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A MULTIPLICATIVE CALCULUS APPROACH TO THE NAVIER-STOKES EQUATIONS

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Abstract This paper introduces and analyzes a novel formulation of the Navier-Stokes equations within the framework of multiplicative calculus. By replacing conventional differential operators with their multiplicative analogs, we construct the Multiplicative Navier-Stokes (MNS) equations, where nonlinear terms such as convection and diffusion are reformulated using logarithmic transformations and exponential operators. This approach leads to a system where the convective term exhibits improved quasi-linearity in log-space, and the diffusion term acts on the logarithm of the velocity field, offering enhanced regularity and stability, particularly in high Reynolds number regimes. We establish foundational properties of the MNS equations, including the boundedness and smoothing behavior of the multiplicative Laplacian. A series of theorems is proved to demonstrate local well-posedness and stability of solutions under minimal regularity assumptions. Furthermore, we conjecture a regularity result within this framework and support it with detailed energy estimates in logarithmic space. Our findings suggest that the multiplicative formulation may circumvent some of the key difficulties associated with blow-up and turbulence in classical fluid dynamics.

1 Introduction

The Navier–Stokes equations are partial differential equations that describe the motion of viscous fluid substances. They are named after Claude-Louis Navier and George Gabriel Stokes, who developed the theories over several decades (1822–1850). These equations mathematically express momentum balance for Newtonian fluids and incorporate the principle of conservation of mass. They are derived by applying Isaac Newton's second law to fluid motion, assuming that the stress in the fluid is a sum of a diffusing viscous term and a pressure term. This distinguishes them from the Euler equations, which only model inviscid flow. The Navier–Stokes equations are elliptic, possessing better analytic properties but lacking the complete integrability of some simpler systems.

The Navier–Stokes equations are crucial in various scientific and engineering fields, including modeling weather, ocean currents, water flow in pipes, and airflow around aircraft wings. They are fundamental in designing aircraft and cars, studying blood flow, designing power stations, and analyzing pollution. Despite their widespread practical applications, the Navier–Stokes equations pose significant mathematical challenges. One of the most prominent is the Navier–Stokes existence and smoothness problem. The primary difficulties with the Navier–Stokes equations stem from their non-linearity and the challenge of understanding the behavior of their solutions, particularly in three dimensions.

The theory of weak solutions was pioneered by Leray, who introduced the concept of Leray-Hopf weak solutions and established the global existence in three dimensions, without proving uniqueness and smoothness. Extensions of Leray's work were undertaken in the 20th century using energy estimates and functional analysis. Johansson [1] presented a detailed analysis of

well-posedness in the generalized version that showed that the incompressible Navier–Stokes system admits stable and unique solutions with specific boundary conditions by providing sufficient regularity in the initial data. Subsequent studies have shown the local-in-time existence of strong solutions for smooth data, with global results only being available in two-dimensional domains because of the absence of vorticity stretching.

Recent developments in the study of well-posedness include results in anisotropic and critical spaces. The authors in [2] employed Littlewood–Paley theory to establish local and global well-posedness for small initial data in critical Besov spaces, which opened a new pathway to study large data with partial regularity conditions. Peng et al. [3] combined classical techniques with refined energy estimates to investigate global existence in three-dimensional domains for a class of initial data varying slowly in two directions. Similarly, Lai et al. [4] worked on compressible Navier–Stokes systems with vacuum states and proved global-in-time existence under smallness assumptions and appropriate compatibility conditions. These contributions show the depth and importance of classical regularity techniques based on energy methods, Sobolev embeddings, and a priori estimates.

Alongside traditional approaches, alternative formulations based on non-Newtonian calculi have been proposed in recent years. Several works investigated multiplicative calculus as a non-Newtonian calculus that replaces additive structures with multiplicative analogues. In this setting, the multiplicative derivative of a positive function $f(x)$ is defined as

$$f^*(x) = e^{\frac{f'(x)}{f(x)}}. \tag{1.1}$$

leading to a calculus that naturally respects proportional change and multiplicative scaling. The foundational theory was laid out by Grossman and Katz [5], and later formalized by Bashirov et al. [6]. Their work established basic rules of differentiation and integration in a multiplicative setting, along with applications to dynamical systems and mathematical modeling.

Later on, the multiplicative theory was extended to complex variables, systems of differential equations, and matrix calculus. The authors in [7, 8] constructed complex multiplicative differential operators and found their implications for analytic functions, showing that many classical results, such as the Cauchy–Riemann equations, have natural analogues in the multiplicative case. Applications of these implications have emerged in various domains, such as economics, biology, and engineering. For example, Bashirov et al. [9] applied multiplicative modeling to elasticity problems, while Florack and van Assen [10] employed multiplicative operators in medical imaging, demonstrating improved feature extraction when analyzing exponentially structured data. Cubillos [11] investigated wave propagation models formulated via multiplicative calculus and showed improvement in numerical behavior for problems with exponential solutions.

Recent studies have begun to explore the mathematical implications of this formulation. For instance, Riza and Özyapıcı [12] investigated multiplicative evolution equations and proved local existence theorems using Banach fixed-point techniques adapted to the multiplicative setting. These results hint at a broader potential for establishing well-posedness and regularity in multiplicative Navier–Stokes (MNS) systems. Building on this foundation, the current research introduces energy estimates and smoothing theorems formulated entirely in multiplicative space, revealing a possible resolution pathway to the classical blow-up problem in 3D NSE. Preliminary results suggest that the multiplicative diffusion term controls the exponential growth of the velocity field, while the convective term remains bounded through logarithmic damping effects.

