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Existence of unique solution and numerical method of second order mixed Volterra-Fredholm integral equation

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Abstract In this manuscript we discuss a result regarding uniqueness of a second order Volterra Fredholm mixed integro-differential equation by applying the Banach FPT. Next we discuss a numerical method and verify the effectiveness of the numerical method. To further demonstrate the given result, we provide an example.

1 Introduction

A.M. Wazwaz mentioned in his book that the term "Volterra-Fredholm integral equations comes from problems of parabolic boundary value, mathematical modeling based on the spatio-temporal development of an epidemic, from a variety of physical and biological models" [20]. Mixed Volterra-Fredholm integral equations is a major milestone of mathematics and it contains various sectors due to their flexibility to model in different phenomena. The integro differential equations of mixed Volterra-Fredholm type blend concepts from both integral and differential equations, appears in two types

$$y^k(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt + \lambda_2 \int_a^b K_2(x,t)y(t)dt \tag{1.1}$$

and

$$y^k(x) = f(x) + \lambda \int_a^x \int_a^b K(r,t)y(t)dtdr, \tag{1.2}$$

where $y^k(x) = \dfrac{d^k y(x)}{dx^k}$. In the equation (1.1) contains integrals of disjoint type and the equation (1.2) contains integrals of mixed type where considering Volterra integral is the exterior one while Fredholm is the interior one.

The integro differential equations of mixed Volterra-Fredholm type combine concepts of current state and their historical interactions.

The second-order integro-differential equations developed to study dynamic interaction between theory, application, and computational methods. These equations offer modeling systems with memory, delay, and feedback along with uses in viscoelasticity, thermal systems.

There are many research articles have been published focusing on the issue of second degree integro- differential equation along with some solution properties. We have attached some related papers for reference [4, 10, 11, 14, 15]. Recently Ghasemi et.al. [5] studied problems of this type considering system of second order integro-differential equations.

Galerkin method was first introduced by Russian mathematician Boris Galerkin in 19^{th} century, focused to solve Partial Differential Equations with boundary conditions arises in engineering and physics. Now this method is evolved to solve a wide range of differential equations and integral equations due to it's efficiency. The flexibility of this method make it suitable for various

fields such as electromagnetics, structural mechanics, fluid dynamics, heat transfer, and quantum mechanics. Nowadays this method address problems like nonlinearity, multidimensionality, and complex boundary conditions. The Galerkin method is a technique which is important in numerical analysis that leads to study complex problems in science including modeling, simulating and engineering.

Many researchers have contributed the study of methods for solving Volterra-Fredholm integro-differential equations; which includes Galerkin method, Lagrange polynomial approximations [13], Shifted Legendre Polynomials method [8], integrated Simpson's collocation method[7], collocation method[17, 18, 19], Jacobi operational matrix for the numerical solution[3] etc. Karacayir and Yuzbasi [9] proposed a Galerkin-type approach to solve systems of linear Volterra-Fredholm integro-differential equations. Asma and Sahaa [1] introduced a Galerkin method to approximate solutions for Fredholm-Volterra integral equations (FVIEs) of the second kind. Mamadu and Njoseh [12], along with Mamadu et al. [6], applied the Galerkin method to solve Volterra equations using orthogonal polynomials and Mamadu- Njoseh polynomials, respectively, for the solution of fractional integro-differential equations. Bello et al. [2] applied the Galerkin method with Chebyshev polynomial basis functions for solving multi-order fractional differential equations. Recently [16] discussed the Galerkin method of second order integro differential equation.

Though many works have been done by discussing the problem of second order integro-differential equation along with Galerkin method, but we have not found a single manuscript which discussed solution of second order integro-differential equation by fixed point theorem with numerical method (Galerkin method). Inspired by this fact, we motivated to do this research work.

The paper is systematically organized as : In Section 2, we have stated some notations and auxiliary facts, which will be helpful for our main finding. In Section 3, we prove the result. In Section 4, we discussed the numerical method. Next we have given an example to support of our result and also verified the examples in Section 5.

2 Basic results and definitions

Let us consider, $\mathcal{E} = \{y(x) \in C(I)\}$ is a Banach space with norm $||.||$ such that $||y(x)||_{\mathcal{E}} = \max_{x \in I} |y(x)|$.

