U$, we have $x_n \in U$ for all $n > N$. Hence, (x_n) converges to x in the topology.

To address uniqueness, suppose $y \in X$, $y \neq x$, is another limit in the topology. Then, for every open set V containing y, there exists $M \in \mathbb{N}$ such that $x_n \in V$ for all $n > M$. Since $(M, \oplus, \preccurlyeq)$ is densely ordered, by Theorem 6.1 there exists an open ball $B(x, \delta)$ containing x but not y. Since (x_n) converges to x, there exists N_1 such that $x_n \in B(x, \delta)$ for $n > N_1$, but since $y \notin B(x, \delta)$, $x_n \notin B(y, \varepsilon')$ for any ε' with $B(y, \varepsilon') \subseteq V$, contradicting convergence to y. Thus, the limit is unique in T_0 spaces. Therefore, every Cauchy sequence converges to a unique limit in the topology induced by open balls, completing the proof. □

Definition 7.10. Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$, where $(X, \leq_X)$ is a partial ordered set. The metric $d : X \times X \to M$ is said to be order-preserving if, for $x, y, u, v \in X$ with $x \leq_X y$ and $u \leq_X v$, the following holds: $d(x, u) \preccurlyeq d(y, v)$.

Theorem 7.11. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$, where $(X, \leq_X)$ is a chain. If a sequence (x_n) in X forms an ascending chain with respect to the partial order $\leq_X$ in X which converges to x in (X, d, M) and the metric d is order-preserving then x will be the supremum of this ascending chain (x_n).*

Proof. If possible, let x is not an upper bound of ascending chain sequence (x_n), then there exists a $k \in \mathbb{N}$ such that $x <_X x_k$. Therefore $O_M \prec d(x, x_k)$, since $x \neq x_k$.
Take $\varepsilon = d(x, x_k)$. Then $O_M \prec \varepsilon$ and the sequence (x_n) converges to x, so there exists $k' \in \mathbb{N}$ such that

$$d(x, x_n) \prec \varepsilon, \forall n \geq k'. \tag{7.3}$$

Now take $N = \max\{k, k'\}$.
Then $x_k \leq_X x_n$ for all $n \geq N$, since the sequence (x_n) is an ascending chain and $k \leq N$.
Now $x \leq_X x$ and $x_k \leq_X x_n$ for all $n \geq N$ and hence by order-preserving property,
$d(x, x_k) \preccurlyeq d(x, x_n)$ for all $n \geq N \Rightarrow \varepsilon \preccurlyeq d(x, x_n)$ for all $n \geq N$, which contradicts (7.3), since $N \geq k'$. Therefore, $x <_X x_k$ is not possible for any $k \in \mathbb{N}$ and hence $x_n \leq_X x$ for all $n \in \mathbb{N}$. This proves that x is an upper bound for the ascending chain sequence (x_n).
Now we shall prove that x is the supremum of the ascending chain sequence (x_n). If the sequence (x_n) is a constant sequence, then there is nothing to prove. So, we consider (x_n) is a non-constant sequence. If possible, let $x' \neq x$ be the supremum of the ascending chain sequence (x_n). Since, x' is the supremum and x is an upper bound of the ascending chain sequence (x_n), $x' <_X x$. Since, x is an upper bound of the ascending chain sequence (x_n), $x_n \leq_X x$ for all $n \in \mathbb{N}$. Therefore, by order-preserving property, $d(x', x_n) \preccurlyeq d(x, x)$ for all $n \in \mathbb{N} \Rightarrow d(x', x_n) \preccurlyeq O_M$ for all $n \geq N$. Again $O_M \preccurlyeq d(x', x_n)$ for all $n \geq N$, by metric assumption. Since the partial order $\preccurlyeq$ is anti-symmetric, we have $d(x', x_n) = O_M$ for all $n \geq N \Rightarrow x_n = x'$ for all $n \geq N$. This shows that (x_n) is a constant sequence, a contradiction. Therefore, x is the supremum of the ascending chain sequence (x_n). This completes the proof. □