Recent developments in multiplicative calculus have significantly extended its analytical framework and opened new avenues for solving nonlinear problems in applied mathematics. Du et al. [13] introduced multiplicative fractional HH-type inequalities via multiplicative Atangana–Baleanu (AB) fractional integral operators, providing powerful tools for handling nonlinear systems involving fractional multiplicative dynamics. Meftah et al. [14] established a companion of Ostrowski-type inequalities for multiplicatively convex functions, offering improved bounds for integrals in multiplicative settings. The authors in [15, 16] presented a comprehensive error estimation framework for a broad class of fractional multiplicative quadrature formulas, enhancing the accuracy of numerical integration techniques involving multiplicative operators. Frioui et al. [17] proposed parametrized multiplicative integral inequalities that enable more flexible control of proportional growth phenomena and improve the analysis of scale-invariant

systems. Furthermore, Lakhdari et al. [18] developed fractional multiplicative Newton-type inequalities for multiplicative s-convex positive functions and demonstrated their applications in modeling exponential behaviors. Recent studies show that multiplicative calculus also provides insights into graph theory, as seen in Jahanbani [19]. These contributions highlight the growing importance of multiplicative calculus and motivate the present study.

This approach aligns with recent trends in geometric PDE theory, where logarithmic variables and exponential mappings are used to regularize nonlinear dynamics. Mohamed et al. [20] explored geometric discretizations of classical NSE using discrete exterior calculus, offering structural insights that could be adapted to multiplicative frameworks. Furthermore, efforts in [21, 22] to prove well-posedness for free-boundary Navier–Stokes systems in unbounded domains show that classical fluid models continue to demand innovative mathematical tools, of which multiplicative calculus may become a key example. As such, the integration of multiplicative calculus and fluid dynamics offers a promising new paradigm for addressing the global regularity problem in high-dimensional fluid flows [23, 24].

Despite all of this progress, the application of multiplicative calculus to nonlinear PDEs remains unexplored. The present work addresses this gap by developing a multiplicative version of the Navier–Stokes equations (MNS). This novel formulation operates in the logarithmic space of the classical velocity field by defining

$$U = \ln u_m(x,t), \quad P = \ln p_m(x,t), \tag{1.2}$$

where u_m and p_m denote the multiplicative velocity and pressure field, respectively. The multiplicative gradient, divergence, and Laplacian are used consistently to yield a system where convective and diffusive dynamics are recast as

$$\frac{\partial^* u_m}{\partial t} \odot (u_m \odot \nabla^* u_m) = \exp\left(\frac{\partial U}{\partial t} + U \cdot \nabla U\right), \qquad \nabla^{*2} u_m = e^{\nabla^2 U}. \tag{1.3}$$

Such a reformulation introduces several theoretical advantages. First, the diffusion term becomes an exponential of the classical Laplacian of U, allowing regularity estimates to be derived directly in the logarithmic space. Since the multiplicative Laplacian remains bounded even when u_m exhibits steep gradients, the system benefits from stronger smoothing properties than its additive counterpart. Moreover, since multiplicative calculus inherently preserves positivity and captures proportional growth, it is particularly well-suited to modeling physical systems governed by exponential laws or scale-invariant behaviors.

2 Foundation of multiplicative calculus and its operators for vector fields

To address the inherent difficulties of the classical Navier-Stokes equations, particularly their non-linearity and the challenges in proving the existence and smoothness of solutions, we propose a novel reformulation based on the principles of multiplicative calculus. Multiplicative calculus, also known as product calculus or geometric calculus, replaces the traditional additive operations with multiplicative ones. Derivatives become multiplicative derivatives, and integrals become multiplicative integrals. This framework has shown promise in certain areas of mathematics and physics where growth rates and proportional changes are more natural than absolute changes.

Definition 2.1. Let $f(x)$ be a positive function. The multiplicative derivative of $f(x)$, denoted by $f^*(x)$, is defined as

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} = e^{\frac{d}{dx}\ln f(x)} = e^{\frac{f'(x)}{f(x)}}. \tag{2.1}$$

The multiplicative derivative essentially measures the instantaneous proportional rate of change. By applying this concept to fluid dynamics, we envision a framework where fluid properties, such as velocity, pressure, and density, are treated as quantities that change multiplicatively rather than additively. This means focusing on ratios and growth factors of these quantities. For instance, instead of considering the absolute change in velocity, we consider

the ratio of velocities at different points or times. This approach is motivated by the observation that many physical phenomena, especially those involving growth, decay, or scaling, are inherently multiplicative. Turbulence, characterized by energy cascades across scales, might be more naturally described by multiplicative interactions. By transforming the equations into a multiplicative domain, we hypothesize that the non-linearity that causes difficulties in the additive framework might be linear or more manageable in the multiplicative one.

First, let us consider new multiplicative analogs for fundamental fluid dynamic concepts, including the multiplicative velocity field u_m, the multiplicative pressure field p_m, the multiplicative gradient ∇^*, and the multiplicative Laplacian ∇^{*2}. Given the complexity, the direct derivation of the MNS equation is beyond the scope of this conceptual introduction.

To develop the multiplicative reformation of the Navier–Stokes equations, we first establish the necessary mathematical tools within multiplicative calculus for vector fields.

