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CERTAIN RESULTS AND APPLICATIONS OF THE GENERALIZED τ–GAUSS WRIGHT HYPERGEOMETRIC MATRIX FUNCTION

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Abstract. In this paper, we introduce a generalized τ–Gauss Wright hypergeometric matrix function (GτGWHMF) by employing the Pochhammer matrix symbol together with the extended beta matrix function (EBMF). We then investigate some fundamental properties of the τ–Gauss Wright hypergeometric matrix function, including its integral representations, differential formulas, and transformation relations. Furthermore, we define the incomplete τ–Gauss Wright hypergeometric matrix function (IτGWHMF) and establish various properties of the associated incomplete generalized Wright hypergeometric matrix function. Finally, as an application, we formulate and analyze a fractional kinetic equation (FKE) involving the GτGWHMF.

1 Introduction

In modern mathematics, special functions play a crucial role in various disciplines. Special versions of these functions have proven invaluable in many fields, including probability theory, computer science, mathematical physics, engineering and many other areas [4, 6, 10, 17, 23, 24, 25, 29, 31, 32, 33]. In the history of hypergeometric functions, Gauss first summarized his studies of hypergeometric functions which have been of great significance for the mathematical modeling of physical phenomena and other applications [6, 13, 17, 28, 29, 32].
The theory of special matrix functions was first developed by Jódar and Cortés [21, 22], who introduced matrix analogues of the gamma and beta functions and established several fundamental properties, including a limit representation for the matrix gamma function. Building on their pioneering work, various researchers have extended this framework to encompass generalized gamma, beta, and Gauss hypergeometric matrix functions [21, 22]. These developments have generated considerable interest in the study of special functions with matrix arguments [1, 3, 5, 9, 14, 26], given their growing relevance in both theoretical investigations and applied contexts.

Special matrix functions now play a critical role in diverse fields such as mathematical and theoretical physics, where they are instrumental in solving physical models and systems of matrix differential equations. They also find applications in statistics [11], Lie group theory [28], and the emerging theory of orthogonal matrix polynomials [16, 18, 20]. As a result, the matrix extensions of classical special functions have become a vibrant area of contemporary mathematical research, offering rich structures and wide-ranging applications.
Throughout this paper, let $\mathbb{I}$ and $\mathbb{O}$ denote the identity matrix and the zero matrix, respectively, in the complex matrix space $\mathbb{C}^{\eta\times\eta}$, which consists of all $\eta \times \eta$ matrices with complex entries. For any matrix $\mathbb{U} \in \mathbb{C}^{\eta\times\eta}$, its spectrum, denoted by $\sigma(\mathbb{U})$, is defined as the set of all eigenvalues of $\mathbb{U}$. A matrix $\mathbb{U} \in \mathbb{C}^{\eta\times\eta}$ is called positive stable if the real part of each of its eigenvalues is strictly positive, that is, $\mathrm{Re}(\xi) > 0$ for all $\xi \in \sigma(\mathbb{U})$.

The Pochhammer matrix symbol (PMS) is defined as [22]:

$$(\mathbb{U})_\eta = \begin{cases} \mathbb{I}, & \eta = 0, \\ \mathbb{U}(\mathbb{U}+I)...(\mathbb{U}+(\eta-1)\mathbb{I}), & \eta \geq 1. \end{cases} \tag{1.1}$$

The gamma matrix function (GMF) for a positive stable matrix (PSM) $\mathbb{U}$ is defined as [21]:

$$\Gamma(\mathbb{U}) = \int_0^\infty e^{-\delta}\, \delta^{\mathbb{U}-\mathbb{I}} d\delta. \tag{1.2}$$

If $\mathbb{U}+\eta\mathbb{I}$ is invertible $\forall\ \eta \geq 0$, then the relation between GMF and the PMS is defined by [21]:

$$\frac{\Gamma(\mathbb{U}+\eta\mathbb{I})}{\Gamma(\mathbb{U})} = (\mathbb{U})_\eta \qquad \eta \geq 1. \tag{1.3}$$

Let $\mathbb{U}, \mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$, then the beta matrix function (BMF) is introduced as [21]:

$$\mathrm{B}(\mathbb{U},\mathbb{V}) = \int_0^1 \delta^{\mathbb{U}-\mathbb{I}}(1-\delta)^{\mathbb{V}-\mathbb{I}}\, d\delta. \tag{1.4}$$