Corollary 7.12. *If the sequence (x_n) in X forms a descending chain with respect to the partial order $\leq_X$ in X which converges to x in (X, d, M) and the metric d is order-preserving then by a similar argument as in the proof of Theorem 7.11 it can be proved that x will be the infimum of the descending chain of the sequence (x_n).*

Definition 7.13. Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$, where $(X, \leq_X)$ is a partial ordered set. The metric $d : X \times X \to M$ is said to be order-coherent if for any $a, b, c \in X$ with $a \leq_X b \leq_X c$ and for any $p \in X$, the following holds

$$d(b, p) \preccurlyeq d(a, p) \vee d(c, p).$$

Theorem 7.14 (Generalized Sandwich theorem). *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$, where $(X, \leq_X)$ is a partially ordered set and the metric d is order-coherent. Also let (x_n), (y_n) and (z_n) be three sequences in X such that*

(i) There exists a natural number $m \in \mathbb{N}$ such that $x_n \leq_X y_n \leq_X z_n$ for all $n \geq m$ and

(ii) $\lim x_n = \lim z_n = p$ for some $p \in X$.

Then $\lim y_n = p$.

Proof. By the given hypothesis $x_n \leq_X y_n \leq_X z_n$ for all $n \geq m$. Since the metric d is order-coherent, for any $n \geq m$,

$$d(y_n, p) \preccurlyeq d(x_n, p) \vee d(z_n, p). \tag{7.4}$$

Now let $\varepsilon \in M$ with $O_M \prec \varepsilon$. Since $\lim x_n = p$ and $O_M \prec \varepsilon$, there exists a natural number $m_1 \in \mathbb{N}$ such that $d(x_n, p) \prec \varepsilon$ for all $n \geq m_1$. Again since $\lim z_n = p$ and $O_M \prec \varepsilon$, there exists a natural number $m_2 \in \mathbb{N}$ such that $d(z_n, p) \prec \varepsilon$ for all $n \geq m_2$. Let $k = \max\{m, m_1, m_2\}$. Then $d(x_n, p) \prec \varepsilon$ and $d(z_n, p) \prec \varepsilon$ for all $n \geq k$. Therefore, $\sup\{d(x_n, p), d(z_n, p)\} \prec \varepsilon$ for all $n \geq k$, i.e., $d(x_n, p) \vee d(z_n, p) \prec \varepsilon$ for all $n \geq k$. Again by (7.4) for all $n \geq k$ we have $d(y_n, p) \preccurlyeq d(x_n, p) \vee d(z_n, p)$. Since the relation " $\preccurlyeq$ " is transitive on M so we have $d(y_n, p) \prec \varepsilon$ for all $n \geq k$. Hence $\lim y_n = p$. This completes the proof. □

Cantor's Intersection Theorem in LOM-Metric Spaces
To establish a generalized version of Cantor's theorem, we first need to define the diameter of a set and introduce a new property for the LOM that ensures limits are well-behaved.

Definition 7.15 (**Diameter of a Set**). Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ where $(M, \preccurlyeq)$ is a **complete lattice**. For any non-empty subset $A \subseteq X$, the **diameter** of A is defined as: $\text{diam}(A) = \mathbf{sup}\{d(x, y) : x, y \in A\}$. (The supremum exists because M is a complete lattice and the set of distances is a subset of M)

Definition 7.16 (**Positive-Meet Property**). A lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ has the **positive-meet property** if the meet of all elements strictly greater than the identity O_M is the identity itself, that is $\bigwedge\{\varepsilon \in M : O_M \prec \varepsilon\} = O_M$.

The above property ensures that if an element is smaller than every "positive" element, it must be the zero element O_M. This is a crucial condition for ensuring the uniqueness of the intersection point.