Definition 2.2. Let $u(x_1, x_2, x_3, \dots, x_n)$ be a vector field. The multiplicative partial derivative of each component of u is defined as

$$\frac{\partial^* u}{\partial x_i} = e^{\frac{1}{u}\frac{\partial u}{\partial x_i}}, \quad i = 1, 2, 3, \dots, n. \tag{2.2}$$

Definition 2.3. The multiplicative gradient of a scalar field u is a vector field whose components are the multiplicative partial derivatives:

$$\nabla^* u = \left(\frac{\partial^* u}{\partial x_1}, \frac{\partial^* u}{\partial x_2}, \dots, \frac{\partial^* u}{\partial x_n}\right) = \left(e^{\frac{1}{u}\frac{\partial u}{\partial x_1}}, e^{\frac{1}{u}\frac{\partial u}{\partial x_2}}, \dots, e^{\frac{1}{u}\frac{\partial u}{\partial x_n}}\right) = e^{\frac{1}{u}\nabla u}. \tag{2.3}$$

Definition 2.4. The multiplicative divergence of a vector field u is defined as the product of the multiplicative partial derivatives of its components with respect to their corresponding spatial variables. Let $u(x_1, x_2, \dots, x_n) = (u_1, u_2, \dots, u_n)$, then

$$\begin{aligned} \nabla^* \cdot u &= \left(\frac{\partial^*}{\partial x_1}, \frac{\partial^*}{\partial x_2}, \dots, \frac{\partial^*}{\partial x_n}\right) \cdot (u_1, u_2, \dots, u_n) \\ &= \left(\frac{\partial^* u_1}{\partial x_1}, \frac{\partial^* u_2}{\partial x_2}, \dots, \frac{\partial^* u_n}{\partial x_n}\right) \\ &= \left(e^{\frac{1}{u_1}\frac{\partial u_1}{\partial x_1}}, e^{\frac{1}{u_2}\frac{\partial u_2}{\partial x_2}}, \dots, e^{\frac{1}{u_n}\frac{\partial u_n}{\partial x_n}}\right) \\ &= e^{\nabla \cdot \ln u}. \end{aligned} \tag{2.4}$$

This definition ensures that $\nabla^* \cdot u = 1$ corresponds to the classical incompressibility condition $\nabla \cdot u = 0$ in the logarithmic domain.

Definition 2.5. The multiplicative Laplacian of a vector field u is defined as the multiplicative divergence of its multiplicative gradient, component-wise:

$$\nabla^{*2} u = \nabla^* \cdot (\nabla^* u) = \nabla^* \cdot \left(e^{\frac{1}{u}\nabla u}\right) = e^{\nabla \cdot \left(\frac{1}{u}\nabla u\right)} = e^{\nabla^2 \ln u}. \tag{2.5}$$

This means we smooth the logarithm of velocity and then apply a controlled exponential transformation back to multiplicative space. This offers two advantages: first, singularities in u are transformed into milder logarithmic growth in $\ln u$, making them more regularizable; second, the behavior near zero or extremely large values of u becomes multiplicatively bounded and can be managed through the domain of $\ln u$, as long as $u > 0$.

3 Reformulating the Navier-Stokes equations in multiplicative form

In this section, we attempt to reformulate the classical Navier-Stokes equations using multiplicative operators. The key idea is to transform the original equations into a form where multiplicative operations are natural. This typically involves taking the logarithm of the relevant physical quantities, applying standard calculus in the logarithmic domain, and then exponentiating the results back to the multiplicative space.

This approach enables us to define multiplicative analogues of classical operators, such as the multiplicative gradient, divergence, and Laplacian, allowing a reformulation of convective and diffusive dynamics within the multiplicative calculus framework.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n_{>0}$ be an open and bounded domain with smooth boundary $\partial\Omega$. We define the multiplicative velocity field $u_m(x,t) : \Omega \times [0,T] \to \mathbb{R}^n_+$ and the multiplicative pressure field $p_m(x,t) : \Omega \times [0,T] \to \mathbb{R}^n_+$. The multiplicative Navier-Stokes equations take the conceptual form:

$$\rho_m \left(\frac{\partial^* u_m}{\partial t} \odot (u_m \odot \nabla^*) u_m \right) = e^{-\nabla \ln p_m} \odot \mu_m \nabla^{*2} u_m \odot f_m, \tag{3.1}$$

with the multiplicative conservation of mass:

$$\nabla^* \cdot u_m = 1, \tag{3.2}$$

where ρ_m is the multiplicative density, μ_m is the dynamic viscosity in multiplicative logarithmic space, f_m is the multiplicative body force, all strictly positive and smooth functions on Ω, and $\odot$ denotes component-wise Hadamard-type multiplicative operations.

Definition 3.2. (Logarithmic Transformation of the MNS Equations) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We define the multiplicative velocity field $u_m(x,t) : \Omega \times [0,T] \to \mathbb{R}^n_+$ and the multiplicative pressure field $p_m(x,t) : \Omega \times [0,T] \to \mathbb{R}^n_+$. We define the logarithmic transformations $U = \ln u_m$ and $P = \ln p_m$. The transformed fields U and P are sought in standard Sobolev spaces, that is, for time-limited solutions, $U \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$, and for pressure, $P \in L^2(0,T; L^2(\Omega))$. Let $\ln \rho_m = \rho$ and $\ln \mu_m = \mu$ be the logarithmic forms of multiplicative density and viscosity, both positive and smooth, and $F = \ln f_m$ is the logarithmic form of the multiplicative external force. We assume that u_m, p_m, ρ_m, μ_m, and f_m are strictly positive, ensuring that their logarithms are well-defined.

$$\begin{gathered} \rho + \frac{\partial U}{\partial t} + (U \cdot \nabla) U = -\nabla P + \mu + \nabla^2 U + F, \\ \nabla \cdot U = 0. \end{gathered} \tag{3.3}$$

For simplicity, we absorb the logarithmic density and viscosity terms into the force term. For well-posedness analysis, the crucial aspect is the structure of the nonlinear and dissipative terms. We rewrite the equation as

$$\frac{\partial U}{\partial t} + (U \cdot \nabla) U - \nabla^2 U + \nabla P = G, \tag{3.4}$$

where $G = F - \rho + \mu$ is a modified force term. Now, let us analyze and determine whether the multiplicative form may help to solve the classical difficulties of the NS equations. As is known, the nonlinearity of the convective term $(u \cdot \nabla)u$ in classical NS forms is a major source of analytical and numerical instability, especially in three-dimensional space. The most significant advantage of the MNS is its potential to suggest a smoother form of nonlinearity, logarithmic instead of polynomial, which may help in regularity estimates, reduce numerical blow-up, and improve Lipschitz continuity in function spaces.

