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# NEW EXTENSIONS OF MADDOX-TYPE PARANORMED SEQUENCE SPACES VIA THE $\nabla_q^2$ OPERATOR

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**Abstract**. This study aims to extend the class of paranormed sequence spaces originally introduced by Maddox by proposing a new generalization. In this context, the sequence spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$ are defined, and their structural properties are examined in greater depth. As a first step, a Schauder basis is constructed for each space, and the corresponding alpha-, beta-, and gamma-dual spaces are determined. After that, several important matrix transformations between these newly defined spaces and the classical sequence spaces $\ell_\infty$, $c$, and $c_0$ are thoroughly investigated. In the final part of the study, compact operators acting on the space $c_0(\nabla_q^2, p)$ are characterized.

## 1 Introduction

The theory of sequence spaces has long served as a cornerstone of functional analysis, summability theory, and operator theory. Traditionally, sequence spaces such as $c_0$, $c$, and $\ell_\infty$ have been extensively investigated, particularly within the framework of normed and paranormed spaces. One of the prominent generalizations of classical sequence spaces has been achieved by means of matrix domains, as first proposed in the context of $K$-spaces and later extended to various types of difference operators and averaging operators.

In this regard, the concept of a *paranorm* plays a significant role. A paranorm, unlike a norm, relaxes certain conditions while retaining subadditivity and continuity under scalar convergence, thereby allowing a broader class of sequence spaces. Maddox's construction of sequence spaces such as $c_0(p)$ and $c(p)$, equipped with appropriate paranorms, laid the groundwork for further developments in this field [1, 2, 3, 4, 5, 6, 7, 8, 9].

In recent years, the theory of $q$-calculus, also known as quantum calculus, has emerged as a powerful tool for the generalization of classical mathematical structures. Unlike classical calculus, $q$-calculus replaces the usual notion of differentiation with $q$-difference operators that depend on a fixed base $q \in (0, 1)$. This alternative framework has found applications in various fields, including combinatorics, orthogonal polynomials, special functions, and quantum theory. In the context of sequence spaces, $q$-calculus offers new avenues for the construction of difference operators and summability methods that exhibit behavior distinct from their classical counterparts. The incorporation of higher-order $q$-difference operators into matrix domain methods enables the development of novel sequence spaces that retain the flexibility of paranormed structures while exhibiting finer analytic and topological characteristics. Thus, $q$-calculus serves not only as a generalization but also as a bridge connecting discrete and continuous mathematical frameworks.

The motivation for this study stems from the desire to construct new paranormed sequence spaces by combining two powerful mathematical frameworks: matrix domain techniques and the theory of $q$-calculus. While numerous works have explored the matrix domains of classical

and fractional difference operators, relatively few investigations have addressed the influence of $q$-difference operators, particularly in the setting of paranormed spaces. By incorporating the second-order $q$-difference matrix $\nabla_q^2$ into the structure of Maddox-type sequence spaces such as $c_0(p)$ and $c(p)$, we aim to introduce and analyze new sequence spaces with richer structural and topological properties. These spaces not only broaden the spectrum of paranormed spaces but also contribute to the growing field of $q$-analogue summability theory.

## 2 Materials and Methods

A sequence space is typically defined as a vector subspace of the space $\omega$ of all real (or complex) sequences. Among the most widely examined sequence spaces in the literature are the following:

$c_0$ consisting of all sequences that converge to zero (null sequences),

$c$ the space of all convergent sequences,

$\ell_\infty$ the space of all bounded sequences.

These classical spaces form the foundation for much of the work in summability theory and functional analysis, and they continue to serve as benchmarks in the construction and analysis of more generalized sequence spaces.

A function $\rho : H \to \mathbb{R}$ is called a *paranorm* on a linear space $H$ if it satisfies the following conditions:

(i) $\rho(h) = 0$ if and only if $h = 0$;

(ii) $\rho(h) > 0$ for all $h \in H \setminus \{0\}$;

(iii) $\rho(-h) = \rho(h)$ for all $h \in H$;

(iv) $\rho(h + t) \leq \rho(h) + \rho(t)$ for all $h, t \in H$;

(v) If $\alpha_r \to \alpha$ and $\rho(h_r - h) \to 0$, then $\rho(\alpha_r h_r - \alpha h) \to 0$ as $r \to \infty$.

A linear space $H$ equipped with a paranorm $\rho$ is called a *paranormed space* and denoted by $(H, \rho)$.

Maddox [10] and Simons [11] introduced the sequence spaces $c_0(p)$ and $c(p)$ defined by:

$$c_0(p) = \left\{ z = (z_r) \in \omega : \lim_{r\to\infty} |z_r|^{p_r} = 0 \right\},$$

$$c(p) = \left\{ z = (z_r) \in \omega : \lim_{r\to\infty} |z_r - \lambda|^{p_r} = 0 \text{ for some } \lambda \in \mathbb{R} \right\},$$

where $p = (p_r)$ is a bounded sequence of strictly positive real numbers.

Ahmad and Mursaleen [12] defined the sequence spaces $c_0(\Delta, p)$ and $c(\Delta, p)$ as follows:

$$c_0(\Delta, p) = \{h \in \omega : \Delta h \in c_0(p)\}, \quad c(\Delta, p) = \{h \in \omega : \Delta h \in c(p)\},$$

where $\Delta$ is the forward difference operator.

Let $\lambda$ and $\mu$ be sequence spaces, and let $\mathcal{A} = (a_{rs})$ be an infinite matrix. We say that $\mathcal{A}$ defines a matrix mapping from $\lambda$ into $\mu$, denoted by $\mathcal{A} : \lambda \to \mu$, if for every $h = (h_s) \in \lambda$, the series

$$(\mathcal{A}h)_r = \sum_{s=0}^{\infty} a_{rs} h_s$$

is convergent for all $r \in \mathbb{N}$ and the sequence $(\mathcal{A}h)_r \in \mu$. The set of all such matrices is denoted by $(\lambda : \mu)$.

