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A STUDY OF ITERATIVE LEARNING CONTROL FOR RANDOM DIFFERENTIAL EQUATIONS WITH INCLUSIONS

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Abstract. This paper investigates iterative learning control (ILC) for a class of nonlinear random differential inclusion system (RDIS) involving set-valued random structures and uncertain evolution processes. By employing measurable selection theory and convex analytical techniques, the inclusion system is converted into a random differential formulation. The ILC laws are constructed to improve tracking performance over repeated iterations. Sufficient asymptotic convergence criteria are derived in the λ-norm concepts using Lipschitz conditions and Grönwall-type inequalities. Finally, the RDIS examples validate the theoretical findings and demonstrate the effectiveness of the proposed random iterative learning methodology.

1 Introduction

Random differential equations (RDEs) with inclusions constitute an important mathematical concepts for modeling uncertain systems involving randomness, multifunctions, and nonunique evolution processes. These systems generalize classical differential equations through random effects and set-valued mappings, leading to extensive applications in optimization, economics, control theory, engineering, and biological sciences. In this direction, Andres et al. (Refn. [5]) investigated random topological degree theory for random differential inclusions using nonlinear topological methods and contributed to measurable multifunction analysis in infinite-dimensional random systems. In a related development, Ivanovich (Refn. [25]) developed strong graphical laws for random semicontinuous mappings through probabilistic convergence methods. Furthermore, Papageorgiou (Refns. [36, 37]) established existence results for random differential inclusions and random fixed point concepts in Banach spaces. Raczynski (Refn. [42]) discussed uncertainty treatment in economic growth using differential inclusions. Aubin et al. (Refn. [7]) introduced foundational concepts of differential inclusion theory and advanced viability analysis for constrained nonlinear systems. Aumann (Refn. [8]) formulated integrals for set-valued functions through measurable multifunction techniques. Bharucha-Reid (Refn. [9]) established probabilistic fixed point principles for random systems. Ponosov (Refn. [38]) presented fixed-point approaches for stochastic equations and local operators. Smirnov (Refn. [48]) discussed theoretical foundations of differential inclusion systems. Moreover, De Blasi et al. (Refn. [17]) investigated nonconvex boundary value problems (BVPs), while contributed structural properties for nonlinear inclusion systems. Schilling (Refn. [45]) developed computational methods for BVPs in differential inclusions. Palimkar (Refns. [34, 35]) discussed theoretical aspects of random differential inclusions and contributed existence analysis for random multivalued systems. Caraballo et al. (Refn. [11]) analyzed random inclusion systems and investigated random evolution concepts, and clarified relationships between stochastic and

random differential inclusions. Finally, Zhang et al. (Refn. [60]) discussed random network coding through differential inclusion techniques, while contributed analytical concepts for random communication systems.

ILC is increasingly applied to RDEs and inclusions to handle uncertainties and randomness in system trajectories and initial conditions. ILC strategies seek to improve system performance over repeated operations by adjusting controls based on previous executions, which is critical for systems where random factors disturb the expected outcomes. Recent studies have proposed ILC strategies for high relative degree systems with random initial shifts, demonstrating that these controllers can achieve accurate tracking despite random disturbances, by properly selecting control gains and using difference equations in the control law. ILC is used in multi-agent systems, robotics, industrial process control, and networked control systems subject to random noise and variability. There is extensive literature available to understand ILC. Interested readers may refer to the following works. Uchiyama (Refn. [53]) introduced trial-based learning concepts for high-speed robotic motion, forming an early foundation of iterative improvement in repetitive control tasks. Arimoto et al. (Refn. [6]) introduced learning-based robotic control for repetitive operations and introduced mathematical foundations for iteration-based error compensation in robotic systems. Bien et al. (Refn. [10]) developed systematic analytical and design concepts for ILC with convergence-oriented methodologies. Furthermore, Chen et al. (Refn. [14]) provided rigorous convergence and robustness analysis for repetitive systems. Ahn et al. (Refns. [4, 32]) established ILC robustness and monotonic convergence conditions under system uncertainties. De Wijdeven et al. (Refn. [18]) strengthened monotonic convergence guarantees for uncertain systems. Moreover, Saab et al. (Refn. [44]) provided early support for adaptive tracking of slowly varying trajectories. In contrast, Chen et al. (Refn. [13]) addressed nonrepetitive uncertainty effects in iterative learning performance. Furthermore, Xu et al. (Refn. [58]) addressed time-varying uncertainty in tracking tasks, while Xu (Refn. [57]) further strengthened nonlinear ILC theory through a comprehensive survey of convergence and robustness results.

Advanced algorithms incorporate adaptation and robustness to address the challenges posed by random and uncertain environments typical in RDEs and inclusions. Despite notable advancements in ILC for systems governed by RDEs and for various forms of differential inclusions, there remains a marked lack of research explicitly addressing ILC for RDEs with inclusions. Existing literature with ILC for impulsive or stochastic systems and separately, with inclusions or random effects, but there are very limited works exploring the synergy of these two challenging aspects within a unified ideas. Recognizing this research gap, the present study is motivated to systematically develop the concept of ILC specifically tailored for RDEs with inclusions. In a related development, Chen et al. (Refn. [12]) studied ILC for high relative-degree discrete-time systems with random initial shifts using convergence and control-theoretic analysis, improving learning performance in nonlinear discrete-time settings. Qiu et al. (Refns. [39, 40]) investigated ILC for differential inclusion systems with random fading channels using averaging techniques, ensuring convergence and stability, and further studied convergence analysis of impulsive differential inclusion systems with randomly varying trial lengths. Moreover, Qiu et al. (Refn. [41]) investigated quantized ILC under data dropouts and communication constraints. Liu et al. (Refn. [30]) proposed ILC for multi-agent systems with random noises and measurement limitations, addressing convergence in the presence of communication constraints. Gu et al. (Refns. [21, 22]) developed D-type ILC for one-sided Lipschitz nonlinear systems, while further extending results to consensus problems in multi-agent concepts with convergence and coordination guarantees. In addition, Li et al. (Refns. [28, 29]) analyzed and developed ILC for nonlinear and linear systems with randomly varying trial lengths, establishing robustness and convergence. Finally, Yu et al. (Refn. [59]) extended D-type and norm-optimal ILC to high-speed train systems, achieving improved tracking performance and robustness in practical high-speed railway applications.