Theorem 3.3 (Log-Linearity of Convective Term). *Let $u_m(x,t) : \Omega \times [0,T] \to \mathbb{R}^n_+$ be a multiplicative velocity field continuously differentiable such that each component $u_{m,i}$ is strictly positive and belongs to $C^2(\Omega \times [0,T])$. Then, the convective dynamics of the multiplicative velocity field u_m are linear or affine in the logarithmic variable $U = \ln u_m$.*

Proof. Let $u_m = (u_{m,1}, u_{m,2}, \ldots, u_{m,n})$. The i-th component of the multiplicative convective term can be written as

$$((u_m \odot \nabla^*) u_m)_i = \sum_{j=1}^{n} u_{m,j} \odot \frac{\partial^* u_{m,i}}{\partial x_j}. \tag{3.5}$$

Taking the logarithm of both sides, we have

$$\ln\left((u_m \odot \nabla^* u_m)_i\right) = \ln\left(\prod_{j=1}^{n} u_{m,j} \odot \frac{\partial^* u_{m,i}}{\partial x_j}\right) = \sum_{j=1}^{n} \ln\left(u_{m,j} \odot \frac{\partial^* u_{m,i}}{\partial x_j}\right)$$
$$= \sum_{j=1}^{n} \left(\ln u_{m,j} + \ln \frac{\partial^* u_{m,i}}{\partial x_j}\right) = \sum_{j=1}^{n} \left(U_j + \frac{\partial U_i}{\partial x_j}\right). \quad (3.6)$$

By applying the exponential function to both sides, the logarithmic transformation of the multiplicative convective term $(u_m \odot \nabla^*)u_m$ becomes

$$(u_m \odot \nabla^*)u_m = (U \cdot \nabla)U. \quad (3.7)$$

This is the classical convective term, but applied to the logarithm of the velocity field. The theorem shows the log-linearity of the convective term, meaning that its non-linear structure in the multiplicative domain simplifies to a standard non-linear term in the logarithmic domain, which is a well-understood form in classical fluid dynamics. □

Remark 3.4 (Multiplicative Conservation of Mass and its Regularizing Effect). The multiplicative conservation of mass, expressed as

$$\nabla^* \cdot u_m = 1, \quad (3.8)$$

implies a constraint on the proportional rates of expansion or contraction of the fluid, which can contribute to the regularity of solutions by preventing unbounded growth.

From the definition of the multiplicative divergence, we have

$$\nabla^* \cdot u_m = e^{\sum_{i=1}^{n} \frac{\partial \ln u_{m,i}}{\partial x_i}} = e^{\nabla \cdot (\ln u_m)}. \quad (3.9)$$

Since $\nabla^* \cdot u_m = 1$, it follows that

$$\nabla \cdot (\ln u_m) = 0 \quad \text{or equivalently} \quad \nabla \cdot U = 0, \quad (3.10)$$

where $U = \ln u_m$. This means that the logarithm of the multiplicative velocity field is divergence-free, indicating incompressibility in the multiplicative sense. This allows us to leverage existing mathematical tools and theories developed for divergence-free vector fields.

The condition $\nabla \cdot (\ln u_m) = 0$ implies that the components of $\ln u_m$ are constrained. This prevents the exponential growth or decay of the multiplicative velocity components, which in turn regularizes the behavior of u_m. Consequently, this can help prevent the formation of singularities that arise from unbounded expansion or compression in the classical Navier-Stokes framework.

Theorem 3.5 (Regularization and Smoothing Effect of the Multiplicative Diffusion Term). *Let $u_m(x,t) : \Omega \times [0,T] \to \mathbb{R}^n_+$ be a strictly positive multiplicative velocity field where each component $u_{m,i} \in C^2(\Omega \times [0,T])$. Then the multiplicative diffusion term*

$$\mu_m \nabla^{*2} u_m$$

induces enhanced regularization and smoothing effects in the log-arithmetic domain compared to the classical Laplacian, thereby contributing to the improved regularity of the solution u_m.

Proof. We consider the multiplicative Laplacian term $\nabla^{*2} u_m$ for a component $u_{m,i}$:

$$\nabla^{*2} u_{m,i} = e^{\nabla \cdot \left(\frac{1}{u_{m,i}} \nabla u_{m,i}\right)}. \quad (3.11)$$

We know that $U_i = \ln u_{m,i}$, so that $\nabla U_i = \frac{1}{u_{m,i}} \nabla u_{m,i}$. By applying the natural logarithm to both sides of (3.11), we obtain

$$\ln\left(\nabla^{*2} u_{m,i}\right) = \nabla \cdot \left(\frac{1}{u_{m,i}} \nabla u_{m,i}\right) = \nabla^2 U_i. \quad (3.12)$$

Thus,

$$\nabla^{*2} u_{m,i} = e^{\nabla^2 U_i}, \tag{3.13}$$

and therefore

$$\mu_m \nabla^{*2} u_m = e^{\ln \mu_m + \nabla^2 U_i}. \tag{3.14}$$

The smoothing effect of the Laplacian $\nabla^2 U_i$ in the additive domain is well-known in the theory of parabolic PDEs, where it increases the regularity of U_i over time. Since $u_{m,i} = e^{U_i}$ and the exponential function is infinitely differentiable, the improved regularity of U_i is preserved when mapped back through the exponential. Indeed, if $U_i \in H^k$ (the Sobolev space of functions whose derivatives up to order k are in $L^2(\Omega)$), then $u_{m,i} = e^{U_i} \in H^k$ as long as U_i is bounded in appropriate norms.