Theorem 7.17 (**Cantor's Intersection Theorem for LOM-Metric Spaces**). *Let (X, d, M) be a complete metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ such that:*

(i) $(M, \preccurlyeq)$ is a complete lattice.

(ii) M has the positive-meet property.

*Let also $\{F_n\}$ be a sequence of non-empty, **closed** subsets of X such that they are nested i.e.,$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$. If the limit of their diameters is the monoid identity, i.e., $\lim_{n\to\infty} diam(F_n) = O_M$, then the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. First, we show that the intersection is non-empty. Let us construct a sequence (x_n) by choosing a point $x_n \in F_n$ for each $n \in \mathbb{N}$. We will show that (x_n) is a Cauchy sequence. Let $\varepsilon \in M$ with $O_M \prec \varepsilon$. Since $\lim_{n\to\infty} \text{diam}(F_n) = O_M$, by the definition of convergence, there exists a natural number $k \in \mathbb{N}$ such that for all $n \geq k$, we have $\text{diam}(F_n) \prec \varepsilon$. Now, consider any two integers $n, m \geq k$. Without loss of generality, assume $m \geq n$. Because the sets are nested, $F_n \supseteq F_m$. Therefore, both x_n and x_m are elements of F_n. By the definition of diameter, the distance between any two points in F_n is bounded by its diameter:$d(x_n, x_m) \preccurlyeq \text{diam}(F_n)$. Since $n \geq k$, we have $\text{diam}(F_n) \prec \varepsilon$. By the transitivity of the partial order $\preccurlyeq$, it follows that $d(x_n, x_m) \prec \varepsilon$ for all $m, n \geq k$. This satisfies the definition of a Cauchy sequence. Since the space (X, d, M) is complete, the sequence (x_n) must converge to some point $x \in X$. Next, we must show that this limit x lies in every F_n. For any fixed $k \in \mathbb{N}$, the tail of the sequence, $(x_n)_{n\geq k}$, is entirely contained within the set F_k. Since F_k is a closed set, it contains all of its limit points. As x is the limit of the sequence (x_n), it is also the limit of the tail $(x_n)_{n\geq k}$. Therefore, $x \in F_k$. Since this holds for every $k \in \mathbb{N}$, $x \in \bigcap_{n=1}^{\infty} F_n$. Thus, the intersection is non-empty. Finally, we show that the intersection contains exactly one point. Suppose x and y are two points in the intersection, i.e., $x, y \in \bigcap_{n=1}^{\infty} F_n$. This implies that for every $n \in \mathbb{N}$, both x and y belong to F_n . By the definition of diameter, this means $d(x, y) \preccurlyeq \text{diam}(F_n)$ for all n. Since $\lim_{n\to\infty} \text{diam}(F_n) = O_M$, for any $\varepsilon \in M$ with $O_M \prec \varepsilon$, there is a k such that $\text{diam}(F_n) \prec \varepsilon$ for $n \geq k$. This implies that $d(x, y) \preccurlyeq \text{diam}(F_n) \prec \varepsilon$, and thus by the transitivity of the partial order $\preccurlyeq$, it follows that $d(x, y) \prec \varepsilon$ for every $\varepsilon \in M$ with $O_M \prec \varepsilon$. Because the LOM M has the positive-meet property, we have $\bigwedge\{\varepsilon \in M : O_M \prec \varepsilon\} = O_M$. Since $d(x, y)$ is a lower bound for the set $\varepsilon \in M : O_M \prec \varepsilon$, it must be that $d(x, y) \preccurlyeq O_M$. Combining this with the non-negativity axiom of the metric, $O_M \preccurlyeq d(x, y)$, by the anti-symmetric property of the partial order $\preccurlyeq$, it follows that $d(x, y) = O_M$. By the identity axiom of the metric, this means $x = y$. Therefore, the intersection contains exactly one point. This completes the proof. □

The Baire Category Theorem for LOM-Metric Spaces The Baire Category Theorem is a fundamental result in topology and functional analysis, describing a key property of complete metric spaces. Its generalization here highlights the robustness of the LOM-metric structure. First, we recall the definition of a Baire space and introduce a helpful property for our monoid.