Current research on fractional differential equations, ILC with random impulses and functional differential structures focuses on modeling uncertainty, memory effects, and BVPs in infinite-dimensional Banach spaces. These concepts extend classical differential equations by incorporating fractional-order operators, random perturbations, delay effects, and set-valued or random influences, leading to rich analytical challenges in existence, stability, controllability, and numerical approximation of solution sets. In this direction, Li et al. (Refn. [27]) investigated second-order random impulsive differential equations RIDEs with BVPs using upper and lower solution techniques, establishing existence results and analytical conditions within impulsive

random concepts. In a related development, Ferhat (Refn. [19]) established existence results for coupled random fractional-order functional differential systems with infinite delay using fixed-point techniques and functional analysis, contributing to theoretical developments in systems with memory, delay, and stochastic coupling. Tamilarasi et al. (Refn. [52]) studied approximate controllability for fractional semilinear delay systems with random impulses, strengthening controllability theory in impulsive fractional concepts and second-order RIDEs using analytical techniques, establishing existence and qualitative results within impulsive stochastic differential concepts and extending theoretical understanding of random nonlinear evolution systems. In addition, Vivek et al. (Refn. [55]) developed ILC for Hilfer-type fractional quaternion-valued impulsive systems, ensuring convergence and stability in nonlinear fractional system. Abdulwasaa et al. (Refn. [2]) formulated fractal-fractional models for COVID-19 outbreak prediction, providing analytical forecasting concepts for epidemic dynamics. S. Ahmad et al. (Refn. [3]) analyzed nonlinear epidemic models for COVID-19 transmission dynamics. Furthermore, Shah et al. (Refns. [46, 47]) established Hyers-Ulam stability criteria for impulsive fractional Cauchy problems and collocation methods for fractional COVID-19 models, providing efficient numerical schemes for complex epidemic systems. In the existing literature, Vivek et al. (Refn. [54]) investigated convergence analysis for nonlinear impulsive pantograph equations involving Hilfer fractional derivatives under ILC strategy, while Vivek et al. (Refn. [56]) also developed a P-type ILC approach for impulsive pantograph equations and derived suitable convergence conditions. Gayathri et al. (Refn. [20]) developed an ILC concepts for nonlinear random impulsive systems with application to soft robotic actuators under random perturbations and impulsive phenomena. Moreover, Sunmitha et al. (Refn. [50]) investigated ILC for switched fractional systems with non-instantaneous impulses, while Sunmitha et al. (Refn. [51]) studied P-type ILC for quaternion-valued systems with applications to soft robotics. Finally, Louakar et al. (Refn. [31]) examined ILC for Hilfer-type fractional stochastic differential systems in robotic applications, addressing convergence and robustness in stochastic fractional control concepts.

Motivated by the aforementioned studies, we consider a nonlinear RDIS given by

$$\begin{cases} \mathscr{D}_t z_l(t,w) \in \overline{co}\mathscr{G}(t,\omega,z_l(t,\omega)) + \mathbb{M}v_l(t,\omega), t \in J := [0,T], \\ r_l(t,\omega) = \mathbb{N}z_l(t,\omega), \end{cases} \tag{1.1}$$

where $l \in N$ denotes the iteration index. The mapping $\mathscr{G} : J \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded set-valued mapping. Here $\overline{co}\mathscr{G}(t,\omega,z_l(t,\omega))$ denotes the closure of the convex hull of the set $\mathscr{G}(t,\omega,z_l(t,\omega))$. Moreover, $z_l(t,\omega) \in \mathbb{R}^n, v_l(t,\omega) \in \mathbb{R}^n$ and $r_l(t,\omega) \in \mathbb{R}^n$ represent the state, control input and output of system Eq. (1.1), respectively. The matrices $\mathbb{M}, \mathbb{N} \in \mathbb{R}^{n\times n}$ are given constant real matrices.

The desired output trajectory is denoted by $r_d(t,\omega) \in \mathbb{R}^n$, and the tracking error is defined as $e_l(t,\omega) := r_d(t,\omega) - r_l(t,\omega)$. A random process $z_l(\cdot,\omega) \in \mathbb{N}(J,\mathbb{R}^n,\Omega)$ is said to ba a solution of system Eq. (1.1) if there exists a measurable selection $g_l : J \times \Omega \to \mathbb{R}^n$ such that $g_l(t,\omega) \in \overline{co}\mathscr{G}(t,\omega,z_l(t,\omega))$ for almost every $t \in J$, (Refn. [34], Theorem 3.4.2 , Page 36) and the following integral equation holds for all $t \in J$,

$$z_l(t,\omega) = \alpha_0(\omega) + \int_0^t (g_l(s,\omega,z_l(s,\omega)) + \mathbb{M}v_l(s,\omega))ds, \tag{1.2}$$

where $\alpha_0(\omega)$ denotes the random initial condition.

For the system Eq. (1.1), we introduce the following ILC laws

(**L1**) D-type open-loop ILC law (Refn. [15], Eq. 7 , Page 3)

$$\begin{aligned} v_{l+1}(t,\omega) &= v_l(t,\omega) + S_d\dot{e}_l(t,\omega), \\ z_{l+1}(0,\omega) &= z_l(0,\omega) + \zeta e_l(0,\omega), \end{aligned} \tag{1.3}$$

where $S_d, \zeta \in \mathbb{R}^{n\times n}$ are constant learning gain matrices.
Without loss of generality, we assume $z_1(0,\omega) = 0$.

For convenience, define the incremental variables.

$$\begin{aligned} \Delta z_l(t,\omega) &:= z_{l+1}(t,\omega) - z_l(t,\omega), \\ \Delta v_l(t,\omega) &:= v_{l+1}(t,\omega) - v_l(t,\omega). \end{aligned}$$

(**L2**) PD-type open-loop ILC law (Refn. [23], Eq. 2, Page 1759)

$$\begin{aligned} v_{l+1}(t,\omega) &= v_l(t,\omega) + S_p e_l(t,\omega) + S_d \dot{e}_l(t,\omega), \\ z_{l+1}(0,\omega) &= z_l(0,\omega) + \zeta e_l(0,\omega), \end{aligned} \tag{1.4}$$

where S_p, S_d and $\zeta \in \mathbb{R}^{n\times n}$ are constant learning gain matrices.
(**L3**) PD-type closed-loop ILC law (Refn. [61], Section 3, Page 4506)

$$\begin{aligned} v_{l+1}(t,\omega) &= v_l(t,\omega) + S_p e_{l+1}(t,\omega) + S_d \dot{e}_{l+1}(t,\omega), \\ z_{l+1}(0,\omega) &= z_l(0,\omega) + \zeta e_l(0,\omega), \end{aligned} \tag{1.5}$$

where S_p, S_d and $\zeta \in \mathbb{R}^{n\times n}$ are constant learning gain matrices.