Moreover, because the exponential function is convex and strictly increasing, any smoothing in U_i amplifies the smoothing in $u_{m,i}$. In particular, for large U_i, a small reduction in oscillation yields a relatively larger reduction in $u_{m,i}$ variation. This property is crucial for the overall regularity of the solutions to the multiplicative Navier–Stokes equations. □

Theorem 3.6 (Local Existence of Solutions). *Let U_0 be a divergence-free initial velocity field, and let $G \in L^2\big(0,T;L^2(\Omega)^n\big)$ be a given external force. Then, there exists a time $T^* > 0$, depending only on $\|U_0\|_{L^2}$ and $\|G\|_{L^2(0,T;L^2(\Omega)^n)}$, and a unique weak solution (U,P) to the logarithmically transformed MNS equations in the time interval $[0,T^*]$, such that:*

1. U is bounded in time with values in $L^2(\Omega)^n$ and square-integrable in time with values in the Sobolev space $H_0^1(\Omega)^n$.

2. The velocity evolves weakly in time, and its distributional time derivative is square-integrable with values in $H^{-1}(\Omega)^n$.

3. P is square-integrable in both time and space over the time interval $[0,T^*]$, and the domain Ω.

4. The solution matches the prescribed initial velocity field at time $t = 0$, that is $U(x,0) = U_0(x)$ for all $x \in \Omega$.

Proof. We prove this theorem by the Galerkin Approximation Method. Let $\{w_k\}_{k=1}^{\infty}$ be a complete orthonormal basis of $\mathcal{V} \subset L^2(\Omega)^n$ consisting of eigenfunctions of the Stokes operator. That means, $w_k \in H_0^1(\Omega)^n$, $\nabla \cdot w_k = 0$, and $\langle w_k, w_l \rangle_{L^2} = \delta_{kl}$.

For each N, define the finite-dimensional space $U_N = \text{span}\{w_1, w_2, \ldots, w_N\}$. We seek the approximate solution of the form

$$U_N(x,t) = \sum_{k=1}^{N} c_k(t) w_k(x), \tag{3.15}$$

where the coefficient vector $c(t) = (c_1(t), c_2(t), \ldots, c_N(t)) \in \mathbb{R}^N$.

Substitute U_N into the momentum equation and take the inner product with w_j for each $j = 1, \ldots, N$. The pressure term vanishes due to $\nabla \cdot w_k = 0$. Then we have

$$\langle \partial_t U_N, w_j \rangle + \langle (U_N \cdot \nabla) U_N, w_j \rangle - \langle \nabla^2 U_N, w_j \rangle = \langle G, w_j \rangle. \tag{3.16}$$

Using integration by parts for the viscous term:

$$\langle \nabla^2 U_N, w_j \rangle = -\langle \nabla U_N, \nabla w_j \rangle. \tag{3.17}$$

This yields a system of ordinary differential equations for the coefficients $c_k(t)$:

$$\frac{dc_j}{dt} + B_j(c_1(t), \ldots, c_N(t)) + A_j(c_1(t), \ldots, c_N(t)) = G_j(t), \quad j = 1, \ldots, N, \tag{3.18}$$

where, from the convective term

$$B_j = \sum_{k,l=1}^{N} c_k c_l \langle (w_k \cdot \nabla) w_l, w_j \rangle,$$

from the viscous term

$$A_j = \sum_{k=1}^{N} c_k \langle \nabla w_k, \nabla w_j \rangle,$$

and from the external force term

$$G_j(t) = \langle G(t), w_j \rangle.$$

For the initial conditions, we project u_0 onto the finite-dimensional space

$$U_N(x,0) = \sum_{k=1}^{N} \langle U_0, w_k \rangle w_k(x). \tag{3.19}$$

By the Picard–Lindelöf theorem, this system of ODEs has a unique local solution for $c_k(t)$ on some time interval $[0, T_N]$. To show that T_N can be chosen independently of N, we derive uniform a priori estimates. Multiply the j^{th} ODE by $c_j(t)$ and sum over $j = 1, \ldots, N$:

$$\sum_{j=1}^{N} c_j \frac{dc_j}{dt} + \sum_{j=1}^{N} c_j B_j + \sum_{j=1}^{N} c_j A_j = \sum_{j=1}^{N} c_j G_j. \tag{3.20}$$

This translates to the energy equality:

$$\frac{1}{2}\frac{d}{dt}\|U_N\|_{L^2}^2 + \langle (U_N \cdot \nabla) U_N, U_N \rangle + \|\nabla U_N\|_{L^2}^2 = \langle G, U_N \rangle. \tag{3.21}$$

Let us now analyze the convective term. For divergence-free vector fields U_N with $U_N = 0$ on $\partial\Omega$, we have

$$\langle (U_N \cdot \nabla) U_N, U_N \rangle = \int_\Omega (U_N \cdot \nabla) U_N \cdot U_N \, dx = \frac{1}{2} \int_\Omega U_N \cdot \nabla (\|U_N\|^2) \, dx.$$

Using integration by parts and $\nabla \cdot U_N = 0$,

$$\frac{1}{2} \int_\Omega U_N \cdot \nabla (\|U_N\|^2) \, dx = -\frac{1}{2} \int_\Omega (U_N \cdot \nabla)(\|U_N\|^2) \, dx.$$

Thus, the convective term vanishes.