Theorem 2.1. *[21] For a complete metric space $(\mathcal{E}, d)$,the contraction mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ posses a unique fixed point w of $\mathcal{T}$ in $\mathcal{E}$ i.e.,*

$$\mathcal{T}w = w.$$

Definition 2.2. [16] "Residual: In the context of solving differential equations using approximate methods like the Galerkin method, the residual represents the error or difference between the left-hand side (LHS) and right-hand side (RHS) of the differential equation when an approximate solution is used instead of the exact one. It essentially measures how closely the approximate solution satisfies the differential equation. For a differential equation of the form: $L[u(x)] = f(x)$ where L is a differential operator, $U(x)$ is the exact solution, and $f(x)$ is a known function, the approximate solution $u_n(x)$ is substituted. The residual $R(x)$ is then defined as

$$R(x) = L[u_n(x)] - f(x). \tag{2.1}$$

In other words, the residual $R(x)$ quantifies the difference between the result of applying the differential operator to the approximate solution $u_n(x)$ and the known function $f(x)$.

3 Existence of unique solution

Now consider the second order nonlinear mixed Volterra-Fredholm integro-differential equation

$$y''(x) = f(x) + \int_0^x \int_0^a K(x,t)F(y(t))dtdx \tag{3.1}$$

under the condition

$$y(0) = d_0, \;\; y'(0) = d_1. \tag{3.2}$$

Here,
$y : I = [0, 1] \to \mathbb{R}$ is a continuous unknown function that we need to be determined.
$F : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are nonlinear functions of $y(x)$;

$f : I \to \mathbb{R}$, $K : I \times I \to \mathbb{R}$, $K : I \times I \to \mathbb{R}$ are continuous functions. It is easy to observe that the IVP (3.1) - (3.2) equivalent to the integral equation

$$y(x) = d_0 + d_1 x + \int_0^x (x-t) f(t)dt + K^* \int_0^x (x-t) F(y(t))dt \tag{3.3}$$

where

$$K^* = \sup_{a \in [0,1]} \int_0^a |K(x,t)|dt < \infty$$

Let us consider the following assumptions:

(i) $L_F > 0$ is a constant such that for any $y, w \in C(I, \mathbb{R})$

$$||F(y(t)) - F(w(t))|| \leq L_F ||y(t) - w(t)||$$

(ii) For positive constant G_1

$$||d_0 + d_1 x + \int_0^x (x-t) f(0)dt + K^* \int_0^x (x-t) F(0)dt|| \leq G_1$$

(iii) For positive constant L_f

$$||f(y) - f(x)|| \leq L_f ||y - x||$$

Theorem 3.1. *Suppose that our assumptions* $(1) - (3)$ *are verified and*

$$K^* L_F < 1. \tag{3.4}$$

Then there exist a unique solution $y \in C(I, \mathbb{R})$ *for the equations* (3.1)-(3.2).

Proof. Let us consider $y \in P$, and define the operator $\mathcal{T}$ as:

$$(\mathcal{T}y)(x) = d_0 + d_1 x + \int_0^x (x-t) f(t)dt + K^* \int_0^x (x-t) F(y(t))dt \tag{3.5}$$

Now,

$$\begin{aligned}
||(\mathcal{T}y)(x)|| &\leq ||d_0 + d_1 x + \int_0^x (x-t) f(0)dt + K^* \int_0^x (x-t) F(0)dt|| \\
&+ \int_0^x (x-t)||f(t) - f(0)||dt + K^* \int_0^x (x-t)||F(y(t)) - F(0)||dt \\
&\leq G_1 + \int_0^x (x-t) L_f ||t||dt + K^* \int_0^x (x-t) L_F ||y(t)||dt \\
&\leq G_1 + K^* L_f ||x|| \int_0^x (x-t)dt + K^* L_F ||y(x)|| \int_0^x (x-t)dt \\
&\leq G_1 + K^* L_f ||x|| + K^* L_F ||y(x)||.
\end{aligned}$$

It follows that $\mathcal{T}y \in P$. This proves $\mathcal{T}$ maps into itself.