In addition, if $\mathbb{U}\mathbb{V} = \mathbb{V}\mathbb{U}$, then the relation of beta and gamma matrix function is defined as:

$$\mathrm{B}(\mathbb{U},\mathbb{V}) = \Gamma(\mathbb{U})\Gamma(\mathbb{V})\Gamma^{-1}(\mathbb{U}+\mathbb{V}). \tag{1.5}$$

In 2015, Abul-Dahab et al. [12] introduced a generalized PMS $(\mathbb{U};\mathbb{W})_\eta$. Let $\mathbb{U}$ and $\mathbb{W}$ be PSM in $\mathbb{C}^{\eta\times\eta}$, then

$$(\mathbb{U};\mathbb{W})_\eta = \begin{cases} \Gamma^{-1}(\mathbb{U})\Gamma(\mathbb{U}+\eta\mathbb{I},\mathbb{W}), & \mathbb{W} \neq \mathbb{O}, \\ (\mathbb{U})_\eta & \mathbb{W} = \mathbb{O}, \end{cases} \tag{1.6}$$

and the generalized GMF defined as [12]:

$$\Gamma(\mathbb{U};\mathbb{W}) = \int_0^\infty \delta^{\mathbb{U}-\mathbb{I}}\, e^{-(\mathbb{I}\delta+\frac{\mathbb{W}}{\delta})}\, d\delta. \tag{1.7}$$

The integral representation for the generalized PMS is

$$(\mathbb{U};\mathbb{W})_\eta = \Gamma^{-1}(\mathbb{U}) \int_0^\infty \delta^{\mathbb{U}+(n-1)\mathbb{I}}\, e^{-(\mathbb{I}\delta+\frac{\mathbb{W}}{\delta})}\, d\delta. \tag{1.8}$$

In 1998, L. Jodar and J. C. Cortes [22] first time introduced the hypergeometric matrix function (HMF) as:

$$\mathrm{F}(\mathbb{U},\mathbb{V};\mathbb{W};z) = \sum_{\eta=0}^{\infty} \frac{(\mathbb{U})_\eta(\mathbb{V})_\eta}{(\mathbb{W})_\eta}\frac{z^\eta}{\eta!}, \tag{1.9}$$

where $\mathbb{U}, \mathbb{V}$ and $\mathbb{W}$ are PSM $\in \mathbb{C}^{\eta\times\eta}$ and $z \in \mathbb{C}$ such that $\mathbb{W}+\eta\mathbb{I}$ is invertible for all integer $\eta \geq 0$.

Later, in 2018 Abdalla and Bakhet [2] investigated the extended Gauss hypergeometric (EGHMF) and Kummer hypergeometric matrix functions (EKHMF) as:

$$\mathrm{F}^{(\mathbb{P})}(\mathbb{U},\mathbb{V};\mathbb{W};z) = \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \sum_{\eta=0}^{\infty} (\mathbb{U})_\eta \mathrm{B}(\mathbb{V}+\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P})\frac{z^\eta}{\eta!}, \tag{1.10}$$

$$\Phi^{(\mathbb{P})}(\mathbb{V};\mathbb{W};z) = \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \sum_{\eta=0}^{\infty} \mathrm{B}(\mathbb{V}+\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P})\frac{z^\eta}{\eta!}, \tag{1.11}$$

where the extended BMF [1] $\mathrm{B}(\mathbb{U},\mathbb{V};\mathbb{P})$ is defined by

$$\mathrm{B}(\mathbb{U},\mathbb{V};\mathbb{P}) = \int_0^1 \delta^{\mathbb{U}-\mathbb{I}}(1-\delta)^{\mathbb{V}-\mathbb{I}} \exp\left(\frac{-\mathbb{P}}{\delta(1-\delta)}\right) d\delta, \tag{1.12}$$

and $\mathbb{P}, \mathbb{U}, \mathbb{V}$ and $\mathbb{W}$ are PSM $\in \mathbb{C}^{\eta\times\eta}$.

Bakhet et al. [8] defined the Wright hypergeometric (WHMF) and Wright Kummer hypergeometric matrix functions (WKHMF) as:

$$ {}_2\mathrm{R}_1^{(\tau)}(\mathbb{U},\mathbb{V};\mathbb{W};z) = \Gamma^{-1}(\mathbb{V})\Gamma(\mathbb{W})\sum_{\eta=0}^{\infty}(\mathbb{U})_\eta\Gamma^{-1}(\mathbb{W}+\tau\eta\mathbb{I})\Gamma(\mathbb{V}+\tau\eta\mathbb{I})\frac{z^\eta}{\eta!}, \tag{1.13} $$

and

$$ {}_1\mathrm{R}_1^{(\tau)}(\mathbb{V};\mathbb{W};z) = \Gamma^{-1}(\mathbb{V})\Gamma(\mathbb{W})\sum_{\eta=0}^{\infty}\Gamma^{-1}(\mathbb{W}+\tau\eta\mathbb{I})\Gamma(\mathbb{V}+\tau\eta\mathbb{I})\frac{z^\eta}{\eta!}, \quad (\tau\in\mathbb{R}^+). \tag{1.14} $$