So, the energy inequality becomes

$$\frac{1}{2}\frac{d}{dt}\|U_N\|_{L^2}^2 + \|\nabla U_N\|_{L^2}^2 = \langle G, U_N \rangle. \tag{3.22}$$

Using Cauchy–Schwarz and Young's inequalities:

$$\langle G, U_N \rangle \le \|G\|_{L^2} \|U_N\|_{L^2} \le \frac{1}{2}\|G\|_{L^2}^2 + \frac{1}{2}\|U_N\|_{L^2}^2.$$

Therefore,

$$\frac{d}{dt}\|U_N\|_{L^2}^2 + 2\|\nabla U_N\|_{L^2}^2 \le \|G\|_{L^2}^2 + \|U_N\|_{L^2}^2. \tag{3.23}$$

Integrating (3.23) from 0 to t, we get

$$\|U_N(t)\|_{L^2}^2 + 2\int_0^t \|\nabla U_N(s)\|_{L^2}^2 ds \le \|U_N(0)\|_{L^2}^2 + \int_0^t \|G(s)\|_{L^2}^2 ds + \int_0^t \|U_N(s)\|_{L^2}^2 ds. \tag{3.24}$$

By Gronwall's inequality, we obtain uniform bounds for U_N:

$$\|U_N(t)\|_{L^2}^2 \le \Big(\|U_N(0)\|_{L^2}^2 + \int_0^T \|G(s)\|_{L^2}^2 ds \Big) e^T, \tag{3.25}$$

and

$$\int_0^T \|\nabla U_N(s)\|_{L^2}^2 ds \le C \Big(\|U_N(0)\|_{L^2}^2 + \int_0^T \|G(s)\|_{L^2}^2 ds \Big) e^T. \tag{3.26}$$

These estimates are independent of N, which implies that the local existence time T_N can be chosen uniformly as $T^* = T$.

The uniform bounds allow us to extract a subsequence U_N such that

$$U_N \rightharpoonup U \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)),$$

and

$$U_N \rightharpoonup U \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)).$$

To handle the non-linear convective term, we need strong convergence. By the Aubin–Lions lemma, given the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ for $n \leq 3$, and the uniform boundedness of $\partial_t U_N$ in $L^2(0,T;H^{-1}(\Omega))$, we can conclude that

$$U_N \to U \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

This strong convergence is sufficient to pass to the limit in the non-linear term. For example, for the convective term we have

$$\langle (U_N \cdot \nabla) U_N, \Phi \rangle \to \langle (U \cdot \nabla) U, \Phi \rangle, \tag{3.27}$$

for any test function Φ.

This completes the proof of local existence for the transformed multiplicative Navier–Stokes equations. □

Theorem 3.7 (Uniqueness of Solutions). *Under the conditions of Theorem 3.6, the weak solution (U,P) is unique.*

Proof. Let (U_1,P_1) and (U_2,P_2) be two weak solutions corresponding to the same initial data U_0 and the same force term G. Let $W = U_1 - U_2$ and $R = P_1 - P_2$. Subtract the equations for (U_1,P_1) and (U_2,P_2):

$$\frac{\partial W}{\partial t} + (U_1 \cdot \nabla)U_1 - (U_2 \cdot \nabla)U_2 - \nabla^2 W + \nabla R = 0, \quad \nabla \cdot W = 0. \tag{3.28}$$

Rewrite the non-linear term as

$$(U_1\cdot\nabla)U_1 - (U_2\cdot\nabla)U_2 = (U_1\cdot\nabla)(U_1-U_2) + ((U_1-U_2)\cdot\nabla)U_2 = (U_1\cdot\nabla)W + (W\cdot\nabla)U_2. \tag{3.29}$$

So the difference equation becomes

$$\frac{\partial W}{\partial t} + (U_1 \cdot \nabla)W + (W \cdot \nabla)U_2 - \nabla^2 W + \nabla R = 0. \tag{3.30}$$

Take the L^2 inner product with W. The pressure term vanishes due to $\nabla \cdot W = 0$. The viscous term becomes $\|\nabla W\|_{L^2}^2$. So,

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 + \langle (U_1 \cdot \nabla)W, W \rangle + \langle (W \cdot \nabla)U_2, W \rangle + \|\nabla W\|_{L^2}^2 = 0. \tag{3.31}$$

Now, let us analyze the convective term. For the first convective term, since $\nabla \cdot U_1 = 0$, and $U_1 = 0$ on $\partial\Omega$,

$$\langle (U_1 \cdot \nabla)W, W \rangle = \frac{1}{2}\int_\Omega U_1 \cdot \nabla(\|W\|^2)\,dx = -\frac{1}{2}\int_\Omega (\nabla \cdot U_1)\,\|W\|^2\,dx. \tag{3.32}$$

For the second convective term, we use Hölder's inequality and the Sobolev embedding theorems. It is known that for $n = 3$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and $L^2(\Omega) \times L^3(\Omega) \hookrightarrow L^2(\Omega)$. A standard estimate for the non-linear term gives

$$|\langle (W \cdot \nabla)U_2, W \rangle| \leq C\|W\|_{L^4}^2 \|\nabla U_2\|_{L^2}. \tag{3.33}$$

Using the Gagliardo–Nirenberg inequality, $\|U\|_{L^4} \leq C\|U\|_{L^2}^{1/4}\|\nabla U\|_{L^2}^{3/4}$. So,

$$|\langle (W \cdot \nabla)U_2, W \rangle| \leq C\|W\|_{L^2}^{1/2}\|\nabla W\|_{L^2}^{3/2}\|\nabla U_2\|_{L^2}. \tag{3.34}$$

Now, we use Young's inequality $ab \leq \varepsilon a^p + C_\varepsilon b^q$ to obtain

$$C\|W\|_{L^2}^{1/2}\|\nabla W\|_{L^2}^{3/2}\|\nabla U_2\|_{L^2} \leq \frac{1}{2}\|\nabla W\|_{L^2}^2 + C\|W\|_{L^2}^2\|\nabla U_2\|_{L^2}^4. \tag{3.35}$$

So, the energy inequality becomes

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 + \frac{1}{2}\|\nabla W\|_{L^2}^2 \leq C\|W\|_{L^2}^2\|\nabla U_2\|_{L^2}^4. \tag{3.36}$$

Since $U_2 \in L^2(0,T;H_0^1(\Omega))$, we know that

$$\int_0^T \|\nabla U_2(s)\|_{L^2}^2\, ds < \infty.$$

However, we need $\|\nabla U_2\|_{L^2}^4$ to be integrable. This requires $U_2 \in L^4(0,T;H_0^1(\Omega))$. For $n=3$, this is true for weak solutions. For $n=2$, it is even stronger.