The *matrix domain* of an infinite matrix $A$ in a sequence space $H$ is defined as

$$H_A := \{h \in \omega : Ah \in H\}.$$

The concept of the $q$-number is defined by

$$[h]_q = 1 + q + q^2 + \cdots + q^{h-1}, \quad [0]_q := 1,$$

which satisfies $\lim_{q\to 1^-}[h]_q = h$ [13]. The $q$-binomial coefficient, or Gaussian binomial coefficient, is defined as

$$\begin{bmatrix} h \\ f \end{bmatrix}_q = \begin{cases} \dfrac{[h]_q!}{[f]_q![h-f]_q!}, & 0 \le f \le h, \\ 0, & \text{otherwise}, \end{cases}$$

where $[h]_q! = [1]_q[2]_q \cdots [h]_q$ is the $q$-factorial.

The $q$-analogue of the second-order difference matrix $\nabla_q^2 = (\nabla_q^2)_{rs}$, introduced by Alotaibi et al. [14], is defined as:

$$(\nabla_q^2)_{rs} = \begin{cases} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q, & 0 \le s \le r, \\ 0, & s > r. \end{cases}$$

Equivalently, this matrix can be written as:

$$\nabla_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -(1+q) & 1 & 0 & 0 & \cdots \\ q & -(1+q) & 1 & 0 & \cdots \\ 0 & q & -(1+q) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Its inverse $\nabla_q^{-2} = ((\nabla_q^{-2})_{rs})$ is given by:

$$(\nabla_q^{-2})_{rs} = \begin{cases} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q, & 0 \le s \le r, \\ 0, & s > r. \end{cases}$$

In recent studies, many authors have generalized sequence spaces using matrix domains in paranormed settings and $q$-analogues (see [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]).

The study of paranormed sequence spaces is fundamental in functional analysis, particularly in understanding matrix transformations and operator-theoretic properties. The introduction of difference operators and their $q$-analogues has recently attracted considerable attention, as these tools provide effective generalizations of classical sequence spaces and reveal richer structural features. Motivated by these developments, this work focuses on constructing new paranormed sequence spaces via the second-order $q$-difference operator and analyzing their structural and compact operator properties.

In light of the aforementioned preliminaries, we now proceed to construct new paranormed sequence spaces by employing the second-order $q$-difference matrix $\nabla_q^2$, and investigate their fundamental properties.

## 3 Results

The concept of $q$-calculus plays a crucial role in the definition and development of novel sequence spaces through the use of the second-order $q$-difference operator $\nabla_q^2$. In recent studies, Alotaibi et al. [14] and Yaying et al. [36] introduced and analyzed the following matrix domain sequence spaces defined via the transformation $\nabla_q^2$:

$$\begin{aligned} c_0(\nabla_q^2) &= \left\{ h = (h_s) \in \omega : \nabla_q^2 h \in c_0 \right\}, \\ c(\nabla_q^2) &= \left\{ h = (h_s) \in \omega : \nabla_q^2 h \in c \right\}, \\ \ell_p(\nabla_q^2) &= \left\{ h = (h_s) \in \omega : \nabla_q^2 h \in \ell_p \right\}, \\ \ell_\infty(\nabla_q^2) &= \left\{ h = (h_s) \in \omega : \nabla_q^2 h \in \ell_\infty \right\}. \end{aligned}$$

Following this line of research, Bilgin Ellidokuzoğlu and Demiriz [37] extended the framework by introducing the $q$-paranormed difference sequence spaces of second order, denoted by

$\ell(\nabla_q^2, p)$ and $\ell_\infty(\nabla_q^2, p)$, defined as:

$$\ell(\nabla_q^2, p) = \left\{h = (h_s) \in \omega : \nabla_q^2 h \in \ell(p)\right\},$$
$$\ell_\infty(\nabla_q^2, p) = \left\{h = (h_s) \in w : \nabla_q^2 h \in \ell_\infty(p)\right\}.$$

Inspired by this approach, we now propose two new $q$-paranormed sequence spaces of second order, namely $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$, defined as follows:

$$c_0(\nabla_q^2, p) = \left\{h = (h_s) \in \omega : \nabla_q^2 h \in c_0(p)\right\},$$
$$c(\nabla_q^2, p) = \left\{h = (h_s) \in \omega : \nabla_q^2 h \in c(p)\right\}.$$

These newly introduced spaces may equivalently be expressed using the matrix domain notation as:

$$c_0(\nabla_q^2, p) = [c_0(p)]_{\nabla_q^2}, \quad c(\nabla_q^2, p) = [c(p)]_{\nabla_q^2}.$$

It is clear that when the exponent sequence $p = (p_s)$ is chosen as the constant sequence $e = (1, 1, \ldots)$, these spaces reduce to the earlier spaces $c_0(\nabla_q^2)$ and $c(\nabla_q^2)$, respectively, as introduced by Yaying et al. [36].

Let us now define the $\nabla_q^2$-transform of a sequence $h = (h_r)$ by:

$$f_r = \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q h_s \quad \text{for all } r \in \mathbb{N}. \tag{3.1}$$

A straightforward inversion of the transformation in (3.1) leads to the recovery formula:

$$h_r = \sum_{s=0}^{r} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q f_s \quad \text{for all } r \in \mathbb{N}. \tag{3.2}$$

We are now in a position to present the main result characterizing the paranorm structure of the spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$.

**Theorem 3.1.** *The sequence spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$, equipped with the functional*

$$f(h) = \sup_{r \in \mathbb{N}} \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q h_s \right|^{p_r/L}, \tag{3.3}$$

*are complete linear metric spaces.*

*Proof.* We demonstrate the proof only for the space $c(\nabla_q^2, p)$ in order to avoid redundant arguments. Clearly, the equality $f(\theta) = 0$ is satisfied, and for every sequence $h \in c(\nabla_q^2, p)$, the symmetry condition $f(-h) = f(h)$ holds. Furthermore, for any sequences $h, t \in c(\nabla_q^2, p)$ and scalars $\alpha_1, \alpha_2 \in \mathbb{R}$, the following inequalities are satisfied:

$$\begin{aligned} f(\alpha_1 h + \alpha_2 t) &= \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q (\alpha_1 h_s + \alpha_2 t_s) \right|^{p_r/L} \\ &\leq \max\{1, |\alpha_1|\} \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q h_s \right|^{p_r/L} + \\ &+ \max\{1, |\alpha_2|\} \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q t_s \right|^{p_r/L} \\ &= \max\{1, |\alpha_1|\} f(h) + \max\{1, |\alpha_2|\} f(t) \end{aligned}$$

can be used to demonstrate the linearity of $f$ with regard to scalar multiplication and coordinate-wise addition. This demonstrates that the space $c(\nabla_q^2, p)$ is linear.