Halil Gezer and Cem Kaanoglu [15] have introduced an extended Wright hypergeometric matrix function (EWHMF) and the extended Wright Kummer hypergeometric matrix function (EWKHMF) as follows:

$$ {}_2\mathrm{R}_1^{(\tau)}((\mathbb{U},\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P}) $$

$$ = \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U},\mathbb{R})_\eta\mathrm{B}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P})\frac{z^\eta}{\eta!}, \tag{1.15} $$

and

$$ {}_1\mathrm{R}_1^{(\tau)}(\mathbb{V};\mathbb{W};z;\mathbb{P}) $$

$$ = \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}\mathrm{B}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P})\frac{z^\eta}{\eta!}, \ |z|\le 1, \ \tau\in(0,\infty). \tag{1.16} $$

Recently Khan et al. [27] introduced the following extension of BMF is defined as:

$$ \mathrm{B}_{(\alpha,\beta)}(\mathbb{U},\mathbb{V};\mathbb{P},\mathbb{Q}) = \int_0^1 \delta^{\mathbb{U}-\mathbb{I}}(1-\delta)^{\mathbb{V}-\mathbb{I}}\exp\left(-\frac{\mathbb{P}}{\delta^\alpha}-\frac{\mathbb{Q}}{(1-\delta)^\beta}\right)d\delta, \tag{1.17} $$

where $\mathbb{U},\mathbb{V},\mathbb{P},\mathbb{Q}$ are positive stable matrices $\in \mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\ge 0$.

Motivated by the above research, in this paper, we introduce the generalized τ–Gauss and τ–Kummer Wright hypergeometric matrix functions by utilizing the Pochhammer matrix symbol (1.6) and the extended BMF (1.17). Further, we defined the incomplete generalized τ–Gauss Wright hypergeometric matrix function and discussed several important properties of these functions such as integral representations, derivative formulae, Mellin transform etc. Finally, as an applications of the generalized τ–Gauss Wright hypergeometric matrix function, we have defined the fractional kinetic equation.

2 The generalized τ–Gauss (GτGWHMF) and τ–Kummer (GτKWHMF) Wright hypergeometric matrix functions

Definition 2.1

Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\ge 0$, then the generalized τ–Gauss Wright hypergeometric matrix function (GτGWHMF) is defined as:

$$ {}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right] $$

$$ = \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U},\mathbb{R})_\eta\mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^\eta}{\eta!}, \ (z\in\mathbb{C}). \tag{2.1} $$

Here equation (2.1) converges absolutely for $|z|<1$ and for $z=1$.

Remarks 2.1

(a). If, we substitute $\mathbb{P} = \mathbb{Q}$ and $\alpha = \beta = 1$ in (2.1), we get the extended Wright hypergeometric matrix function (1.15):

$$ {}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{P};1,1\right] = {}_2\mathrm{R}_1^{(\tau)}((\mathbb{U},\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P}). $$

(b). Putting $\mathbb{P} = \mathbb{Q} = \mathbb{R} = \mathbb{O}$ and $\alpha = \beta = 1$ in (2.1), we get the Wright hypergeometric matrix function (1.13):

$$ {}_2\mathrm{G}_1^{(\tau)}\left[(\mathbb{U};\mathbb{O}),\mathbb{V};\mathbb{W};z;\mathbb{O},\mathbb{O};1,1\right] = {}_2\mathrm{R}_1^{(\tau)}(\mathbb{U},\mathbb{V};\mathbb{W};z). $$

(c). If, we take $\mathbb{P} = \mathbb{Q}$; $\alpha = \beta = \tau = 1$ and $\mathbb{R} = \mathbb{O}$ in (2.1), we get the extended Gauss hypergeometric matrix function (1.10):

$$ {}_2\mathrm{G}_1^{(1)}\left[(\mathbb{U};\mathbb{O}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{P};1,1\right] = \mathrm{F}^{(\mathbb{P})}(\mathbb{U},\mathbb{V};\mathbb{W};z). $$