Definition 7.18. A topological space is called a Baire space if the intersection of any countable collection of dense open subsets is itself a dense set.

Theorem 7.19 (**Baire Category Theorem for LOM-Metric Spaces**). *Let (X, d, M) be a complete metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ that satisfies the following conditions:*

(i) M is residually closed and densely ordered.

(ii) There is a sequence in M that converges to O_M.

(iii) The conditions for Cantor's Intersection Theorem (7.17) hold (i.e., M is a complete lattice with the positive-meet property).

Then (X, d, M) is a Baire space.

Proof. Let $\{G_n\}$ be a countable collection of dense open subsets of X. To show that X is a Baire space, we must prove that their intersection $G = \bigcap_{n=1}^{\infty} G_n$ is dense in X. According to Theorem 5.14, this is equivalent to showing that G has a non-empty intersection with every non-empty open subset of X. Let U be any non-empty open subset of X. We will construct a sequence of nested closed balls whose intersection lies within $U \cap G$.
Step 1: Construction of the first ball B_1: Since G_1 is dense and U is a non-empty open set, their intersection $U \cap G_1$ is also a non-empty open set. Let $x_1 \in U \cap G_1$. Since this set is open, there exists an element $\varepsilon'_1 \in M$ with $O_M \prec \varepsilon'_1$ such that $B(x_1, \varepsilon'_1) \subseteq U \cap G_1$. Let (δ_n) be a sequence in M converging to O_M (by condition 2). We can choose a δ_{k_1} from this sequence such that $\delta_{k_1} \prec \varepsilon'_1$. Let $\varepsilon_1 = \delta_{k_1}$. Now consider the closed ball $B_1 = B[x_1, \varepsilon_1]$. Since $\varepsilon_1 \prec \varepsilon'_1$, it follows that $B_1 \subseteq B(x_1, \varepsilon'_1) \subseteq U \cap G_1$.
Step 2: Inductive Construction: Assume we have constructed a sequence of non-empty closed balls $B_1, B_2, ..., B_{n-1}$ such that $B_k = B[x_k, \varepsilon_k]$ and $B_k \subseteq B(x_{k-1}, \varepsilon_{k-1}) \bigcap G_k$ for $k = 2, ..., n-1$. We also ensure that $\varepsilon_k = \delta_{k_j}$ for some $k_j > k_{j-1}$, which will guarantee $\varepsilon_k \to O_M$. Now we construct B_n. The ball $B(x_{n-1}, \varepsilon_{n-1})$ is a non-empty open set (since M is residually closed, open balls are open sets). G_n is also dense and open. Therefore, their intersection $B(x_{n-1}, \varepsilon_{n-1}) \cap G_n$ is a non-empty open set. We can choose a point x_n in this intersection. As before, there exists ε'_n with $O_M \prec \varepsilon'_n$ such that $B(x_n, \varepsilon'_n) \subseteq B(x_{n-1}, \varepsilon_{n-1}) \bigcap G_n$. We then choose a term $\varepsilon_n = \delta_{k_n}$ from our sequence such that $k_n > k_{n-1}$ and $\varepsilon_n \prec \varepsilon'_n$. We define the closed ball $B_n = B[x_n, \varepsilon_n]$. This construction ensures $B_n \subseteq B_{n-1}$ and that the sequence of radii (ε_n) converges to O_M.
Step 3: Applying Cantor's Theorem: This inductive process gives us a nested sequence of non-empty closed balls $B_1 \supseteq B_2 \supseteq B_3 \supseteq ....$ The diameter of each ball B_n satisfies $\operatorname{diam}(B_n) \preccurlyeq \varepsilon_n \oplus \varepsilon_n$. Since $\varepsilon_n \to O_M$ and $\oplus$ is monotone, $\varepsilon_n \oplus \varepsilon_n \to O_M$, which implies $\lim_{n\to\infty} \operatorname{diam}(B_n) = O_M$. The space (X, d, M) is complete and the conditions for **Theorem 7.17** are met. Therefore, the intersection of these closed balls is non-empty and contains exactly one point. Let this point be x such that $x \in \bigcap_{n=1}^{\infty} B_n$. By our construction, we have $x \in B_1 \subseteq U$ and for every $n \geq 1, x \in B_n \subseteq G_n$. This implies $x \in U$ and $x \in \bigcap_{n=1}^{\infty} G_n = G$. So, $x \in U \cap G$, which shows their intersection is non-empty. Since we have shown that G intersects every arbitrary non-empty open set U, G is dense in X. Therefore, (X, d, M) is a Baire space. This completes the proof. □