2 Prerequisites

We adopt the some ideas and auxillary results from (Refns. [1], Section 1, Page 129, [16], Section 2, Page 317, [43], Section 2, Page 79, and [49], Section 1, Page 539)
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space.
Let $\mu : \Omega \to \mathbb{R}^n$ be a random vector defined by

$$\mu(\omega) = \Big(\mu_1(\omega), \mu_2(\omega), ..., \mu_n(\omega)\Big)^T \in \mathbb{R}^n,$$

with the Euclidean norm

$$||\mu(\omega)|| = \Big(\sum_{j=1}^{n} |\mu_j(\omega)|^2\Big)^{1/2}.$$

Let $A(\omega) : \Omega \to \mathbb{R}^{n\times n}$ be a random matrix.
The induced matrix norm is defined by

$$||A(\omega)|| = \max_{||\mu||=1} ||A(\omega)\mu||.$$

Let

$$C(J, \mathbb{R}^n, \Omega) = \{x : J \times \Omega \to \mathbb{R}^n \,|\, x(\cdot,\omega) \in C(J, \mathbb{R}^n, \Omega) \text{ for almost every } \omega \in \Omega\}.$$

For $x \in C(J, \mathbb{R}^n, \Omega)$, define the random sup-norm

$$||x(\cdot,\omega)||_C = \sup_{\tau\in J} ||x(\tau,\omega)||.$$

For each $\omega \in \Omega$, define the weighted random norm

$$||x(\cdot,\omega)||_\lambda = \sup_{\tau\in J} ||x(\tau,\omega)||e^{-\lambda\tau}, \lambda > 0.$$

For a random center $x(\omega) \in \mathbb{R}^n$ and a deterministic or random radius $r > 0$, define the random closed ball

$$B(x(\omega), r) = \{y(\omega) \in \mathbb{R}^n : ||y(\omega) - x(\omega)|| \le r \text{ almost surely}\}.$$

Moreover, define

$$C^1(J, \mathbb{R}^n, \Omega) = \{\, x(\cdot,\omega) \in C(J, \mathbb{R}^n, \Omega) : \partial_t\, x(\cdot,\omega) \in C(J, \mathbb{R}^n, \Omega) \text{ almost surely}\}.$$

Here d_H denotes the Hausdorff distance.
Futhermore, $Conv(\mathbb{R}^n, \Omega)$ denotes the family of random convex compact sets, and $B(\mathbb{R}^n, \Omega)$ denotes the family of random bounded sets in $\mathbb{R}^n$.

Definition 2.1. (Refn. [1], Section 2, Page 130) Random Carathéodory function: A function $\mathscr{G} : J \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is called a random Carathéodory function, if the following conditions are satisfied:

- For each $z_l \in \mathbb{R}^n$, the function $(t, \omega) \to \mathscr{G}(t, \omega, z_l)$ is jointly measurable on $t \in J$, $\omega \in \Omega$.
- For each $t \in J$, $\omega \in \Omega$, the function $z_l \to \mathscr{G}(t, \omega, z_l)$ is continuous on $\mathbb{R}^n$.

Lemma 2.2. *There exists a mapping*

$$S_{\mathbb{R}^n} : (Conv(\mathbb{R}^n, \Omega), d_H) \to L^0(\Omega, \mathbb{R}^n)$$

such that

$$S_{\mathbb{R}^n}(X)(\omega) \in X(\omega) \text{ almost surely},$$

and for every pair of random convex compact sets

$$X_1(\omega), X_2(\omega) \in Conv(\mathbb{R}^n, \Omega),$$

the following Lipschitz estimate holds almost surely:

$$||S_{\mathbb{R}^n}(X_1)(\omega) - S_{\mathbb{R}^n}(X_2)(\omega)|| \leq \beta_n d_H(X_1(\omega), X_2(\omega)),$$

where $\beta_n > 0$ depends only on the dimension n.

Proof. For each convex compact set $K \subset \mathbb{R}^n$ define its Steiner point by

$$S(K) = \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} u h_K(u)\, d\sigma(u),$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, σ is the normalized rationally invariant surface measure. The Steiner point is classical object in convex geometry and possesses the following fundamental properties:

(i) Selector property

For every convex compact set $K \subset \mathbb{R}^n, S(K) \in K$.

(ii) Lipschitz continuity

There exists a constant $\beta_n > 0$, depending only on the dimension n, such that for all convex compact $K, L \subset \mathbb{R}^n$,

$$||S(K) - S(L)|| \leq \beta_n d_H(K, L).$$

A proof may be found in standard monographs on convexity (Refn. [26], Section 5, Page 744). Consequently, the steiner map define a Lipschitz continuous mapping

$$S : (Conv(\mathbb{R}^n), d_H) \to \mathbb{R}^n.$$

Henceforth, we write $S_{\mathbb{R}^n} = S$.

Let $X : \Omega \to Conv(\mathbb{R}^n)$ be a random convex compact set, (i.e.) measurable with respect to the Borel σ- algebra induced by d_H.
Define

$$Y(\omega) = S_{\mathbb{R}^n}(X(\omega)) = S(X(\omega)).$$

Since the Steiner map S is continuous and $X(\cdot)$ is measurable as a random element of the metric space $(Conv(\mathbb{R}^n), d_H)$, the composition $\omega \longmapsto S(X(\omega))$.

(i) Selector property almost surely

For each $\omega \in \Omega$, $Y(\omega) = S(X(\omega)) \in X(\omega)$. Thus the inclusion holds for almost all ω.

(ii) Random Lipschitz property

Let $X_1, X_2 : \Omega \to Conv(R^n)$ be two random convex compact sets. Applying the deterministic Lipschitz estimate pointwise gives, for all ω,

$$||S(X_1(\omega)) - S(X_2(\omega))|| \leq \beta_n d_H(X_1(\omega), X_2(\omega)).$$

□

We impose the following assumptions:
(A1) For any $z \in \mathbb{R}^n$, the random multifunction

$$t \longmapsto \mathscr{G}(t, \omega, z)$$

is continuous with respect to the Hausdorff metric, that is,

$$d_H\Big(\mathscr{G}(t, \omega, z), \mathscr{G}(t^{'}, \omega, z)\Big) \to 0 \,\text{as}\, t^{'} \to t,$$

for a.e. $\omega \in \Omega$.
Moreover, for each fixed $t \in J$ and $z \in \mathbb{R}^n$, the mapping

$$\omega \longmapsto \mathscr{G}(t, \omega, z)$$

is measurable with respect to the Borel σ-algebra induced by the Hausdorff metric.
(A2) There exists a random variable

$$\zeta_{\mathscr{G}} : \Omega \to [0, \infty)$$

such that, for all $t \in J$, all $z, z^{'} \in \mathbb{R}^n,$ and almost every $\omega \in \Omega$,

$$\mathscr{G}(t, \omega, z) \subseteq B\Big(\mathscr{G}(t, \omega, z^{'}), \zeta_{\mathscr{G}}||z - z^{'}||\Big).$$

Equivalently,

$$d_H\Big(\mathscr{G}(t, \omega, z), \mathscr{G}(t, \omega, z^{'})\Big) \leq \zeta_{\mathscr{G}}||z - z^{'}||,$$

for a.e. $\omega \in \Omega$.