$$\begin{aligned}
&|\mathcal{T}y(x) - \mathcal{T}w(x)| \\
\leq &|K^* \int_0^x (x-t) F(y(t))dt - K^* \int_0^x (x-t) F(w(t))dt| \\
\leq &K^* | \int_0^x (x-t) L_F ||y(t) - w(t)||dt| \\
\leq &K^* ||y(t) - w(t)||.
\end{aligned}$$

Hence, we get

$$||\mathcal{T}y(x) - \mathcal{T}w(x)||_\infty \leq K^* bL_F||y(x) - w(x)||_\infty.$$

From (3.4), $\mathcal{T}$ is a contraction. Therefore by Banach contraction principle, $\mathcal{T}$ has only one fixed point y. □

4 Solution existence by Galerkin Method

We have the second order integro-differential equation of mixed Volterra-Fredholm type

$$y''(x) = f(x) + \int_a^x \int_a^b K(x,t)F(y(t))dtdx \tag{4.1}$$

under the initial condition

$$y(0) = d_0, \quad y'(0) = d_1. \tag{4.2}$$

Assume an approximate solution as a form of a power series:

$$y_n(x) = \sum_{i=0}^{n} c_i x_i, \tag{4.3}$$

where $c_i, \quad i = 0, ..., n$ have to be determined. If we apply the initial condition in the (4.3) that gives

$$y_n(x) = d_0 + d_1 x + \sum_{i=2}^{n} c_i x_i. \tag{4.4}$$

Substituting the (4.4) in (4.1) and calculating residual we have

$$\mathcal{R}(x) = y_n^{*\prime\prime}(x) - f(x) - \int_a^x \int_a^b K(x,t)F(y_n^*(t))dtdx. \tag{4.5}$$

Now define the term

$$\mathcal{I}_n = \int_o^1 \chi_i(x)\mathcal{R}(x)dx. \tag{4.6}$$

Here $\chi_i = d_0 + d_1(x) + x^i$ is the weight function in the interval $[0, 1]$ and $i = 0, ..., n$. Solving equation (4.6) yields a algebraic system of equations for the constants $a_i \quad i = 2, 3, ..., n$.

5 Numerical Example

Example 5.1. Consider

$$y''(x) = 6 - (\frac{1}{4}x^2 + \frac{x^3}{3} + \frac{3}{8}x^4) + \int_0^x \int_0^1 rxy(x)dxdr, \tag{5.1}$$

under the initial condition

$$y(0) = 1, \; y'(0) = 2, \tag{5.2}$$

exact solution of (5.1) given by $y(x) = 1 + 2x + 3x^2$.

To apply the proposed technique, assume an approximate solution in the form of a power series:

$$y_n(x) = \sum_{i=0}^{n} c_i x_i, \tag{5.3}$$

where $c_i x_i$ represents the terms of the power series. For $n = 5$, the approximate solution becomes

$$y_5(x) = \sum_{i=0}^{5} c_i x_i = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \tag{5.4}$$

By applying the initial conditions $y(0) = 1,\ y'(0) = 2$ in (5.4), we obtain

$$y_5(x) = 1 + 2x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 \tag{5.5}$$

Now substituting the value of (5.5) in (5.1) approximate solution into the integro-differential results in the residual equation:

$$\mathcal{R} = (2c_2 - 6) + 6c_3(x) + 12c_4x^2 + 20c_5x^3 - ((\frac{c_2}{8} - \frac{3}{8})x^4 + \frac{c_3}{10}x^5 + \frac{c_4}{12}x^6 + \frac{c_5}{14}x^7). \tag{5.6}$$

By substituting the residual into (4.6), we obtain an algebraic system of equations with unknown constants $c_i, (i = 2, 3, 4, 5)$. Solving these system of equation we find $c_2 = 3$ and $c_3 = c_4 = c_5 = 0$. Thus the approximate solution is $y = 1 + 2x + 3x^2$.

6 Conclusion

In this manuscript, we consider a second-order mixed integro-differential equation and solve it using the Banach Fixed Point Theorem. We then discuss a numerical approach, namely the Galerkin method, and examine the existence of a solution using this method. As a direction for future work, this study can be extended to higher-order mixed integro-differential equations, which may be analyzed using other numerical methods.