Definition 7.20 (**Monoid absorption property**). A lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ is said to have the monoid absorption property if for any $a \in M$ with $O_M \prec a$, there exists $b \in M$ with $O_M \prec b \prec a$ such that $a \oplus b = a$.

Example 7.21. The LOM $(P(X), \cup, \subseteq)$ has the monoid absorption property for any non-empty set X but the LOM $(\mathbb{R}, +, \leq)$ do not have the monoid absorption property.

Theorem 7.22. *Let (X, d, M) be a metric space over a lattice-ordered monoid $(M, \oplus, \preccurlyeq)$ having the monoid absorption property. If a sequence (x_n) in X converges to $x \in X$ and there exists $y \in X$ such that $d(x_n, y) \preccurlyeq d(x, y)$ for all $n \in \mathbb{N}$, then $x = y$.*

Proof. Assume that $d(x_n, y) \preccurlyeq d(x, y)$ for all $n \in \mathbb{N}$. We need to show that $x = y$. Suppose for a contradiction that $x \neq y$. Then by metric axioms, $O_M \prec d(x, y)$. So, by the monoid absorption property of $(M, \oplus, \preccurlyeq)$, there exists $\varepsilon \in M$ with $O_M \prec \varepsilon \prec d(x, y)$ such that

$$d(x, y) \oplus \varepsilon = d(x, y). \tag{7.5}$$

Since the sequence (x_n) in X converges to $x \in X$ and $\varepsilon \in M$ with $O_M \prec \varepsilon$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec \varepsilon$ for all $n \geq N$. By the triangle inequality we have,

$$d(x, y) \preccurlyeq d(x, x_n) \oplus d(x_n, y). \tag{7.6}$$

Since $d(x_n, x) \prec \varepsilon$ for all $n \geq N$ and $d(x_n, y) \preccurlyeq d(x, y)$ and $\oplus$ is monotone, for all $n \geq N$ we have,

$$d(x_n, x) \oplus d(x_n, y) \prec \varepsilon \oplus d(x, y). \tag{7.7}$$

Since the relation " $\preccurlyeq$ " is transitive on M so we have from (7.6) and (7.7)

$$d(x, y) \prec \varepsilon \oplus d(x, y) = d(x, y) \text{ [by 7.5]}.$$

Therefore, we have $d(x, y) \prec d(x, y)$ which contradicts the reflexivity of the relation " $\preccurlyeq$ ". Hence $x = y$ and the proof is completed. □

Remark 7.23. It is crucial to note that Theorem 7.22 is a novel result specific to LOM-metric spaces possessing the monoid absorption property and has no direct analogue in classical metric spaces. In a standard metric space (X, d) over $(\mathbb{R}, +, \leq)$, the conditions of the theorem do not force the conclusion that $x = y$.
For a simple counterexample, let $X = \mathbb{R}^2$ with the Euclidean metric. Let $x = (1, 0)$ and $y = (0, 0)$. The distance $d(x, y) = 1$. Now consider the sequence $x_n = (\cos(\frac{1}{n}), \sin(\frac{1}{n}))$, which converges to x. For every n, the distance $d(x_n, y) = 1$. The condition $d(x_n, y) \leq d(x, y)$ is satisfied as $1 \leq 1$, but clearly $x \neq y$.
The conclusion of our theorem is made possible only by the monoid absorption property, which is fundamentally non-classical.