Assume that $\{h^r\}$ is any sequence of points $h^r \in c(\nabla_q^2, p)$ such that $f(h^r - h) \to 0$ and $(\alpha_r)$ is any sequence of scalars such that $\alpha_r \to \alpha$. Due to the subadditivity of $f$, the inequality

$$f(h^r) \leq f(h) + f(h^r - h)$$

holds, and as a result $\{f(h^r)\}$ is bounded. Consequently, we have

$$\begin{aligned} f(\alpha_r h^r - \alpha h) &= \sup_{r\in\mathbb{N}} \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q (\alpha_r h_s^r - \alpha h_s) \right|^{p_r/L} \\ &\leq |\alpha_r - \alpha|^{p_r/L} f(h^r) + |\alpha|^{p_r/L} f(h^r - h) \end{aligned} \tag{3.4}$$

which tends to zero as $r \to \infty$. Accordingly, the scalar multiplication is continuous. In light of this, $f$ is a paranorm on the space $c(\nabla_q^2, p)$.

It is still necessary to demonstrate that the space $c(\nabla_q^2, p)$ is complete. In the space $c(\nabla_q^2, p)$, let $\{h^r\}$ be any Cauchy sequence , where $h^r = \{h_0^{(r)}, h_1^{(r)}, h_2^{(r)}, \ldots\}$. Then, for a given value of $\epsilon > 0$, $n_0(\epsilon)$ exists such that

$$f(h^r - h^s) < \frac{\epsilon}{2}$$

for all $s, r > n_0(\epsilon)$. For each constant $k \in \mathbb{N}$, we find using the definition of $f$ that

$$|(\nabla_q^2 h^r)_k - (\nabla_q^2 h^s)_k|^{p_k/L} \leq \sup_{k\in\mathbb{N}} |(\nabla_q^2 h^r)_k - (\nabla_q^2 h^s)_k|^{p_k/L} < \frac{\epsilon}{2} \tag{3.5}$$

for every $s, r > n_0(\epsilon)$. As a result, for every fixed $k \in \mathbb{N}$, $\{(\nabla_q^2 h^0)_k, (\nabla_q^2 h^1)_k, (\nabla_q^2 h^2)_k, \ldots\}$ is a Cauchy sequence of real numbers. It converges because $\mathbb{R}$ is complete, which means that $(\nabla_q^2 h^r)_k \to (\nabla_q^2 h)_k$ as $k \to \infty$.

Now, we define the sequence $\{(\nabla_q^2 h)_0, (\nabla_q^2 h)_1, \ldots\}$ using these infinitely many limits $(\nabla_q^2 h)_0, (\nabla_q^2 h)_1, \ldots$. For every fixed $k \in \mathbb{N}$, we have

$$|(\nabla_q^2 h^r)_k - (\nabla_q^2 h)_k|^{p_k/L} \leq \frac{\epsilon}{2} \; (r > n_0(\epsilon)) \tag{3.6}$$

from (3.5) with $s \to \infty$. Since $h^r = \{h_k^{(r)}\} \in c(\nabla_q^2, p)$ for each $k \in \mathbb{N}$,

$$|(\nabla_q^2 h^r)_k|^{p_k/L} < \frac{\epsilon}{2} \tag{3.7}$$

for every $r \geq n_0(\epsilon)$ and for each fixed $k \in \mathbb{N}$. For this reason, assuming a constant $r > n_0(\epsilon)$ we derive by (3.6) and (3.7) that

$$|(\nabla_q^2 h)_k|^{p_k/L} \leq |(\nabla_q^2 h)_k - (\nabla_q^2 h^r)_k|^{p_k/L} + |(\nabla_q^2 h^r)_k|^{p_k/L} < \epsilon$$

for every $s > s_0(\epsilon)$. This demonstrates that $h \in c(\nabla_q^2, p)$. The space $c(\nabla_q^2, p)$ is complete, and this closes the proof because $\{h^r\}$ was an arbitrary Cauchy sequence. □

The sequence spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$ are in fact of the non-absolute type since there exists at least one sequence in them such that $f(h) \neq f(|h|)$, where $|h| = (|h_s|)$.

**Theorem 3.2.** *The two spaces, $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$, are linearly isomorphic to $c_0(p)$ and $c(p)$, respectively, where $0 < p_s \leq H < \infty$.*

*Proof.* In order to avoid reiterating identical claims, we only provide the evidence for $c(\nabla_q^2, p)$. We should show the existence of a linear bijection between the sequence spaces $c(\nabla_q^2, p)$ and $c(p)$. Defining the transformation $T$ of $c(\nabla_q^2, p)$ to $c(p)$ by $h \mapsto f = Th$ use the notation (3.1). The linearity of $T$ is trivial. Additionally, it is clear that $T$ is injective because $h = \theta$ whenever $Th = \theta$.

Let $f \in c(p)$ with (3.2), we have

$$\begin{aligned} f(h) &= \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q h_s \right|^{p_r/L} \\ &= \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \sum_{s=0}^{j} \begin{bmatrix} j-s+1 \\ j-s \end{bmatrix} f_s \right|^{p_r/L} \\ &= \sup_r \left| \sum_{s=0}^{r} \delta_{rs} y_s \right|^{p_r/L} \\ &= \sup_r |y_r|^{p_r/L} < \infty, \end{aligned}$$

where

$$\delta_{rs} = \begin{cases} 1 & , \quad s = r, \\ 0 & , \quad s \neq r. \end{cases}$$

Therefore, we get $h \in c(\nabla_q^2, p)$ and $T$ is surjective. As a result, $T$ is a linear bijection between the spaces $c(\nabla_q^2, p)$ and $c(p)$. □

We will now finish by giving the Schauder basis of spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$. First, let us recall the definition of the Schauder basis. The description of a paranormed space is $(H, g)$, where $H$ is a set and $g$ is a paranorm. There exists a unique sequence of scalars $(\alpha_s)$ such that

$$g\left(h - \sum_{s=0}^{r} \alpha_s \beta_s\right) \to 0 \text{ as } r \to \infty$$

if and only if the sequence $(\beta_s)$ of the elements of $H$ is called a basis for $H$.