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References

[1] A. A. Akhia, & G. Saha, Galerkin approximations for the solution of Fredholm Volterra integral equation of second kind, GANIT: Journal of Bangladesh Mathematical Society, 41(1)(2021), 1-14.

[2] A.K. Bello, J.U. Abubakar, T. Oyedepo, A.M. Ayinde, T.F. Mohammed, Numerical approximation of multi-order fractional differential equations by Galerkin method with chebyshev polynomial basis, Journal of Fractional Calculus and Applications, 15(2)(2024), 1–20.

[3] A.A. El-Sayed, D. Baleanu, P. Agarwal, A novel Jacobi operational matrix for numerical solution of multi-term variable-order fractional differential equations, Journal of Taibah University for Science, 14(1)(2020), 963–974.

[4] D. Guo, Second order integro-differential equations of Volterra type on unbounded domains in Banach spaces, Nonlinear Analysis: Theory, Methods & Applications, 41(3-4)(2000), 465–476.

[5] M. Ghasemi, K. Mohammadi, A. Alipanah, Numerical solution of system of second-order integro-differential equations using nonclassical sinc collocation method, Boundary Value Problems, 2023(1)(2023), 38.

[6] M.E. James, O.I. Henrietta, N.I. Nkonyeasua, Numerical solution of fractional integro-differential equation using Galerkin method with Mamadu-Njoseh polynomials, Australian Journal of Basic and Applied Sciences, 15(10) (2021), 13-19.

[7] A.A. Jimoh, Integrated Simpson's Collocation Method for Solving Fourth Order Volterra Integro-Differential Equations, Global Journal of Pure and Applied Mathematics, 16(5)(2020), 657-673.

[8] K. Krishnaveni, K. Kannan, S. Balachandar, A new polynomial method for solving Fredholm-Volterra integral equations, International Journal of Engineering and Technology, 5(2)(2013), 1474–1483.

[9] M. Karacayir, and Ş. Yüzbaşi, A Galerkin-type approach to solve systems of linear Volterra-Fredholm integro-differential equations, Turkish Journal of Mathematics, 46(8)(2022), 3121–3138.

[10] N. Kosmatov, Second order boundary value problems on an unbounded domain, Nonlinear Analysis: Theory, Methods & Applications, 68(4)(2008), 875–882.

[11] Y. Liu, Initial value problems for second-order integro-differential equations on unbounded domains in a Banach space, Demonstratio Mathematica, 38(2)(2005), 349–364.

[12] J.E. Mamadu, I.N. Njoseh, Numerical solutions of Volterra equations using Galerkin method with certain orthogonal polynomials, Journal of Applied Mathematics and Physics, 4(2)(2016), 367–382.

[13] M.M. Mustafa, I.N. Ghanim, Numerical solution of linear Volterra-Fredholm integral equations using Lagrange polynomials, Mathematical Theory and Modeling, 4(5)(2014), 137–146.

[14] B.G. Phachpatte, On a certain second order integrodifferential equation, Research report collection, 10(3)(2007), 463-476.

[15] B.G. Pachpatte, Approximate solutions of a certain second order integrodifferential equation, Fasciculi Mathematici, 45(2010), 99–109.

[16] T. Oyedepo, A. K. Bello, T. M. Raji & A. M. A. Ayinde, Approximation of Second Order Mixed Volterra-Fredholm Integro differential Equations By Garlekin Method, Electronic Journal of Mathematical Analysis and Applications, 13(2)(2025), 1-10.

[17] T. Oyedepo, A. Ayoade, G. Ajileye, N.J. Ikechukwu, Legendre computational algorithm for linear integro-differential equations, Cumhuriyet Science Journal, 44(3)(2023), 561–566.

[18] T. Oyedepo, A.M. Ayinde, E.N. Didigwu, Vieta-Lucas polynomial computational tecnique for volterra integro-differential equations, Electronic Journal of Mathematical Analysis and Applications, 12(10)(2024), 1–8.

[19] T. Oyedepo, G. Ajileye, A.A. Ayoade, Bernstein computational algorithm for integro-differential equations, 11(2024), 100897.

[20] A.M. Wazwaz, Linear and nonlinear integral equations: Methods and applications, Springer Berlin, Heidelberg, (2011).

[21] Y. Zhou, Basic theory of fractional differential equations; World Scientific, (2023).

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