8 Applications

The generalised framework of metric spaces over lattice-ordered monoids (LOMs) introduced in this paper offers significant potential for application across various mathematical and applied domains. By extending the notion of distance beyond real numbers to partially ordered algebraic structures, the theory enables more flexible modeling in contexts where traditional metrics are inadequate. Key application areas include the following:

(i) **Fuzzy Set Theory and Fuzzy Logic**

In fuzzy systems, degrees of membership and truth values naturally form lattice structures. LOM-valued metrics provide a robust way to measure distances between fuzzy elements or fuzzy sets, aligning well with the semantics of **many-valued logic**, **residuated lattices**, and **graded truth systems**. This enables the formulation of more structurally faithful metrics in fuzzy topology and fuzzy clustering.

(ii) **Domain Theory and Theoretical Computer Science**

Computation over partially ordered domains—such as information states, data approximations, or computational resources—is fundamental in **denotational semantics** and **abstract interpretation**. LOM-metric spaces provide a framework for analysing convergence, approximation, and stability in such settings, particularly when elements are only partially comparable or accumulate over algebraic operations.

(iii) **Logic and Algebraic Semantics** The interplay between order and combination in LOMs aligns naturally with algebraic semantics of logic systems, including **intuitionistic**, **substructural**, and **non-classical logics**. Metrics valued in LOMs can capture semantic distances between logical formulas or models, providing a unified algebraic foundation for reasoning about similarity, consistency, or entailment.

(iv) **Topological Data Analysis and Granular Computing**

In areas like **granular computing**, **rough sets**, and **data classification**, data points often reside in discrete or order-enriched spaces. LOM-metric structures enable more nuanced clustering and similarity measures, especially in systems where overlap, hierarchy, or uncertainty plays a key role.

(v) **Mathematical Structures with Non-Linear or Discrete Behaviour**

Structures such as **Boolean algebras**, **power set lattices**, and **finite lattices** naturally fit into the LOM framework. These arise in a wide variety of combinatorial, algebraic, and discrete mathematical settings, where our theory provides tools for defining and analysing convergence, boundedness, and continuity in inherently non-Euclidean spaces.

This broad applicability underlines the value of LOM-metric spaces as a foundational structure bridging algebra, topology, logic, and computation. Future research may further explore application-specific refinements and domain-theoretic interpretations.

9 Conclusion:

In this paper we introduced a broad generalization of metric spaces by allowing the distance function to take values in lattice-ordered monoids. We established a rigorous foundation for this class of spaces, defining core concepts such as open and closed balls, neighbourhood systems, open and closed sets, convergence, and completeness purely from algebraic properties.
By identifying key structural requirements—including residual closure, sub-additivity, dense ordering, and the finite meet property—we demonstrated how classical topological results, particularly the separation axioms, can be recovered and extended in this new setting. The robustness and depth of the LOM-metric framework were further showcased through the formulation and proof of generalized versions of several major theorems of analysis: a Generalised Sandwich Theorem, Cantor's Intersection Theorem, and the Baire Category Theorem. The successful extension of these theorems confirms that LOM-metric spaces retain a rich and consistent analytical structure.
This algebraic and order-theoretic perspective not only subsumes the classical theory but also supports a wide range of applications, from fuzzy systems to logic and computation. Future directions for research include the study of compactness, continuity, fixed-point theory, and homological aspects within LOM-metric spaces, further bridging the gap between abstract algebra and topology.

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