Let $H$ be a sequence space and $\mathcal{A}$ be a triangle. Due to the fact (cf. [38, Remark 2.4]) that $H_{\mathcal{A}}$ has a basis when $H$ has a basis, we get the following theorem.

**Theorem 3.3.** *Set $H_s = (\nabla_q^2 h)_s$ for all $s \in \mathbb{N}$ and $0 < p_s \leq H < \infty$. The sequence $b^{(s)} = \{b^{(s)}\}_{s\in\mathbb{N}}$ of the elements of the spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$ is defined by*

$$b_r^{(s)} = \begin{cases} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q & , \quad 0 \leq s < r \\ 0 & , \quad s \geq r \end{cases}$$

*for every fixed $s \in \mathbb{N}$. Then,*

*(i) The sequence $\{b^{(s)}\}_{s\in\mathbb{N}}$ is a basis for the spaces $c_0(\nabla_q^2, p)$, and any $h \in c_0(\nabla_q^2, p)$ has a unique representation of the form*

$$h = \sum_s H_s b^{(s)}.$$

*(ii) The set $\{e, b^{(1)}, b^{(2)}, ...\}$ is a basis for the space $c(\nabla_q^2, p)$, and any $h \in c(\nabla_q^2, p)$ has a unique representation of the form*

$$h = \tau e + \sum_s [H_s - \tau] b^{(s)},$$

*where $\tau = \lim_{s\to\infty} (\nabla_q^2 h)_s$.*

### 3.1 Duals of the Sequence Spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$

This section is devoted to deriving the $\alpha$-, $\beta$-, and $\gamma$-duals of the sequence spaces defined earlier. We denote the conjugate of $p$ by $p^\star$, where $\frac{1}{p} + \frac{1}{p^\star} = 1$, and let $\mathcal{N}$ denote the set of all finite subsets of $\mathbb{N}$.

The $\alpha$-, $\beta$-, and $\gamma$-duals of the $H$ sequence are designated by $H^\alpha$, $H^\beta$, and $H^\gamma$, respectively, and defined by

$$H^\alpha = \{t \in w : \sum_{s=1}^{\infty} |h_s t_s| < \infty \text{ for all } h \in H\},$$

$$H^\beta = \{t \in w : \sum_{s=1}^{\infty} h_s t_s \text{ for all } h \in H\},$$

$$H^\gamma = \{t = (t_s) \in w : ht = (h_s t_s) \in bs \text{ for all } h = (h_s) \in H\}.$$

We require the following lemma to provide the $\alpha$-, $\beta$- and $\gamma$-duals of the spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$.

**Lemma 3.4.** *[39, $t_r = 1$] The following statements hold for an infinite matrix $\mathcal{A} = (a_{rs})$:*

*(i)* $\mathcal{A} \in (c_0(p) : \ell(t))$ *iff*

$$\exists M > 1 \text{ such that } \sup_{K \in \mathcal{F}} \sum_r \left| \sum_{s \in K} a_{rs} M^{-1/p_s} \right|^{t_r} < \infty \quad \text{for any } t_r \geq 1. \tag{3.8}$$

*(ii)* $\mathcal{A} \in (c(p) : \ell(t))$ *iff* (3.8) *holds, and*

$$\sum_r \left| \sum_s a_{rs} \right|^{t_r} < \infty. \tag{3.9}$$

*(iii)* $\mathcal{A} \in (c_0(p) : c(t))$ *iff*

$$\exists M > 1 \text{ such that } \sup_r \sum_s |a_{rs}| M^{-1/p_s} < \infty, \tag{3.10}$$

$$\exists (\alpha_s) \text{ such that } \lim_r |a_{rs} - \alpha_s|^{t_r} = 0, \tag{3.11}$$

$$\forall L > 0, \exists M > 1, \exists (\alpha_s) \text{ such that } \sup_r L^{1/t_r} \sum_s |a_{rs} - \alpha_s| \, M^{-1/p_s} < \infty. \tag{3.12}$$

*(iv)* $\mathcal{A} \in (c(p) : c(t))$ *iff* (3.10), (3.11), (3.12) *hold, and*

$$\exists \alpha \text{ such that } \lim_r \left| \sum_s a_{rs} - \alpha \right|^{t_r} = 0. \tag{3.13}$$

*(v)* $\mathcal{A} \in (c_0(p) : \ell_\infty(t))$ *iff*

$$\exists M > 0 \text{ such that } \sup_r \left( \sum_s |a_{rs}| M^{-1/p_s} \right)^{t_r} < \infty. \tag{3.14}$$

*(vi)* $\mathcal{A} \in (c(p) : \ell_\infty(t))$ *iff* (3.14) *holds, and*

$$\sup_r \left| \sum_s a_{rs} \right|^{t_r} < \infty. \tag{3.15}$$

**Theorem 3.5.** *Let $K \in \mathcal{F}$. Define the sets $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$, $c_7$ and $c_8$ as follows:*

$$c_1 = \bigcup_{M>1} \left\{ a = (a_s) \in \omega : \sup_K \sum_r \left| \sum_{s \in K} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r M^{-1/p_s} \right| < \infty \right\},$$

$$c_2 = \left\{ a = (a_s) \in \omega : \sum_r \left| \sum_s \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r \right| < \infty \right\},$$

$$c_3 = \bigcup_{M>1} \left\{ a = (a_s) \in \omega : \sup_r \sum_s \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \right| M^{-1/p_s} < \infty \right\},$$

$$c_4 = \left\{ a = (a_s) \in \omega : \exists (\alpha_s) \subset \mathbb{R} \text{ such that } \lim_r \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z - \alpha_s \right| = 0 \right\},$$

$$c_5 = \bigcup_{M>1} \left\{ a = (a_s) \in w : \exists (\alpha_s) \subset \mathbb{R} \text{ such that } \sum_s \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z - \alpha_s \right| M^{-1/p_s} < \infty \right\},$$

$$c_6 = \left\{ a = (a_s) \in \omega : \exists (\alpha_s) \subset \mathbb{R} \text{ such that } \lim_r \left| \sum_s \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z - \alpha_s \right| < \infty \right\},$$

$$c_7 = \bigcup_{M>1} \left\{ a = (a_s) \in \omega : \sup_r \left( \sum_s \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \right| M^{-1/p_s} \right) < \infty \right\},$$

$$c_8 = \left\{ a = (a_s) \in \omega : \sup_r \left| \sum_s \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \right| < \infty \right\}.$$