(d). Putting $\mathbb{P} = \mathbb{Q} = \mathbb{R} = \mathbb{O}$ and $\alpha = \beta = \tau = 1$ in (2.1), we get the hypergeometric matrix function (1.9):

$$ {}_2\mathrm{G}_1^{(1)}\left[(\mathbb{U};\mathbb{O}),\mathbb{V};\mathbb{W};z;\mathbb{O},\mathbb{O};1,1\right] = \mathrm{F}(\mathbb{U},\mathbb{V};\mathbb{W};z). $$

Definition 2.2

Let $\mathbb{P},\mathbb{Q},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta \geq 0$, then the generalized τ–Kummer Wright hypergeometric matrix function (GτKWHMF) defined as:

$$ {}_1\mathrm{KW}_1^{(\tau)}\left[\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right] $$

$$ = \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}\mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^{\eta}}{\eta!}. \tag{2.2} $$

Remarks 2.2

(a). If we substitute $\mathbb{P} = \mathbb{Q}$ and $\alpha = \beta = 1$ in (2.2), we get the extended Wright hypergeometric matrix function (1.16):

$$ {}_1\mathrm{KW}_1^{(\tau)}\left[\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{P};1,1\right] = {}_1\mathrm{R}_1^{(\tau)}(\mathbb{V};\mathbb{W};z;\mathbb{P}). $$

(b). Putting $\mathbb{P} = \mathbb{Q} = \mathbb{O}$ and $\alpha = \beta = 1$ in (2.2), we get the Wright hypergeometric matrix function (1.14):

$$ {}_1\mathrm{KW}_1^{(\tau)}\left[\mathbb{V};\mathbb{W};z;\mathbb{O},\mathbb{O};1,1\right] = {}_1\mathrm{R}_1^{(\tau)}(\mathbb{V};\mathbb{W};z). $$

(c). Take $\mathbb{P} = \mathbb{Q}$ and $\alpha = \beta = \tau = 1$ in (2.2), we get the Kummer hypergeometric matrix function (1.11):

$$ {}_1\mathrm{KW}_1^{(1)}\left[\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{P};1,1\right] = \Phi^{(\mathbb{P})}(\mathbb{V};\mathbb{W};z). $$

Theorem 2.1. *Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta \geq 0$, then the GτGWHMF ${}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]$ holds the following integral representation:*

$$ {}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right] $$

$$ = e^{z\sigma\delta^{\tau}}\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{U})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\int_0^{\infty}\int_0^1 \sigma^{\mathbb{U}-\mathbb{I}}\, e^{-(\mathbb{I}\sigma+\frac{\mathbb{U}}{\sigma})} $$

$$ \times\, \delta^{\mathbb{V}-\mathbb{I}}\,(1-\delta)^{\mathbb{W}-\mathbb{V}-\mathbb{I}}\,\exp\left(-\frac{\mathbb{P}}{\delta^{\alpha}}-\frac{\mathbb{Q}}{(1-\delta)^{\beta}}\right)\,d\delta\,d\sigma. \tag{2.3} $$

Proof. From definition (2.1), we get

$$
{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]
$$

$$
=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U},\mathbb{R})_{\eta}\mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^{\eta}}{\eta!}.
$$

Now, apply (1.17) and (1.8) in the above expression and by simplifying we get our desired result (2.3). □

Theorem 2.2. *Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\geq 0$, then the GτGWHMF ${}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]$ holds another integral representation:*

$$
{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]
$$

$$
=\Gamma^{-1}(\mathbb{U})\int_0^{\infty}\delta^{\mathbb{U}-\mathbb{I}}\,e^{-(\mathbb{I}\delta+\frac{\mathbb{U}}{\delta})}\,{}_1\mathrm{KW}_1^{(\tau)}\left[\mathbb{V};\mathbb{W};z\delta;\mathbb{P},\mathbb{Q};\alpha,\beta\right]\,d\delta. \tag{2.4}
$$

Proof. Apply the definition (2.1), we get

$$
{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]
$$

$$
=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U};\mathbb{R})_{\eta}\,\mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^{\eta}}{\eta!}.
$$

Using (1.8) in the above expression, we get

$$
=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{U})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}\int_0^{\infty}\delta^{\mathbb{U}+(n-1)\mathbb{I}}\,e^{-(\mathbb{I}\delta+\frac{\mathbb{W}}{\delta})}\,d\delta
$$

$$
\times\,\mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^{\eta}}{\eta!}.
$$