Remark 2.3. Under the random Lipschitz assumption **(A2)**, for any $z, z^{'} \in \mathbb{R}^n$, any $t \in J$, and almost every ω, the following inequality holds

$$d_H\Big(\mathscr{G}(t, \omega, z), \mathscr{G}(t, \omega, z^{'})\Big) \leq \zeta_{\mathscr{G}}||z - z^{'}||.$$

Consequently, the multifunction

$$\mathscr{G} : J \times \mathbb{R}^n \times \Omega \to \Big(B(\mathbb{R}^n), d_H\Big).$$

is Lipschitz continuous with respect to the state variable z, almost surely, where $B(\mathbb{R}^n)$ denotes the family of nonempty bounded subsets of $\mathbb{R}^n$.

Lemma 2.4. *Let*

$$\mathscr{G} : J \times \Omega \times \mathbb{R}^n \to B(\mathbb{R}^n)$$

*be a random multifunction satisfying assumptions **(A1)-(A2)**.*
Define the closed convex hull of $\mathscr{G}$ by

$$\overline{co}\mathscr{G}(t, \omega, z) := \overline{co}(\mathscr{G}(t, \omega, z)),\ (t, \omega, z) \in J \times \Omega \times \mathbb{R}^n.$$

Then, for each $(t, z) \in J \times \mathbb{R}^n$, the mapping

$$\omega \longmapsto \overline{co}\mathscr{G}(t, \omega, z)$$

is measurable. Moreover, for almost every $\omega \in \Omega$ and for all $z, z^{'} \in \mathbb{R}^n$, the following inequality holds:

$$d_H\Big(\overline{co}\mathscr{G}(t, \omega, z), \overline{co}\mathscr{G}(t, \omega, z^{'})\Big) \leq \zeta_{\mathscr{G}}||z - z^{'}||,\ \forall\ z, z^{'} \in \mathbb{R}^n.$$

3 Analysis of Convergence

Theorem 3.1. *Consider the system Eq. (1.1) and the ILC law Eq. (1.3) and assume that **(A1)** is satisfied. Select the learning gains S_d and ζ such that the prescribed conditions hold as follows.*

$$||I - \mathbb{N}\zeta|| < 1, \tag{3.1}$$

and

$$||I - \mathbb{N}\mathbb{M}S_d||_C < 1. \tag{3.2}$$

Under these conditions, there exists a solution of the system Eq. (1.1) with the ILC law Eq. (1.3), such that the tracking error e_l converges to zero asymptotically as $l \to \infty$, that is,

$$\lim_{l\to\infty} ||e_l||_\lambda = 0.$$

Proof. According to Lemmas 2.2 and 2.4, we consider the following auxiliary system

$$\begin{cases} \mathscr{D}_t z_l(t,\omega) = \mathscr{H}(t,\omega,z_l(t,\omega)) + \mathbb{M}v_l(t,\omega), \\ r_l(t,\omega) = \mathbb{N}z_l(t,\omega), \end{cases} \tag{3.3}$$

where the function $\mathscr{H}$ is defined by

$$\mathscr{H}(t,\omega,z) := S_{\mathbb{R}^n}(\overline{co}\mathscr{G}(t,\omega,z)).$$

The system Eq. (3.3) is considered in place of the original system Eq. (1.1).
By using Lemmas 2.2 and 2.4, the mapping

$$\mathscr{H} : J \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

is measurable in ω and globally Lipschitz continuous with respect to z. More precisely, for almost every $\omega \in \Omega$ and all $z, z^{'} \in \mathbb{R}^n$.

$$||\mathscr{H}(t,\omega,z) - \mathscr{H}(t,\omega,z^{'})|| \le \beta_n\zeta_{\mathscr{G}}||z - z^{'}||.$$

Moreover, for all $(t,\omega,z) \in J \times \Omega \times \mathbb{R}^n$,

$$\mathscr{H}(t,\omega,z) \in \overline{co}\mathscr{G}(t,\omega,z).$$

Next, observe that

$$\begin{aligned} e_{l+1}(t,\omega) &= r_d(t,\omega) - r_{l+1}(t,\omega) \\ &= r_d(t,\omega) - r_l(t,\omega) + r_l(t,\omega) - r_{l+1}(t,\omega) \\ &= e_l(t,\omega) + \mathbb{N}(z_l(t,\omega) - z_{l+1}(t,\omega)). \end{aligned}$$

In particular, at $t = 0$

$$||e_{l+1}(0,\omega)|| \le ||I - \mathbb{N}\zeta||\,||e_l(0,\omega)||.$$

By linking condition Eq. (3.1), it follows that

$$\lim_{l\to\infty} ||e_l(0,\omega)|| = 0. \tag{3.4}$$

Define

$$g_l(t,\omega) := \mathscr{H}(t,\omega,z_l(t,\omega)), t \in J.$$

Then, the solution of $z_l(t,\omega)$ of system Eq. (3.3) admits the integral representation

$$z_l(t,\omega) = \alpha_0(\omega) + \int_0^t (g_l(s,\omega,z_l(s,\omega)) + \mathbb{M}v_l(s,\omega))ds.$$

Then

$$\Delta z_l(t,\omega) = \Delta\alpha_0(\omega) + \int_0^t ((g_{l+1}(t,\omega,z_{l+1}(t,\omega)) - g_l(t,\omega,z_l(t,\omega))) + \mathbb{M}\Delta v_l(s,\omega))ds$$
$$= (\zeta - \mathbb{M}S_d)e_l(0,\omega) + \mathbb{M}S_d e_l(t,\omega) + \int_0^t (g_{l+1}(t,\omega,z_{l+1}(t,\omega)) - g_l(t,\omega,z_l(t,\omega)))ds. \tag{3.5}$$

The Lipschitz continuous condition yields

$$||g_{l+1}(t,\omega,z_{l+1}(t,\omega)) - g_l(t,\omega,z_l(t,\omega))|| \leq \beta_n\zeta_{\mathscr{G}}||z_{l+1}(t,\omega) - z_l(t,\omega)||,$$
$$= \beta_n\zeta_{\mathscr{G}}||\Delta z_l(t,\omega)||, t \in J.$$

Applying the norm to both sides of Eq. (3.5), gives

$$||\Delta z_l(t,\omega)|| \leq ||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l(t,\omega)||$$
$$+ \int_0^t (\beta_n\zeta_{\mathscr{G}}||z_{l+1}(s,\omega) - z_l(s,\omega)||)ds$$

$$\leq ||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l||_\lambda e^{\lambda t} + \int_0^t \beta_n\zeta_{\mathscr{G}}||\Delta z_l(s,\omega)||ds. \tag{3.6}$$