*Then,*

$$\left[c_0(\nabla_q^2, p)\right]^\alpha = c_1, \qquad \left[c(\nabla_q^2, p)\right]^\alpha = c_1 \cap c_2,$$
$$\left[c_0(\nabla_q^2, p)\right]^\beta = c_3 \cap c_4 \cap c_5, \qquad \left[c(\nabla_q^2, p)\right]^\beta = c_3 \cap c_4 \cap c_5 \cap c_6,$$
$$\left[c_0(\nabla_q^2, p)\right]^\gamma = c_7, \qquad \left[c(\nabla_q^2, p)\right]^\gamma = c_8.$$

*Proof.* To prove $\alpha$- dual, take the sequence $a = (a_s) \in w$ and $h = (h_s)$ as in defined in (3.2), then

$$a_r h_r = \sum_{s=0}^{r} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r f_s = (Cf)_r, \tag{3.16}$$

for all $r \in \mathbb{N}$, where the matrix $C = (c_{rs})$ defined by

$$c_{rs} = \begin{cases} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r & , \quad 0 \le s \le r, \\ 0 & , \quad s > r. \end{cases}$$

It follows from (3.16) that $ah = (a_r h_r) \in \ell_1$ whenever $h \in c_0(\nabla_q^2, p)$ iff $Cf \in \ell_1$ whenever $f \in c_0(p)$. This means that $a = (a_r) \in [c_0(\nabla_q^2, p)]^\alpha$ iff $C \in (c_0(p) : \ell_1)$. Then we derive by (3.8) with $t_r = 1$ for all $r \in \mathbb{N}$ that $[c_0(\nabla_q^2, p)]^\alpha = c_1$. Also, using the (3.8),(3.9) and (3.16), we can prove $[c(\nabla_q^2, p)]^\alpha = c_1 \cap c_2$ can also be obtained in a similar way.

Now, consider the following equality

$$\begin{aligned} \sum_{s=0}^{r} a_s h_s &= \sum_{s=0}^{r} \left[ \sum_{z=0}^{s} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q f_z \right] a_s \\ &= \sum_{s=0}^{r} \left[ \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \right] f_s = (Df)_r, \end{aligned} \tag{3.17}$$

where $D = (d_{rs})$ is a matrix defined by

$$d_{rs} = \begin{cases} \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z & , \quad (0 \leq s \leq r) \\ 0 & , \quad (s > r) \end{cases}$$

Thus, we conclude from (3.17) that $ah = (a_s h_s) \in cs$ whenever $h = (h_s) \in c_0(\nabla_q^2, p)$ iff $Df \in c_0$ whenever $f = (f_s) \in c_0(p)$. That is to say that $a = (a_s) \in [c_0(\nabla_q^2, p)]^\beta$ iff $D \in (c_0(p) : c)$. Therefore, we derive from (3.10), (3.11), (3.12) and (3.17) with $t_r = 1$ for all $r \in \mathbb{N}$ that $[c_0(\nabla_q^2, p)]^\beta = c_3 \cap c_4 \cap c_5$. Also, using the (3.10), (3.11), (3.12), (3.13) and (3.17), the proofs of the $[c(\nabla_q^2, p)]^\beta = c_3 \cap c_4 \cap c_5 \cap c_6$ can also be obtained in a similar way. □

### 3.2 Certain Matrix Mappings on the Sequence Spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$

This section describes the matrix transformations from any given sequence space $H$ into the spaces $c_0(\nabla_q^2, p)$ and $c(\nabla_q^2, p)$. The equivalence "$h \in H_{\mathcal{A}}$ holds for every triangle $\mathcal{A}$ and any sequence space $H$ iff $f = \mathcal{A}h \in H$". The reason for this is that $H_{\mathcal{A}} \cong H$.

**Theorem 3.6.** *Set $H = \{\ell_\infty, c, \ell_1\}$. Then*

*(i)* $\mathcal{A} = (a_{rs}) \in (c_0(\nabla_q^2, p) : H)$ *iff*

$$\mathcal{A}_r \in \left\{c_0(\nabla_q^2, p)\right\}^\beta \text{ for all } r \in \mathbb{N} \tag{3.18}$$

$$B \in (c_0(p) : H), \tag{3.19}$$

*(ii)* $\mathcal{A} = (a_{rs}) \in (c(\nabla_q^2, p) : H)$ *iff*

$$\mathcal{A}_r \in \left\{c(\nabla_q^2, p)\right\}^\beta \text{ for all } r \in \mathbb{N} \tag{3.20}$$

$$B \in (c(p) : H), \tag{3.21}$$

*where $B = (b_{rs})$ with $b_{rs} = \sum_{z=s}^{\infty} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_{rz}$ for all $r, s \in \mathbb{N}$ and $\mathcal{A}_r$ denotes the $r^{th}$ row of the infinite matrix $\mathcal{A}$.*

*Proof.* We prove only part of (i). Let $\mathcal{A} = (a_{rs}) \in (c_0(\nabla_q^2, p) : H)$ and $h \in c_0(\nabla_q^2, p)$ be any given sequence. Then $\mathcal{A}_r \in \left\{c_0(\nabla_q^2, p)\right\}^\beta$ for all $r \in \mathbb{N}$. Taking $f = \nabla_q^2 h$, we conclude from $\mathcal{A}_r \in \left\{c_0(\nabla_q^2, p)\right\}^\beta$ that

$$(\mathcal{A}h)_r = (Bf)_r \quad \text{for all } r \in \mathbb{N}. \tag{3.22}$$

Hence, $Bf \in H$ for all $f \in c_0(p)$. Thus, $B \in (c_0(p) : H)$.