Now, by utilizing (2.2) in the above mathematical expression, we get (2.4). □

Theorem 2.3. *Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\geq 0$, then the GτGWHMF ${}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]$ holds another integral representation:*

$$
{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]
$$

$$
=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\int_0^1\delta^{\mathbb{V}-\mathbb{I}}(1-\delta)^{\mathbb{W}-\mathbb{V}-\mathbb{I}}\exp\left(-\frac{\mathbb{P}}{\delta^{\alpha}}-\frac{\mathbb{Q}}{(1-\delta)^{\beta}}\right) \tag{2.5}
$$

$$
\times\;{}_1\mathrm{F}_0\left[(\mathbb{U};\mathbb{R}),-,z\delta^{\tau}\right]\,d\delta.
$$

Proof. Using (2.1) and (1.17), we get

$$
{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right]
$$

$$
=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\int_0^1\delta^{\mathbb{V}-\mathbb{I}}(1-\delta)^{\mathbb{W}-\mathbb{V}-\mathbb{I}}\exp\left(-\frac{\mathbb{P}}{\delta^{\alpha}}-\frac{\mathbb{Q}}{(1-\delta)^{\beta}}\right)
$$

$$
\times\sum_{\eta=0}^{\infty}(\mathbb{U};\mathbb{R})_{\eta}\frac{(z\delta^{\tau})^{\eta}}{\eta!}d\delta.
$$

Since,

$$
{}_1\mathrm{F}_0\left[(\mathbb{U};\mathbb{R}),-,z\delta^{\tau}\right]=\sum_{\eta=0}^{\infty}(\mathbb{U};\mathbb{R})_{\eta}\frac{(z\delta^{\tau})^{\eta}}{\eta!}. \tag{2.6}
$$

Using (2.6) in the above mathematical expression we get the desired result (2.5). □

Theorem 2.4. *Let $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha, \beta \geq 0$, then the GτGWHMF ${}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}), \mathbb{V}; \mathbb{W}; z; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right]$ holds the following successive differential formula:*

$$\frac{d^k}{dz^k}\left\{{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}), \mathbb{V}; \mathbb{W}; z; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right]\right\}$$

$$= (\mathbb{U})_k\ {}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U}+k\mathbb{I};\mathbb{R}), \mathbb{V}+k\mathbb{I}; \mathbb{W}-\mathbb{V}; z; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right]. \tag{2.7}$$

Proof. Taking derivative of (2.1) w. r. to z, we get

$$\frac{d}{dz}\left\{{}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}), \mathbb{V}; \mathbb{W}; z; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right]\right\}$$

$$= \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=1}^{\infty}(\mathbb{U},\mathbb{R})_\eta \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q})\frac{z^{\eta-1}}{(\eta-1)!}.$$

Replacing, $\eta = \eta + 1$ in the above expression, we get

$$= \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U},\mathbb{R})_{\eta+1} \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau(\eta+1)\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q})\frac{z^{\eta}}{\eta!}.$$

Using the relation $(\mathbb{U};\mathbb{R})_{\eta+1} = \mathbb{U}(\mathbb{U}+\mathbb{I};\mathbb{R})_\eta$ in the above expression and continuing k times derivatives, we obtain the required result (2.7). □

3 Incomplete generalized τ–Gauss (IGτGWHMF) Wright hypergeometric matrix functions

Definition 3.1

Let $\mathbb{P}, \mathbb{Q}, \mathbb{U}, \mathbb{V}$ are PSM in $\mathbb{C}^{\eta\times\eta}$ and $\alpha, \beta \geq 0$, the incomplete extended beta matrix function $\mathrm{B}^{(a)}_{(\alpha,\beta)}(\mathbb{U}, \mathbb{V}; \mathbb{P}, \mathbb{Q})$ is defined as:

$$\mathrm{B}^{(a)}_{(\alpha,\beta)}(\mathbb{U}, \mathbb{V}; \mathbb{P}, \mathbb{Q}) = \int_0^a \delta^{\mathbb{U}-\mathbb{I}}(1-\delta)^{\mathbb{V}-\mathbb{I}}\exp\left(-\frac{\mathbb{P}}{\delta^\alpha} - \frac{\mathbb{Q}}{(1-\delta)^\beta}\right)\, d\delta, \tag{3.1}$$

and

$$[\mathbb{U}, \mathbb{V}; \mathbb{P}, \mathbb{Q}; a]^{(\tau)}_{(\eta)} = \mathrm{B}^{(a)}_{(\alpha,\beta)}(\mathbb{U}+\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}), \tag{3.2}$$