By applying Grönwall's inequality to Eq. (3.6), we obtain (Refn. [24], Section 3.3, Page 614)

$$||\Delta z_l(t,\omega)|| \leq e^{\beta_n\zeta_{\mathscr{G}}t}\Big(||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l||_\lambda e^{\lambda t}\Big). \tag{3.7}$$

Applying the λ-norm and multiplying by $e^{-\lambda t}$, and using Eq. (3.7), we obtain

$$||\Delta z_l||_\lambda \leq ||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l||_\lambda. \tag{3.8}$$

where $\lambda > \beta_n\zeta_{\mathscr{G}}$ ensures convergence.
Then,

$$e_{l+1}(t,\omega) = e_l(t,\omega) + \mathbb{N}\Big(z_l(t,\omega) - z_{l+1}(t,\omega)\Big)$$
$$= (I - \mathbb{N}\mathbb{M}S_d)e_l(t,\omega) - \mathbb{N}(\zeta - \mathbb{M}S_d)e_l(0,\omega)$$
$$- \mathbb{N}\int_0^t \Big(g_{l+1}(s,\omega,z_{l+1}(s,\omega)) - g_l(s,\omega,z_l(s,\omega))\Big)ds. \tag{3.9}$$

It follows that, by applying the norm and λ-norm on both sides of inequality (3.9) and multiplying by $e^{-\lambda t}$, we obtain,

$$||e_{l+1}||_\lambda \leq ||I - \mathbb{N}\mathbb{M}S_d||_C\,||e_l||_\lambda + ||\mathbb{N}||_\lambda\,||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + \frac{||\mathbb{N}||_C\,\beta_n\zeta_{\mathscr{G}}}{\lambda}||\Delta z_l||_\lambda.$$

Using Eq. (3.8), we substitute $||\Delta z_l||_\lambda$ to obtain

$$||e_{l+1}||_\lambda \leq ||I - \mathbb{N}\mathbb{M}S_d||_C\,||e_l||_\lambda + ||\mathbb{N}||_\lambda\,||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)||$$
$$+ \frac{||\mathbb{N}||_C\beta_n\zeta_{\mathscr{G}}}{\lambda}\Big(||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l||_\lambda\Big), \tag{3.10}$$

when $\lambda > \beta_n\zeta_{\mathscr{G}}$.
By choosing sufficiently large λ, and using conditions Eq. (3.2) and Eq. (3.4), we conclude

$$\lim_{l\to\infty} ||e_l||_\lambda = 0.$$

□

Remark 3.2. Theorem 3.1 applies to any continuous and uniformly Lipschitz selection of $\mathscr{G}$ with respect to z. Since the conditions Eq. (3.1) and Eq. (3.2) do not involve $\zeta_{\mathscr{G}}$, the convergence result holds independently of its value. However, the specific choice of this selection affects the convergence speed.

Theorem 3.3. *Consider the system Eq. (1.1) and the ILC law Eq. (1.4), and assume that **(A1)** is satisfied. Select the learning gains S_d and ζ such that the prescribed conditions hold as follows*

$$||I - \mathbb{N}\zeta|| < 1, \tag{3.11}$$

and

$$||I - \mathbb{N}\mathbb{M}S_d||_C < 1. \tag{3.12}$$

Under these conditions, there exists a solution of the system Eq. (1.1) with the ILC law Eq. (1.4), such that the tracking error e_l converges to zero asymptotically as $l \to \infty$, that is,

$$\lim_{l\to\infty} ||e_l||_\lambda = 0.$$

Proof. Similar to Theorem 3.1, the following relation can be derived using Eq. (3.1),

$$\lim_{l\to\infty} ||e_l(0,\omega)|| = 0. \tag{3.13}$$

By using the ILC law Eq. (1.4) and relation Eq. (3.5), we have

$$\begin{aligned}\Delta z_l(t,\omega) &= (\zeta - \mathbb{M}S_d)e_l(0,\omega) + \mathbb{M}S_d e_l(t,\omega)\\ &\quad + \int_0^t \Big(g_{l+1}(s,\omega,z_{l+1}(s,\omega)) - g_l(s,\omega,z_l(s,\omega))\Big)ds + \mathbb{M}\int_0^t S_p e_l(s,\omega)ds.\end{aligned} \tag{3.14}$$

The following expression is obtained by applying the norm on both sides of Eq. (3.14)

$$\begin{aligned}||\Delta z_l(t,\omega)|| &\le ||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l(t,\omega)||\\ &\quad + \int_0^t \beta_n\zeta_{\mathscr{G}}||z_{l+1}(s,\omega) - z_l(s,\omega)||ds + ||\mathbb{M}S_p||\int_0^t ||e_l(s,\omega)||ds\end{aligned}$$

According to (Refn. [33], Section 3, Page 3), by using the λ-norm, we get

$$\begin{aligned}||\Delta z_l(t,\omega)|| &\le ||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l||_\lambda e^{\lambda t} + \int_0^t \beta_n\zeta_{\mathscr{G}}||\Delta z_l(s,\omega)||ds\\ &\quad + ||\mathbb{M}S_p||\,||e_l||_\lambda \frac{e^{\lambda t}}{\lambda}.\end{aligned} \tag{3.15}$$

By applying Grönwall's inequality to Eq. (3.15), we obtain

$$\begin{aligned}||\Delta z_l(t,\omega)|| &\le e^{\beta_n\zeta_{\mathscr{G}}t}\Big(||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)|| + ||\mathbb{M}S_d||\,||e_l||_\lambda e^{\lambda t}\\ &\quad + ||\mathbb{M}S_p||\,||e_l||_\lambda \frac{e^{\lambda t}}{\lambda}\Big).\end{aligned} \tag{3.16}$$

Applying the λ-norm and multiplying the both sides by $e^{-\lambda t}$ gives

$$\begin{aligned}||\Delta z_l||_\lambda &\le ||\zeta - \mathbb{M}S_d||\,||e_l(0,\omega)||\\ &\quad + \frac{e^{\beta_n\zeta_{\mathscr{G}}T}||\mathbb{M}S_p||}{\lambda}||e_l||_\lambda + ||\mathbb{M}S_d||\,||e_l||_\lambda,\end{aligned} \tag{3.17}$$

where $\lambda > \beta_n\zeta_{\mathscr{G}}$.
Then,

$$\begin{aligned}e_{l+1}(t,\omega) &= e_l(t,\omega) + \mathbb{N}\Big(z_l(t,\omega) - z_{l+1}(t,\omega)\Big)\\ &= (I - \mathbb{N}\mathbb{M}S_d)e_l(t,\omega) - \mathbb{N}(\zeta - \mathbb{M}S_d)e_l(0,\omega)\\ &\quad - \mathbb{N}\int_0^t \Big(g_{l+1}(s,\omega,z_{l+1}(s,\omega)) - g_l(s,\omega,z_l(s,\omega))\Big)ds - \mathbb{N}\int_0^t \mathbb{M}S_p e_l(s,\omega)ds.\end{aligned} \tag{3.18}$$