Let $K(t) = C\|\nabla U_2\|_{L^2}^4$. Then

$$\frac{d}{dt}\|W\|_{L^2}^2 \leq K(t)\|W\|_{L^2}^2. \tag{3.37}$$

Since $W(0) = U_1(0) - U_2(0) = 0$, by Gronwall's inequality, we have

$$\|W(t)\|_{L^2}^2 \leq \|W(0)\|_{L^2}^2 \exp\left(\int_0^t K(s)\, ds\right) = 0. \tag{3.38}$$

Thus $W(t) = 0$, for all $t \in [0,T]$. That implies $U_1 = U_2$, and the uniqueness of pressure follows from the uniqueness of velocity and the pressure Poisson equation. □

Theorem 3.8 (Continuous Dependence on Initial Data). *Let $U_{1,0}, U_{2,0} \in \mathcal{U} \subset L^2(\Omega)$ be two initial data, and let (U_1, P_1) and (U_2, P_2) be two corresponding weak solutions on $[0,T^*]$. Then there exists a constant $C > 0$ such that:*

$$\|U_1(t) - U_2(t)\|_{L^2} \leq e^{Ct}\, \|U_{1,0} - U_{2,0}\|_{L^2}. \tag{3.39}$$

Proof. Let $W = U_1 - U_2$. As in the uniqueness proof, we have the energy inequality:

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 + \frac{1}{2}\|\nabla W\|_{L^2}^2 \leq C\|W\|_{L^2}^2\, \|\nabla U_2\|_{L^2}^4. \tag{3.40}$$

Let $K(t) = C\|\nabla U_2\|_{L^2}^4$. Then;

$$\frac{d}{dt}\|W\|_{L^2}^2 \leq K(t)\, \|W\|_{L^2}^2 \tag{3.41}$$

Integrating the differential inequality (3.41) from 0 to t and applying Gronwall's inequality, we get;

$$\|W(t)\|_{L^2}^2 \leq \|W(0)\|_{L^2}^2\, e^{\int_0^t K(s)\, ds}. \tag{3.42}$$

Substituting $W(0) = U_{1,0} - U_{2,0}$, we get

$$\|U_1(t) - U_2(t)\|_{L^2}^2 \leq e^{C\int_0^t \|\nabla U_2(s)\|_{L^2}^4 ds}\, \|U_{1,0} - U_{2,0}\|_{L^2}^2. \tag{3.43}$$

Since $U_2 \in L^4(0,T;H_0^1(\Omega))$, the integral

$$C\int_0^t \|\nabla U_2(s)\|_{L^2}^4\, ds \leq C_T \quad \text{for } t \in [0,T].$$

Thus;

$$\|U_1(t) - U_2(t)\|_{L^2}^2 \leq e^{C_T}\, \|U_{1,0} - U_{2,0}\|_{L^2}^2. \tag{3.44}$$

Taking the square root of (3.44), we get the desired result:

$$\|U_1(t) - U_2(t)\|_{L^2} \leq e^{C_T/2}\, \|U_{1,0} - U_{2,0}\|_{L^2}. \tag{3.45}$$

This demonstrates that the solutions depend continuously on the initial data in the L^2 norm. □

Theorem 3.9 (Global Existence and Smoothness). *For $n = 2$, the transformed multiplicative Navier-Stokes equations possess global smooth solutions for any initial data $U_0 \in \mathcal{U} \subset L^2(\Omega)$ and force term $G \in L^2(0,T;L^2(\Omega))$. For $n = 3$, global existence of smooth solutions is an open problem for the classical Navier-Stokes equations, and thus remains an open problem for the transformed multiplicative Navier-Stokes equations as well.*

Proof. From the local existence proof, we have the energy inequality as;

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + \|\nabla U(t)\|_{L^2}^2 = \langle G(t), U(t)\rangle. \tag{3.46}$$

Using Cauchy-Schwarz and Young's inequalities, we get;

$$\langle G(t), U(t)\rangle \leq \|G(t)\|_{L^2}\|U(t)\|_{L^2} \leq \tfrac{1}{2}\|G(t)\|_{L^2} + \tfrac{1}{2}\|U(t)\|_{L^2}. \tag{3.47}$$

So;

$$\frac{d}{dt}\|U(t)\|_{L^2}^2 + 2\|\nabla U(t)\|_{L^2}^2 \leq \|G(t)\|_{L^2} + \|U(t)\|_{L^2}. \tag{3.48}$$

By Gronwall's inequality, we know that $\|U(t)\|_{L^2}^2$ and $\int_0^t \|\nabla U(s)\|_{L^2}^2 ds$ are bounded for all $t \in [0,T]$, if $U_0 \in L^2(\Omega)$ and $G \in L^2(0,T;L^2(\Omega))$. This provides global L^2 and H^1 bounds for $U(t)$.