$$\{\mathbb{U}, \mathbb{V}; \mathbb{P}, \mathbb{Q}; a\}^{(\tau)}_{(\eta)} = \mathrm{B}^{(1-a)}_{(\alpha,\beta)}(\mathbb{W}-\mathbb{V}, \mathbb{U}+\mathbb{I}; \mathbb{P}, \mathbb{Q}) \qquad 0 \leq a < 1, \tag{3.3}$$

and

$$[\mathbb{U}, \mathbb{V}; \mathbb{P}, \mathbb{Q}; a]^{(\tau)}_{(\eta)} + \{\mathbb{U}, \mathbb{V}; \mathbb{P}, \mathbb{Q}; a\}^{(\tau)}_{(\eta)} = \mathrm{B}_{(\alpha,\beta)}(\mathbb{U}+\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}). \tag{3.4}$$

Defintion 3.2

Let $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{W}-\mathbb{V}$ are PSM in $\mathbb{C}^{\eta\times\eta}$ and $\alpha, \beta \geq 0$, then the IGτGWHMF is defined as:

$${}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R}); [\mathbb{V}, \mathbb{W}; \mathbb{P}, \mathbb{Q}; a]^{\tau}_{\eta}; z; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right)$$

$$= \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U},\mathbb{R})_\eta \mathrm{B}^{(a)}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q})\frac{z^{\eta}}{\eta!}, \tag{3.5}$$

and

$${}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R}); \{\mathbb{V}, \mathbb{W}; \mathbb{P}, \mathbb{Q}; a\}^{\tau}_{\eta}; z; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right)$$

$$= \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^{\infty}(\mathbb{U},\mathbb{R})_\eta \mathrm{B}^{(1-a)}_{(\alpha,\beta)}(\mathbb{W}-\mathbb{V}, \mathbb{V}+\tau\eta\mathbb{I}; \mathbb{P}, \mathbb{Q})\frac{z^{\eta}}{\eta!}. \tag{3.6}$$

It follows that

$$
\begin{aligned}
&{}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)\\
&+\ {}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});\{\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a\}_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)\\
&=\ {}_2\mathbf{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}),\mathbb{V};\mathbb{W};z;\mathbb{P},\mathbb{Q};\alpha,\beta\right].
\end{aligned}
\tag{3.7}
$$

Theorem 3.1. *Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are PSM in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\geq 0$, then the IGτGWHMF ${}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)$ holds the following integral representation:*

$$
\begin{aligned}
&{}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)\\
&=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{U})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})a^{\mathbb{V}}\ e^{\sigma z(a\delta)^\tau}\\
&\times\int_0^\infty\int_0^1\sigma^{\mathbb{U}-\mathbb{I}}e^{-(\mathbb{I}\sigma+\frac{\mathbb{R}}{\sigma})}\delta^{\mathbb{V}-\mathbb{I}}(1-a\delta)^{\mathbb{W}-\mathbb{V}-\mathbb{I}}\exp\left(-\frac{\mathbb{P}}{(a\delta)^\alpha}-\frac{\mathbb{Q}}{(1-a\delta)^\beta}\right)d\delta d\sigma.
\end{aligned}
\tag{3.8}
$$

Proof. Using (3.1), (1.7) in (3.5) and after some simple calculation, we get the required result (3.8). □

Theorem 3.2. *Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are PSM in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\geq 0$, then the IGτGWHMF ${}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});\{\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a\}_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)$ holds the another integral representation:*

$$
\begin{aligned}
&{}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});\{\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a\}_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)\\
&=(1-a)\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{U})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^\infty\int_0^\infty\int_0^1\sigma^{\mathbb{U}+(\eta-1)\mathbb{I}}\ e^{-(\mathbb{I}\sigma+\frac{\mathbb{R}}{\sigma})}\\
&\times\{\delta(1-a)\}^{\mathbb{W}-\mathbb{V}-\mathbb{I}}\{1-\delta(1-a)\}^{\mathbb{V}+(\tau\eta-1)\mathbb{I}}\\
&\times\exp\left(-\frac{\mathbb{P}}{\{\delta(1-a)\}^\alpha}-\frac{\mathbb{Q}}{\{1-\delta(1-a)\}^\beta}\right)d\delta d\sigma.
\end{aligned}
\tag{3.9}
$$

Proof. Utilizing the Definition (3.2) by applying (1.7), (3.1) and using suitable substitutions we obtain the required result (3.9). □