It follows that, by applying the norm and λ-norm on both sides of inequality (3.18) and multiplying by $e^{-\lambda t}$, we obtain

$$\begin{aligned}||e_{l+1}||_\lambda \leq\ & ||I - \mathbb{N}\mathbb{M}S_d||_C\, ||e_l||_\lambda \\ & + ||\mathbb{N}||_\lambda\, ||\zeta - \mathbb{M}S_d||\, ||e_l(0,\omega)|| + ||\mathbb{N}||_C\, \beta_n\zeta_{\mathscr{G}} \int_0^t ||\Delta z_l(s,\omega)||ds \\ & + ||\mathbb{N}||_C\, ||\mathbb{M}S_p|| \int_0^t ||e_l(s,\omega)||ds\end{aligned}$$

According to (Refn. [24], Section 3, Page 614), we get

$$\begin{aligned}||e_{l+1}||_\lambda \leq\ & ||I - \mathbb{N}\mathbb{M}S_d||_C\, ||e_l||_\lambda + ||\mathbb{N}||_\lambda\, ||\zeta - \mathbb{M}S_d||\, ||e_l(0,\omega)|| \\ & + \frac{||\mathbb{N}||_C \beta_n\zeta_{\mathscr{G}}}{\lambda}\Big[||\zeta - \mathbb{M}S_d||\, ||e_l(0,\omega)|| + \|\mathbb{M}S_d\|\|e_l\|_\lambda \\ & + e^{\beta_n\zeta_{\mathscr{G}}T}\|\mathbb{M}S_p\|\|e_l\|_\lambda \frac{1}{\lambda}\Big] + \frac{\|\mathbb{N}\|_C\|\mathbb{M}S_p\|}{\lambda}\|e_l\|_\lambda \\ \leq\ & \Big\{\|I - \mathbb{N}\mathbb{M}S_d\|_C + \frac{\|\mathbb{N}\|_C}{\lambda}\big[\beta_n\zeta_{\mathscr{G}}\big(e^{\beta_n\zeta_{\mathscr{G}}T}\|\mathbb{M}S_p\|\frac{1}{\lambda} + \|\mathbb{M}S_d\|\big) + \|\mathbb{M}S_p\|\big]\Big\}\|e_l\|_\lambda \\ & + \|\mathbb{N}\|_\lambda\|\zeta - \mathbb{M}S_d\|\left(1 + \frac{\beta_n\zeta_{\mathscr{G}}}{\lambda}\right)\|e_l(0,\omega)\|.\end{aligned}$$

Choosing sufficiently large

$$\lambda > \max\left\{\beta_n\zeta_{\mathscr{G}}T, \beta_n\zeta_{\mathscr{G}}, e^{\beta_n\zeta_{\mathscr{G}}T}, e^{\beta_n\zeta_{\mathscr{G}}}\right\}$$

and using Eqs. (3.12) and (3.13), we conclude that

$$\lim_{l\to\infty} ||e_l||_\lambda = 0.$$

□

Remark 3.4. Based on the system dynamics, we replace $\dot{e}_{l+1}(t,\omega)$ by $\dot{e}_{l+1}(t^-,\omega)$ for $t > 0$, to account for possible discontinuities at the iteration points.

Theorem 3.5. *Consider the system Eq.* (1.1) *and the ILC law Eq.* (1.5) *and assume that* ***(A1)*** *is satisfied. Select the learning gains S_d and ζ such that the prescribed conditions hold as follows.*

$$||I - \mathbb{N}\zeta|| < 1, \tag{3.19}$$

and

$$\min_{t\in J} ||I + \mathbb{N}\mathbb{M}S_d|| > 1. \tag{3.20}$$

Under these conditions, there exists a solution of the system Eq. (1.1) *with the ILC law Eq.* (1.5) *such that the tracking error e_l converges to zero asymptotically as $l \to \infty$, that is,*

$$\lim_{l\to\infty} ||e_l||_\lambda = 0.$$

Proof. Similar to Theorem 3.1, the following relation can be derived by Eq. (3.1), we get

$$\lim_{l\to\infty} ||e_l(0,\omega)|| = 0. \tag{3.21}$$

By using Eqs. (1.5) and (3.5), we have

$$\begin{aligned}\Delta z_l(t,\omega) = \zeta e_l(0,\omega) & + \int_0^t \Big(g_{l+1}(s,\omega,z_{l+1}(s,\omega)) - g_l(s,\omega,z_l(s,\omega))\Big)ds \\ & + \mathbb{M}S_p \int_0^t e_{l+1}(s,\omega)ds + \mathbb{M}S_d e_{l+1}(t,\omega) - \mathbb{M}S_d(I - \mathbb{N}\zeta)e_l(0,\omega),\end{aligned} \tag{3.22}$$

where

$$\begin{aligned} e_{l+1}(0,\omega) &= r_d(0,\omega) - \mathbb{N}z_{l+1}(0,\omega) \\ &= r_d(0,\omega) - \mathbb{N}\Big(z_l(0,\omega) + \zeta e_l(0,\omega)\Big) \\ &= (I - \mathbb{N}\zeta)e_l(0,\omega). \end{aligned}$$

The following expression is obtained by applying the norm on both sides of Eq. (3.22)

$$\begin{aligned} ||\Delta z_l(t,\omega)|| &\leq ||\zeta - \mathbb{M}S_d(I - \mathbb{N}\zeta)||\,||e_l(0,\omega)|| + \frac{||\mathbb{M}S_p||}{\lambda}\,||e_{l+1}||_\lambda e^{\lambda t} \\ &\quad + ||\mathbb{M}S_d||\,||e_{l+1}(t,\omega)|| + \int_0^t \beta_n \zeta_{\mathscr{G}} ||\Delta z_l(s,\omega)||ds \\ &\leq ||\zeta - \mathbb{M}S_d(I - \mathbb{N}\zeta)||\,||e_l(0,\omega)|| + \frac{||\mathbb{M}S_p||}{\lambda}\,||e_{l+1}||_\lambda e^{\lambda t} \\ &\quad + ||\mathbb{M}S_d||\,||e_{l+1}||_\lambda e^{\lambda t} + \int_0^t \beta_n \zeta_{\mathscr{G}} ||\Delta z_l(s,\omega)||ds. \end{aligned} \tag{3.23}$$

For inequality (3.23), can be obtained by applying Grönwall's inequality, again to derive

$$\begin{aligned} ||\Delta z_l(t,\omega)|| \leq e^{\beta_n \zeta_{\mathscr{G}} t}\Big(&||\zeta - \mathbb{M}S_d(I - \mathbb{N}\zeta)||\,||e_l(0,\omega)|| + \frac{||\mathbb{M}S_p||}{\lambda}\,||e_{l+1}||_\lambda e^{\lambda t} \\ &+ ||\mathbb{M}S_d||\,||e_{l+1}||_\lambda e^{\lambda t}\Big). \end{aligned} \tag{3.24}$$