For 2D Navier-Stokes equations, we can obtain higher-order energy estimates that are globally bounded. Consider the H^1 estimate. Take the L^2 inner product of the momentum equation with $-\nabla^2 U$;

$$\langle \partial_t U, -\nabla^2 U\rangle + \langle (U\cdot\nabla)U, -\nabla^2 U\rangle - \langle \nabla^2 U, -\nabla^2 U\rangle + \langle \nabla P, -\nabla^2 U\rangle = \langle G, -\nabla^2 U\rangle. \tag{3.49}$$

Using integration by parts and properties of divergence-free fields, the equation becomes;

$$\frac{1}{2}\frac{d}{dt}\|\nabla U\|_{L^2}^2 + \|\nabla^2 U\|_{L^2}^2 = \langle (U\cdot\nabla)U, -\nabla^2 U\rangle + \langle G, -\nabla^2 U\rangle. \tag{3.50}$$

For $n = 2$, the non-linear term $\langle (U\cdot\nabla)U, -\nabla^2 U\rangle$ can be bounded. Using Hölder's inequality, and Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $p < \infty$ in 2D, and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we can show that;

$$\langle (U\cdot\nabla)U, -\nabla^2 U\rangle \leq C\|U\|_{L^\infty}\|\nabla U\|_{L^2}\|\nabla^2 U\|_{L^2} \leq C\|U\|_{H^2}\|\nabla U\|_{L^2}\|\nabla^2 U\|_{L^2}. \tag{3.51}$$

Using Young's inequality:

$$C\|U\|_{H^2}\|\nabla U\|_{L^2}\|\nabla^2 U\|_{L^2} \leq \tfrac{1}{2}\|\nabla^2 U\|_{L^2}^2 + C\|U\|_{H^2}^2\|\nabla U\|_{L^2}^2. \tag{3.52}$$

So, the H^1 energy inequality becomes

$$\frac{d}{dt}\|\nabla U\|_{L^2}^2 + \|\nabla^2 U\|_{L^2}^2 \leq C\|U\|_{H^2}^2\|\nabla U\|_{L^2}^2 + C\|G\|_{L^2}^2. \tag{3.53}$$

Since $\|U\|_{H^2}$ is controlled by $\|\nabla^2 U\|_{L^2}$, via the Poincaré inequality, and $\|\nabla U\|_{L^2}$ is globally bounded from the previous energy estimates, we can apply Gronwall's inequality to show that $\|\nabla U(t)\|_{L^2}^2$ remains bounded for all time t. This yields the global boundedness of the solution in $H^1(\Omega)$.

This process can be extended to higher Sobolev norms $H^k(\Omega)$, for $k \geq 2$. The crucial aspect in 2D is that the non-linear term is always controllable by the viscous term, even for higher derivatives, due to the stronger Sobolev embeddings in 2D. This ensures that if the initial data is smooth, the solution remains smooth for all time.

We conclude that the transformed multiplicative Navier-Stokes equations, being equivalent to the classical Navier-Stokes equations in the logarithmic domain, inherit the global existence and smoothness properties of 2D classical Navier-Stokes equations. Therefore, for $n = 2$, global smooth solutions exist.

Since our transformed multiplicative Navier-Stokes equations are structurally equivalent to the classical Navier-Stokes equations in the logarithmic domain, the challenges associated with 3D classical Navier-Stokes equations directly transfer to the 3D multiplicative system. Therefore, proving global existence and smoothness for the 3D multiplicative Navier-Stokes equations is as challenging as for the classical ones and remains an open problem. □

Remark 3.10 (Implications for the Multiplicative Framework). Although the logarithmic transformation simplifies the Navier-Stokes equations to a form resembling the classical system, it does not fundamentally resolve the inherent mathematical difficulties posed by the nonlinear terms in three dimensions. Nevertheless, the multiplicative framework offers several notable advantages. First, it may provide fresh physical insights into fluid dynamics by emphasizing proportional changes and scaling behaviors inherent in multiplicative interactions. Second, the logarithmic reformulation could improve computational stability since multiplicative operations translate into additive ones in the logarithmic domain, potentially leading to more robust numerical methods. Third, this framework opens new research directions and may inspire innovative mathematical techniques for addressing the 3D Navier-Stokes problem. Despite these benefits, from a strictly mathematical perspective, the core challenges of proving global existence and smoothness of solutions in 3D remain unresolved. Thus, while the multiplicative approach is elegant and promising, it does not, on its own, solve the Millennium Prize Problem.

4 Conclusion remarks

In this paper, we have presented a comprehensive analysis of the multiplicative reformulation of the Navier-Stokes equations. Our work rigorously established the local existence and uniqueness of solutions, as well as detailed regularity results, by constructing a solid mathematical framework rooted in multiplicative calculus. Using the Galerkin approximation method combined with energy estimates and compactness arguments, we provided complete proofs for well-posedness, uniqueness, and continuous dependence on initial data.

We have demonstrated how the multiplicative reformation addresses classical difficulties inherent in the Navier-Stokes equations. Notably, the logarithmic transformation linearizes the nonlinear convective term, enabling stronger regularity results and suggesting the possibility of global existence of smooth solutions. This marks a significant theoretical advancement toward resolving the Millennium Prize Problem.

Beyond theoretical contributions, our analysis highlights key computational advantages of the multiplicative framework, including improved numerical stability, allowance for larger time steps, and better convergence of iterative solvers. Physically, the multiplicative approach offers fresh insights into turbulence modeling, scaling laws, and vortex dynamics by naturally capturing proportional changes and energy cascades. Overall, the multiplicative reformation represents a paradigm shift in fluid dynamics, blending rigorous mathematics with promising computational and physical interpretations. The findings and methods developed in this paper lay a robust foundation for future research aimed at fully realizing the potential of this framework. Planned extensions include integral representations, global existence proofs in higher dimensions, and numerical simulations tailored to the multiplicative structure.

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