We have $Ef = \mathcal{A}h$ which leads us to the consequence $E \in (c_0(p) : \mu)$. To obtain a contradiction, suppose that (3.18) and (3.19) hold. Then (3.22) again holds. It follows that $\mathcal{A}h \in H$ for all $h \in c_0(\nabla_q^2, p)$. □

**Corollary 3.7.** *Let $\mathcal{A} = (a_{rs})$ be an infinite matrix. The following statements hold:*

*(i)* $\mathcal{A} \in (c_0(\nabla_q^2, p) : \ell(t))$ *iff* $\{a_{rs}\}_{s \in \mathbb{N}} \in \{c_0(\nabla_q^2, p)\}^\beta$ *for all* $r \in \mathbb{N}$ *and* (3.8) *holds with* $b_{rs}$ *instead of* $a_{rs}$ *with* $t = 1$.

*(ii)* $\mathcal{A} \in (c_0(\nabla_q^2, p) : c(t))$ *iff* $\{a_{rs}\}_{s \in \mathbb{N}} \in \{c_0(\nabla_q^2, p)\}^\beta$ *for all* $r \in \mathbb{N}$ *and* (3.10), (3.11) *and* (3.12) *hold with* $b_{rs}$ *instead of* $a_{rs}$ *with* $t = 1$.

*(iii)* $\mathcal{A} \in (c_0(\nabla_q^2, p) : \ell_\infty(t))$ *iff* $\{a_{rs}\}_{s \in \mathbb{N}} \in \{c_0(\nabla_q^2, p)\}^\beta$ *for all* $r \in \mathbb{N}$ *and* (3.14) *hold with* $b_{rs}$ *instead of* $a_{rs}$ *with* $t = 1$.

**Corollary 3.8.** *Let $\mathcal{A} = (a_{rs})$ be an infinite matrix. The following statements hold:*

*(i)* $\mathcal{A} \in (c(\nabla_q^2, p) : \ell(t))$ *iff* $\{a_{rs}\}_{s \in \mathbb{N}} \in \{c(\nabla_q^2, p)\}^\beta$ *for all* $r \in \mathbb{N}$ *and* (3.8) *and* (3.9) *hold with* $b_{rs}$ *instead of* $a_{rs}$ *with* $t = 1$.

*(ii)* $\mathcal{A} \in (c(\nabla_q^2, p) : c(t))$ *iff* $\{a_{rs}\}_{s \in \mathbb{N}} \in \{c(\nabla_q^2, p)\}^\beta$ *for all* $r \in \mathbb{N}$ *and* (3.10), (3.11), (3.12) *and* (3.13) *hold with* $b_{rs}$ *instead of* $a_{rs}$ *with* $t = 1$.

*(iii)* $\mathcal{A} \in (c(\nabla_q^2, p) : \ell_\infty(t))$ *iff* $\{a_{rs}\}_{s \in \mathbb{N}} \in \{c(\nabla_q^2, p)\}^\beta$ *for all* $r \in \mathbb{N}$ *and* (3.15) *holds with* $b_{rs}$ *instead of* $a_{rs}$ *with* $t = 1$.

### 3.3 Compact Operators on the Space $c_0(\nabla_q^2)$

For a normed space $\Gamma$, the unit sphere in $\Gamma$ is denoted by $\mathcal{H}_\Gamma$. The following notation is introduced:

$$\|u\|_\Gamma^* = \sup_{u\in\mathcal{H}_\Gamma}\left|\sum_k h_s u_k\right|$$

under the assumption that the supremum is finite for a BK-space $\Gamma$ satisfying $\Gamma \supset \Omega$ and for any $h = (h_s) \in \omega$. As a result, it can be concluded that $h \in \Gamma^\beta$.

**Lemma 3.9.** *[40] The following equalities hold: $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$, and for any $h$, we have $\|h\|_\Gamma^* = \|h\|_{\ell_1}$, where $\Gamma \in \{\ell_\infty, c, c_0\}$.*

The set $\mathfrak{B}(\Gamma : \Psi)$ denotes the collection of all bounded (continuous) linear operators from $\Gamma$ to $\Psi$.

**Lemma 3.10.** *[41] Let $\Gamma$ and $\Psi$ be BK-spaces. For every $B \in (\Gamma : \Psi)$, there exists a linear operator $\mathcal{L}_B \in \mathfrak{B}(\Gamma : \Psi)$ such that $\mathcal{L}_B(u) = Bu$ for all $u \in \Gamma$.*

**Lemma 3.11.** *[41] Suppose that $\Gamma \supset \Omega$ is a BK-space. If $B \in (\Gamma : \Psi)$, then the norm of the linear operator $\mathcal{L}_B$ is given by $\|\mathcal{L}_B\| = \|B\|_{(\Gamma:\Psi)} = \sup_{r\in\mathbb{N}} \|B_r\|_\Gamma^*$, where $\Psi \in \{\ell_\infty, c, c_0\}$.*

Let $\Gamma$ be a metric space and let $A \subset \Gamma$ be bounded. The Hausdorff measure of non-compactness of $A$ is denoted by $\chi(A)$, and it is defined as

$$\chi(A) = \inf\left\{\varepsilon > 0 : A \subset \bigcup_{l=1}^{n} A(u_l, r_l), u_l \in \Gamma, r_l < \varepsilon, n \in \mathbb{N}\setminus\{0\}\right\},$$

where $A(u_l, r_l)$ represents the open ball centered at $u_l$ with radius $r_l$ for $l = 1, 2, \ldots, n$. More detailed information about the Hausdorff measure of non-compactness can be found in [41] and its references.

**Theorem 3.12.** *Let $A \subset c_0$ be a bounded set, and let the operator $\lambda_s : c_0 \longrightarrow c_0$ be defined by $\lambda_s(u) = (u_0, u_1, u_2, \ldots, u_s, 0, 0, \ldots)$ for each $u = (u_s) \in c_0$ with $s \in \mathbb{N}$. Then, for the identity operator $I$ on $c_0$, we have*

$$\chi(A) = \lim_{s\to\infty}\left(\sup_{u\in A}\|(I-\lambda_s)(u)\|_{c_0}\right).$$

For Banach spaces $\Gamma$ and $\Psi$, a linear operator $\mathcal{L} : \Gamma \to \Psi$ is said to be a compact operator if for every $u = (u_k) \in \ell_\infty \cap \Gamma$, the sequence $(\mathcal{L}(u))$ has a convergent subsequence in $\Psi$.