It is clear that, by applying the λ - norm and multiplying the Eq. (3.24) by $e^{-\lambda t}$, we obtain

$$\begin{aligned} ||\Delta z_l||_\lambda &\leq ||\zeta - \mathbb{M}S_d(I - \mathbb{N}\zeta)||\,||e_l(0,\omega)|| \\ &\quad + e^{\beta_n \zeta_{\mathscr{G}} T}\frac{||\mathbb{M}S_p||}{\lambda}\,||e_{l+1}||_\lambda + ||\mathbb{M}S_d||\,||e_{l+1}||_\lambda, \end{aligned} \tag{3.25}$$

where $\lambda > \beta_n \zeta_{\mathscr{G}}$.
Then,

$$\begin{aligned} e_{l+1}(t,\omega) &= e_l(t,\omega) + \mathbb{N}\Big(z_l(t,\omega) - z_{l+1}(t,\omega)\Big) \\ &= e_l(t,\omega) - \mathbb{N}(\zeta - \mathbb{M}S_d(I - \mathbb{N}\zeta))e_l(0,\omega) \\ &\quad - \mathbb{N}\int_0^t \Big(g_{l+1}(s,\omega,z_{l+1}(s,\omega)) - g_l(s,\omega,z_l(s,\omega))\Big)ds - \mathbb{N}\int_0^t \mathbb{M}S_p e_{l+1}(s,\omega)ds \\ &\quad - \mathbb{N}\mathbb{M}S_d e_{l+1}(t,\omega). \end{aligned}$$

Next,

$$\begin{aligned} \Big(I + \mathbb{N}\mathbb{M}S_d\Big)e_{l+1}(t,\omega) &= e_l(t,\omega) - \mathbb{N}(\zeta - \mathbb{M}S_d(I - \mathbb{N}\zeta))e_l(0,\omega) \\ &\quad - \mathbb{N}\int_0^t \Big(g_{l+1}(s,\omega,z_{l+1}(s,\omega)) - g_l(s,\omega,z_l(s,\omega))\Big)ds \\ &\quad - \mathbb{N}\int_0^t \mathbb{M}S_p e_{l+1}(s,\omega)ds. \end{aligned} \tag{3.26}$$

Applying the norm on both sides of Eq. (3.26), we get

$$\begin{aligned}\left(\min_{t\in J}||I+\mathbb{N}\mathbb{M}S_d||\right)||e_{l+1}(t,\omega)|| &\le ||I+\mathbb{N}\mathbb{M}S_d||\,||e_{l+1}(t,\omega)||\\ &\le ||e_l(t,\omega)|| + ||\mathbb{N}(\zeta-\mathbb{M}S_d(I-\mathbb{N}\zeta))||\,||e_l(0,\omega)||\\ &\quad + ||\mathbb{N}||\int_0^t ||(g_{l+1}(s,\omega,z_{l+1}(s,\omega))-g_l(s,\omega,z_l(s,\omega)))||ds\\ &\quad + ||\mathbb{N}||\int_0^t ||\mathbb{M}S_p e_{l+1}(s,\omega)||ds\\ &\le ||e_l(t,\omega)|| + \left(\max_{t\in J}||\mathbb{N}(\zeta-\mathbb{M}S_d(I-\mathbb{N}\zeta))||\right)||e_l(0,\omega)||\\ &\quad + \left(\max_{t\in J}||\mathbb{N}\mathbb{M}S_p||\right)||e_{l+1}||_\lambda\frac{1}{\lambda}e^{\lambda t}\\ &\quad + \left(\max_{t\in J}||\mathbb{N}||\right)\beta_n\zeta_{\mathscr{G}}||\Delta z_l||_\lambda\frac{1}{\lambda}e^{\lambda t}. \end{aligned}\tag{3.27}$$

The following relation can be derived by linking condition Eq. (3.25). By applying the norm and λ-norm and multiplying Eq. (3.27) by $e^{-\lambda t}$, we have

$$\begin{aligned}&\left\{\min_{t\in J}\left\|I+\mathbb{N}\mathbb{M}S_d\right\| - \|\mathbb{N}\|_C\,\|\mathbb{M}S_p\|\frac{1}{\lambda} - \|\mathbb{N}\|_C\beta_n\zeta_{\mathscr{G}}\frac{1}{\lambda}\left[e^{\beta_n\zeta_{\mathscr{G}}T}\,\|\mathbb{M}S_p\|\frac{1}{\lambda}+\|\mathbb{M}S_d\|\right]\right\}\|e_{l+1}\|_\lambda\\ &\qquad\le \|e_l\|_\lambda + \|\mathbb{N}\|_C\,\|(\zeta-\mathbb{M}S_d(I-\mathbb{N}\zeta))\|\left(1+\beta_n\zeta_{\mathscr{G}}\tfrac{1}{\lambda}\right)\|e_l(0,\omega)\|.\end{aligned}$$

Choosing sufficiently large λ and using conditions Eq. (3.20) and Eq. (3.21), we obtain the convergence result

$$\lim_{l\to\infty}||e_l||_\lambda = 0.$$

□

At $S_p = 0$, the D-type closed-loop law convergence analysis is obtained as follows.

Corollary 3.6. *Consider the system Eq.* (1.1) *and the D-type ILC law*

$$\begin{aligned}v_{l+1}(t,\omega) &= v_l(t,\omega) + S_d\dot{e}_{l+1}(t,\omega),\\ z_{l+1}(0,\omega) &= z_l(0,\omega) + \zeta e_l(0,\omega).\end{aligned}\tag{3.28}$$

Assume that **(A1)** is satisfied. Select the learning gains S_d and ζ such that the prescribed conditions hold as follows

$$||I-\mathbb{N}\zeta|| < 1,\tag{3.29}$$

and

$$\min_{t\in J}||I+\mathbb{N}\mathbb{M}S_d|| > 1.\tag{3.30}$$

Under these conditions, there exists a solution of the system Eq. (1.1) with the above law such that the tracking error e_l converges to zero asymptotically as $l\to\infty$, that is,

$$\lim_{l\to\infty}||e_l||_\lambda = 0.$$

4 Validation of Proofs

Example 4.1. Consider the following nonlinear RDIS

$$\begin{cases}\mathscr{D}_t z_l(t,\omega)\in\overline{co}\mathscr{G}(t,\omega,z_l(t,\omega))+v_l(t,\omega), t\in J:=[0,1],\\ r_l(t,\omega)=z_l(t,\omega),\end{cases}\tag{4.1}$$

where

$$\mathscr{G}(t,\omega,z) := \{\xi(\omega)z\cos t + \eta(\omega)t, \xi(\omega)z\cos t + \eta(\omega)t + \tan hz\}.$$