Theorem 3.3. *Let $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{U},\mathbb{V},\mathbb{W},\mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha,\beta\geq 0$, then the IGτGWHMF ${}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)$ holds the following successive differential formula:*

$$
\begin{aligned}
&\frac{d^k}{dz^k}\left\{{}_2\mathbf{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)\right\}\\
&=(\mathbb{U})_k\ {}_2\mathbf{GW}_1^{(\tau)}\left\{(\mathbb{U}+k\mathbb{I};\mathbb{R});[\mathbb{V}+k\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q};a];z;\mathbb{P},\mathbb{Q};\alpha,\beta\right\}.
\end{aligned}
\tag{3.10}
$$

Proof. Taking derivative of (3.5) w. r. to z, we get

$$
\begin{aligned}
&\frac{d}{dz}\left\{\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^\infty(\mathbb{U},\mathbb{R})_\eta\mathbf{B}_{(\alpha,\beta)}^{(a)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^\eta}{\eta!}\right\}\\
&=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=1}^\infty(\mathbb{U},\mathbb{R})_\eta\mathbf{B}_{(\alpha,\beta)}^{(a)}(\mathbb{V}+\tau\eta\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^{\eta-1}}{(\eta-1)!}.
\end{aligned}
$$

Replacing, $\eta=\eta+1$ in the above expression, we get

$$
=\Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\sum_{\eta=0}^\infty(\mathbb{U},\mathbb{R})_{\eta+1}\mathbf{B}_{(\alpha,\beta)}^{(a)}(\mathbb{V}+\tau(\eta+1)\mathbb{I},\mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^\eta}{\eta!}.
$$

Using the relation $(\mathbb{U};\mathbb{R})_{\eta+1}=\mathbb{U}(\mathbb{U}+\mathbb{I};\mathbb{R})_\eta$ in the above expression and continuing k times derivatives, we obtain the required result (3.10). □

Definition 3.3 [30]

Consider a function $\mathrm{F}(\mathbb{U})$, where $\mathbb{U}$ is a positive stable matrix $\in \mathbb{C}^{\eta\times\eta}$. Then, the Mellin transform is established in the following manner:

$$\mathcal{M}\left\{\mathrm{F}(\mathbb{U}) : \mathbb{U} \longrightarrow \mathbb{S}\right\} = \int_0^\infty \mathbb{U}^{\mathbb{S}-\mathbb{I}}\, \mathrm{F}(\mathbb{U})\, d\mathbb{U}. \tag{3.11}$$

Lemma 3.1 [30]

Let $\mathbb{U}, \mathbb{R}, \mathbb{S}$ and $\mathbb{U}+\mathbb{R}$ are positive stable matrices in $\in \mathbb{C}^{\eta\times\eta}$. Then the Mellin transform of extended gamma function is defined as:

$$\mathcal{M}\left\{\Gamma(\mathbb{U};\mathbb{R}) : \mathbb{R} \longrightarrow \mathbb{S}\right\} = \Gamma(\mathbb{S})\Gamma(\mathbb{U}+\mathbb{S}). \tag{3.12}$$

Theorem 3.4. *Let $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{W}-\mathbb{V}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha, \beta \geq 0$. The Mellin transform of the incomplete generalized τ–Gauss Wright hypergeometric matrix functions (IGτGWHMF) ${}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)$ is defined as:*

$$\begin{aligned}
&\mathcal{M}\left\{{}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right) : \mathbb{R} \longrightarrow \mathbb{S}\right\} \\
&= \Gamma(\mathbb{W})\Gamma(\mathbb{U}+\mathbb{S})\Gamma(\mathbb{S})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\Gamma^{-1}(\mathbb{U}) \\
&\times\ {}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U}+\mathbb{S};\mathbb{O});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right).
\end{aligned} \tag{3.13}$$

Proof. Apply (3.11) in (3.5), we get

$$\begin{aligned}
&\mathcal{M}\left\{{}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right) : \mathbb{R} \longrightarrow \mathbb{S}\right\} \\
&= \int_0^\infty \mathbb{R}^{\mathbb{S}-\mathbb{I}}\, {}_2\mathrm{GW}_1^{(\tau)}\left((\mathbb{U};\mathbb{R});[\mathbb{V},\mathbb{W};\mathbb{P},\mathbb{Q};a]_\eta^\tau;z;\mathbb{P},\mathbb{Q};\alpha,\beta\right)\, d\mathbb{R} \\
&= \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V})\Gamma^{-1}(\mathbb{U}) \sum_{\eta=0}^{\infty} \mathrm{B}_{(\alpha,\beta)}^{(a)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V};\mathbb{P},\mathbb{Q})\frac{z^\eta}{\eta!} \\
&\times \int_0^\infty \mathbb{R}^{\mathbb{S}-\mathbb{I}}\, \Gamma(\mathbb{U}+\eta\mathbb{I};\mathbb{R})\, d\mathbb{R}.
\end{aligned}$$

Now, using (3.12) in the above expression and after simplification, we obtain the desired result (3.13). □

4 An application : Fractional kinetic equation (FKE) involving GτGWHMF

In this section, we employ the Laplace transform to examine the solvability of the fractional kinetic equations related to GτGWHMF. In dynamic systems, engineering, physics, mathematics, and control systems, these equations play a crucial role in the modeling of dynamic systems.