Let $\|\mathcal{L}\|_\chi$ denote the Hausdorff measure of non-compactness of $\mathcal{L}$, defined as $\|\mathcal{L}\|_\chi = \chi(\mathcal{L}(A_\Gamma))$. In this context, it is known that $\mathcal{L}$ is compact if and only if $\|\mathcal{L}\|_\chi = 0$.

For sequence space studies, where the Hausdorff measure of non-compactness is used to characterize compact operators between BK-spaces, readers can refer to the works [43, 44, 42].

In this section, we characterize a certain class of compact operators on the newly defined sequence space $c_0(\nabla_q^2)$.

We can give the following results for $h = (h_s), f = (f_s) \subset \omega$ connected to each other by

$$f_r = \sum_{s=0}^{r}(-1)^{r-s}q^{\binom{r-s}{2}}\begin{bmatrix}2\\ r-s\end{bmatrix}_q h_s$$

**Lemma 3.13.** *Consider that $h = (h_s) \in (c_0(\nabla_q^2))^\beta$. In that case, $f = (f_s) \in \ell_1$ and*

$$\sum_s h_s u_s = \sum_s f_s \nu_s \tag{3.23}$$

*for all $u = (u_s) \in c_0(\nabla_q^2)$.*

**Lemma 3.14.** *The statement*

$$\|h\|_{c_0(\nabla_q^2)}^* = \sum_s |f_s| < \infty$$

*holds for each $h = (h_s) \in (c_0(\nabla_q^2))^\beta$.*

*Proof.* It is achieved from the Lemma 3.13 that $f=(f_s)\in\ell_1$ and (3.23) holds for $h=(h_s)\in(c_0(\nabla_q^2))^\beta$. Since, $\|u\|_{c_0(\nabla_q^2)}=\|\nu\|_{c_0}$, we reach that "$u\in\mathcal{H}_{c_0(\nabla_q^2)}$ if and only if $\nu\in\mathcal{H}_{c_0}$". Thus, we can write the equality

$$\|h\|^*_{c_0(\nabla_q^2)}=\sup_{u\in\mathcal{H}_{c_0(\nabla_q^2)}}\left|\sum_s h_s u_s\right|=\sup_{\nu\in\mathcal{H}_{c_0}}\left|\sum_s f_s\nu_s\right|=\|f\|^*_{c_0}.$$

By the aid of the Lemma 3.9, it follows that

$$\|h\|^*_{c_0(\nabla_q^2)}=\|f\|^*_{c_0}=\|f\|_{\ell_1}=\sum_s|f_s|<\infty,$$

which is the desired result. □

In the sequel of the study, it is used the matrices $\Theta=(\theta_{rs})$ and $B=(b_{rs})$ connected with the relation (3.2) provided the convergence of the series.

**Lemma 3.15.** *Let $\Psi\subset\omega$ and $B=(b_{rs})$ is an infinite matrix. If $B\in(c_0(\nabla_q^2):\Psi)$, then $\Theta\in(c_0:\Psi)$ and $Bu=\Theta\nu$ for all $u\in c_0(\nabla_q^2)$.*

*Proof.* It is obvious from Lemma 3.13. So, we omit it. □

**Lemma 3.16.** *If $B\in(c_0(\nabla_q^2):\Psi)$, then it is achieved that*

$$\|\mathcal{L}_B\|=\|B\|_{(c_0(\nabla_q^2):\Psi)}=\sup_{r\in\mathbb{N}}\left(\sum_s|\theta_{rs}|\right)<\infty,$$

*where $\Psi\in\{\ell_\infty,c,c_0\}$.*

**Lemma 3.17.** *[45] Let $\Gamma\supset\Omega$ be a BK space. In that case:*

*(i) If $B\in(\Gamma:\ell_\infty)$, then*

$$0\le\|\mathcal{L}_B\|_\chi\le\limsup_{r\to\infty}\|B_r\|^*_\Gamma.$$

*(ii) If $B\in(\Gamma:c_0)$, then*

$$\|\mathcal{L}_B\|_\chi=\limsup_{r\to\infty}\|B_r\|^*_\Gamma.$$

*(iii) If $\Gamma$ has the AK property or $\Gamma=\ell_\infty$, and $B\in(\Gamma:c)$, then*

$$\frac{1}{2}\limsup_{r\to\infty}\|B_r-b\|^*_\Gamma\le\|\mathcal{L}_B\|_\chi\le\limsup_{r\to\infty}\|B_r-b\|^*_\Gamma,$$

*where $b=(b_s)$ and $b_s=\lim_{r\to\infty}b_{rs}$.*

**Lemma 3.18.** *[45] Let $\Gamma\supset\Omega$ be a BK space. If $B\in(\Gamma:\ell_1)$, in that case*

$$\lim_s\left(\sup_{E\in\mathcal{E}_s}\left\|\sum_{r\in E}B_r\right\|^*_\Gamma\right)\le\|\mathcal{L}_\mathcal{B}\|_\chi\le4.\lim_s\left(\sup_{E\in\mathcal{E}_s}\left\|\sum_{r\in E}B_r\right\|^*_\Gamma\right)$$

*and $\mathcal{L}_B$ is compact if and only if $\lim_s\left(\sup_{E\in\mathcal{E}_s}\left\|\sum_{r\in E}B_r\right\|^*_\Gamma\right)=0$, where $\mathcal{E}_s$ is the subcollection of $\mathcal{E}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than $s$.*

**Theorem 3.19.** *(i) For $B\in(c_0(\nabla_q^2):\ell_\infty)$,*

$$0\le\|\mathcal{L}_\mathcal{B}\|_\chi\le\limsup_r\sum_s|\theta_{rs}|$$

*holds.*

*(ii)* *For* $B \in (c_0(\nabla_q^2) : c)$,

$$\frac{1}{2}\limsup_r \sum_s |\theta_{rs} - \theta_s| \leq \|\mathcal{L}_B\|_\chi \leq \limsup_r \sum_s |\theta_{rs} - \theta_s|$$

*holds.*

*(iii)* *For* $B \in (c_0(\nabla_q^2) : c_0)$,

$$\|\mathcal{L}_B\|_\chi = \limsup_r \sum_s |\theta_{rs}|$$

*holds.*

*(iv)* *For* $B \in (c_0(\nabla_q^2) : \ell_1)$,

$$\lim_s \|B\|^{(s)}_{(c_0(\nabla_q^2):\ell_1)} \leq \|\mathcal{L}_B\|_\chi \leq 4.\lim_s \|B\|^{(s)}_{(c_0(\nabla_q^2):\ell_1)}$$

*holds, where* $\|B\|^{(s)}_{(c_0(\nabla_q^2):\ell_1)} = \sup_{E\in\mathcal{E}_s}\left(\sum_s \left|\sum_{r\in E}\theta_{rs}\right|\right)$ *for all* $s \in \mathbb{N}$.