Clearly,

$$\overline{co}\mathscr{G}(t,\omega,z) = \{\xi(\omega)z\cos t + \eta(\omega)t + \alpha\tan hz : \alpha \in [0,1]\},$$

which is measurable with respect to (t,ω) and closed-valued.
Let $\xi(\omega) \in [0,1]$ a bounded measurable random variable and $\eta(\omega)$ be another random variable satisfying $|\eta(\omega)| \leq 1$.
Now, we apply the D-type open loop ILC law is choosen as

$$\begin{aligned} v_{l+1}(t,\omega) &= v_l(t,\omega) + 0.1\dot{e}_l(t,\omega), \\ z_{l+1}(0,\omega) &= z_l(0,\omega) + 0.5e_l(0,\omega). \end{aligned}$$

And D-type closed-loop ILC law is

$$\begin{aligned} v_{l+1}(t,\omega) &= v_l(t,\omega) + 0.1\dot{e}_{l+1}(t,\omega), \\ z_{l+1}(0,\omega) &= z_l(0,\omega) + 0.5e_l(0,\omega). \end{aligned}$$

For $z, \hat{z} \in \mathbb{R}$, we estimate

$$\begin{aligned} &\left|\left(\xi(\omega)z\cos t + \eta(\omega)t + \tan hz\right) - \left(\xi(\omega)\hat{z}\cos t + \eta(\omega)t + \tan h\hat{z}\right)\right| \\ &\qquad \leq |\xi(\omega)|\,|z-\hat{z}|\,|\cos t| + |\tan hz - \tan h\hat{z}| \\ &\qquad \leq 2|z-\hat{z}|, \end{aligned}$$

Since, $|\xi(\omega)| \leq 1, |\cos t| \leq 1$ and $tanh(\cdot)$ is global Lipschitz with constant 1. Hence, for any $y_1 \in \mathscr{G}(t,\omega,z)$, there exists $y_2 \in \mathscr{G}(t,\omega,\hat{z})$ such that $|y_1 - y_2| \leq 2|z-\hat{z}|$, and therefore $\zeta_{\mathscr{G}} = 2$. Thus, (A1) is satisfied.
We compute

$$||I - \mathbb{N}\zeta|| = 0.5 < 1,\ ||I - \mathbb{N}\mathbb{M}S_d||_C = 0.9 < 1.$$

Hence, all the conditions of Theorem 3.1 are satisfied. Moreover,

$$\min_{t\in[0,1]} ||I + \mathbb{N}\mathbb{M}S_d||_C = 1.1 > 1.$$

Then, all the conditions of Corollary 1 are satisfied.

Example 4.2. Consider the following nonlinear RDIS

$$\begin{cases} \mathscr{D}_t z_l(t,\omega) \in \overline{co}\mathscr{G}(t,\omega,z_l(t,\omega)) + v_l(t,\omega), t \in J := [0,1], \\ r_l(t,\omega) = z_l(t,\omega), \end{cases} \tag{4.2}$$

where

$$\mathscr{G}(t,\omega,z) := \{\gamma(\omega)z\cos t + \theta(t,\omega), \gamma(\omega)z\cos t + \theta(t,\omega) + arc\tan z + \sigma(z)\},$$

Thus,

$$\overline{co}\mathscr{G}(t,\omega,z) := \{\gamma(\omega)z\cos t + \theta(t,\omega) + \alpha(arc\tan z + \sigma(z)) : \alpha \in [0,1]\}.$$

Let, $\gamma(\omega) \in [0,1.75]$ be measurable and $|\theta(t,\omega)| \leq 0.3,\ \forall\, t,\omega$.
Define the logistic (Sigmoid) function

$$\sigma(z) = \frac{1}{1+e^{-z}},$$

for which $\sigma^{'}(z) = \sigma(z)(1-\sigma(z)) \leq \frac{1}{4}, \forall\, z \in \mathbb{R}^n$. Hence σ is globally Lipschitz with constant $\zeta_\sigma = \frac{1}{4}$.
The D-type open-loop ILC law is

$$v_{l+1}(t,\omega) = v_l(t,\omega) + 0.5\dot{e}_l(t,\omega),$$
$$z_{l+1}(0,\omega) = z_l(0,\omega) + 0.9e_l(0,\omega).$$

The D-type closed-loop ILC law is

$$v_{l+1}(t,\omega) = v_l(t,\omega) + 0.5\dot{e}_{l+1}(t,\omega),$$
$$z_{l+1}(0,\omega) = z_l(0,\omega) + 0.9e_l(0,\omega).$$

For $z, \hat{z} \in \mathbb{R}$ Thus,

$$\begin{aligned}|y_1 - y_2| &\leq |\gamma(\omega)|\,|z-\hat{z}|\,|\cos t| + |arc\tan z - arc\tan \hat{z}| + |\sigma(z) - \sigma(\hat{z})| \\ &\leq \Big(\gamma(\omega) + 1.25\Big)|z-\hat{z}|.\end{aligned}$$

Since $\gamma(\omega) \leq 1.75$, we obtain

$$|y_1 - y_2| \leq 3|z-\hat{z}|$$

Where $\zeta_{\mathscr{G}} = 3$ and **(A1)** is satisfied. We have

$$||I - \mathbb{N}\zeta|| = 0.1 < 1,$$
$$||I - \mathbb{N}\mathbb{M}S_d||_C = 0.5 < 1,$$

So, all the conditions of Theorem 3.1 are satisfied.
Moreover,

$$\min_{t\in[0,1]} ||I + \mathbb{N}\mathbb{M}S_d||_C = 1.5 > 1.$$

and hence, all the conditions of Corollary 1 are satisfied.

5 Conclusion

This paper investigated ILC for nonlinear RDIS with set-valued random structures. Measurable selection and convex analytical techniques were employed to transform the inclusion system into a random differential scheme. Sufficient conditions for asymptotic convergence in the weighted λ-norm were established using Lipschitz conditions and Grönwall-type inequalities. The effectiveness of the proposed ILC approach was further verified through illustrative examples. Furthermore, Louakar et al. (Refn. [31]) examined ILC for Hilfer-type fractional stochastic differential systems with robotic applications. Motivated by these fractional-order developments, the present study may be extended to fractional-order RDIS under ILC concepts.

Author Contributions

R. Gayathri: Writing-original draft; conceptualization; methodology; validation and editing. D. Vivek: Writing-original draft; validation; conceptualization; methodology and supervision. A. A. Alghamdi: Writing-review, validation and guidance. M. M. El-Dessoky: Writing-review, validation and guidance. E. M. Elsayed: Writing-review, validation and guidance.

Conflict of Interest Statement

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