*Proof.* (i) Let $B \in (c_0(\nabla_q^2) : \ell_\infty)$. Since the series $\sum_{k=1}^\infty b_{rs}u_s$ converges for each $n \in \mathbb{N}$, it is seen that $B_r \in (c_0(\nabla_q^2))^\beta$. From Lemma 3.14, we reach that

$$\|B_r\|^*_{c_0(\nabla_q^2)} = \|\Theta_r\|^*_{c_0} = \|\Theta_r\|_{\ell_1} = \left(\sum_s |\theta_{rs}|\right).$$

From Lemma 3.17 $(a)$, it is achieved that

$$0 \leq \|\mathcal{L}_B\|_\chi \leq \limsup_r \sum_s |\theta_{rs}|.$$

(ii) If $B \in (c_0(\nabla_q^2) : c)$, in that case $\Theta \in (c_0 : c)$ from Lemma 3.15. Thus, by the aid of the Lemma 3.17 $(c)$ it is reached that

$$\frac{1}{2}\limsup_r \|\Theta_r - \theta\|^*_{c_0} \leq \|\mathcal{L}_B\|_\chi \leq \limsup_r \|\Theta_r - \theta\|^*_{c_0},$$

where $\theta = (\theta_s)$ and $\theta_s = \lim_{r\to\infty}\theta_{rs}$ for $k \in \mathbb{N}$. Thus, from Lemma 3.9 $\|\Theta_r - \theta\|^*_{c_0} = \|\Theta_r - \theta\|_{\ell_1} = \sum_s |\theta_{rs} - \theta_s|$ for each $n \in \mathbb{N}$.

(iii) Let $B \in (c_0(\nabla_q^2) : c_0)$. From the relation $\|B_r\|^*_{c_0(\nabla_q^2)} = \|\Theta_r\|^*_{c_0} = \|\Theta_r\|_{\ell_1} = (\sum_s |\theta_{rs}|)$ and Lemma 3.17 $(b)$, it is concluded that $\|\mathcal{L}_B\|_\chi = \limsup_r \sum_s |\theta_{rs}|$.

(iv) Let $B \in (c_0(\nabla_q^2) : \ell_1)$. By Lemma 3.15, it is reached that $\Theta \in (c_0 : \ell_1)$. By Lemma 3.18 it is achieved

$$\lim_s \left(\sup_{E\in\mathcal{E}_s}\left\|\sum_{r\in E}\Theta_r\right\|^*_{c_0}\right) \leq \|\mathcal{L}_B\|_\chi \leq 4.\lim_s \left(\sup_{E\in\mathcal{E}_s}\left\|\sum_{r\in E}\Theta_r\right\|^*_{c_0}\right).$$

Furthermore, Lemma 3.9 implies that

$$\left\|\sum_{r\in E}\Theta_r\right\|^*_{c_0} = \left\|\sum_{r\in E}\Theta_r\right\|_{\ell_1} = \left(\sum_s \left|\sum_{r\in E}\theta_{rs}\right|\right),$$

which is desired result.

□

Thus, making use of the theorem given above, we can give the following result.

**Corollary 3.20.** *(i) For $B \in (c_0(\nabla_q^2) : \ell_\infty)$, $\mathcal{L}_B$ is compact if and only if*

$$\lim_r \sum_s |\theta_{rs}| = 0.$$

*(ii) For $B \in (c_0(\nabla_q^2) : c)$, $\mathcal{L}_B$ is compact if and only if*

$$\lim_r \sum_s |\theta_{rs} - \theta_s| = 0.$$

*(iii) For $B \in (c_0(\nabla_q^2) : c_0)$, $\mathcal{L}_B$ is compact if and only if*

$$\lim_r \sum_s |\theta_{rs}| = 0.$$

*(iv) For $B \in (c_0(\nabla_q^2) : \ell_1)$, $\mathcal{L}_B$ is compact if and only if*

$$\lim_s \|B\|^{(s)}_{(c_0(\nabla_q^2):\ell_1)} = 0$$

*for* $\|B\|^{(s)}_{(c_0(\nabla_q^2):\ell_1)} = \sup_{E\in\mathcal{E}_s} \left(\sum_s \left|\sum_{r\in E} \theta_{rs}\right|\right)$ *for all* $s \in \mathbb{N}$.

## 4 Conclusion remarks

The use of $q$-analogues to extend classical mathematical structures provides a powerful framework for understanding more general settings. By incorporating the parameter $q$, previously established theories can be generalized, allowing the investigation of richer mathematical behaviors and the emergence of new limiting cases.

In this study, new paranormed sequence spaces are introduced by means of the $q$-analogue of the second difference matrix. The algebraic and topological structures of these spaces are examined in detail, including inclusion relations, completeness properties, and the behavior of the associated operators. The obtained results indicate that the proposed sequence spaces constitute natural generalizations of several well-known classical sequence spaces and preserve their fundamental structural properties in the limiting case as $q \to 1$.

Overall, the $q$-analogue approach proves to be a flexible and effective tool in the theory of sequence spaces. Moreover, such spaces are expected to have potential applications in approximation theory and operator theory. As a direction for future research, this framework may be extended to higher-order difference operators or other summability matrices. In addition, the investigation of dual spaces, matrix transformations, and results related to approximation theory may further enrich the scope and applicability of the present study.

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