The fractional kinetic equations defined by Haubold and Mathai [19] as follows:

$$\mathrm{K}(t) - \mathrm{K}_0 = -g^w\, {}_0D_t^{-w}\, \mathrm{K}(t) \qquad g, t \in \mathbb{R}^+; (w) > 0, \tag{4.1}$$

where ${}_0D_t^{-w}$ is defined by

$${}_0D_t^{-w}\phi(t) = \frac{1}{\Gamma(w)}\int_0^\infty (t-u)^{w-1}\phi(u)\, du \qquad t \in \mathbb{R}^+; (w) > 0, \tag{4.2}$$

and the solution of the equation (4.1) is given by

$$\mathrm{K}(t) = \mathrm{K}_0\mathrm{E}_w(-g^w t^w), \tag{4.3}$$

where $\mathrm{E}_w(-g^w t^w)$ denotes the Mittag-Leffler function of one parameter defined as [34]:

$$\mathrm{E}_w(t) = \sum_{\eta=0}^{\infty} \frac{t^\eta}{\Gamma(\eta w + 1)} \tag{4.4}$$

Theorem 4.1. *Let $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{M}$ are positive stable matrices in $\mathbb{C}^{\eta\times\eta}$ and $\alpha, \beta \geq 0$. Then, for $t, w \in \mathbb{C}$ and $(w) > 0$, the fractional kinetic matrix equation*

$$\mathrm{K}(t)\mathbb{I} = \mathrm{K}_0\, {}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}), \mathbb{V}; \mathbb{W}; t; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right] - \mathbb{M}^w\, {}_0D_t^{-w}\, \mathrm{K}(t) \tag{4.5}$$

has a solution of the following formula:

$$\begin{aligned} \mathrm{K}(t)\mathbb{I} = \mathrm{K}_0 \sum_{\eta=0}^{\infty} \Gamma(\mathbb{W})\Gamma(\eta+1)\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \frac{(\mathbb{U};\mathbb{R})_\eta t^\eta}{\eta!} \\ \times \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q})\, \mathrm{E}_{w,\eta+1}(\mathbb{M}^w t^w), \end{aligned} \tag{4.6}$$

where $\mathrm{E}_{w,\eta+1}(\mathbb{M}^w t^w)$ is the generalized Mittag-Leffler function defined as [13]:

$$\mathrm{E}_{w,\eta+1}(\mathbb{M}^w t^w) = \sum_{r=0}^{\infty} \mathbb{M}^{wr} \frac{t^{wr}}{\Gamma(wr+\eta+1)}. \tag{4.7}$$

Proof. By applying the Laplace transform to both sides of (4.5), we get

$$\begin{aligned} &\mathrm{L}\left\{\mathrm{K}(t)\mathbb{I}; \rho\right\} \\ &= \mathrm{K}_0 \mathrm{L}\left\{ {}_2\mathrm{GW}_1^{(\tau)}\left[(\mathbb{U};\mathbb{R}), \mathbb{V}; \mathbb{W}; t; \mathbb{P}, \mathbb{Q}; \alpha, \beta\right]; \rho \right\} - \mathbb{M}^w\, \mathrm{L}\left\{ {}_0D_t^{-w}\, \mathrm{K}(t); \rho \right\}. \end{aligned} \tag{4.8}$$

Using Laplace transform, the Riemann–Liouville for the fractional integral operator is given by [7]:

$$\mathrm{L}\left\{ {}_0D_t^{-w}\, \mathrm{K}(t); \rho \right\} = \rho^{-w}\, \mathrm{K}(\rho). \tag{4.9}$$

Using (4.9) in (4.8) and apply (2.1), we get

$$\begin{aligned} &\mathrm{K}(\rho)\mathbb{I} = \mathrm{K}_0 \int_0^\infty e^{-\rho t}\, \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \\ &\times \sum_{\eta=0}^{\infty} (\mathbb{U},\mathbb{R})_\eta \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}) \frac{t^\eta}{\eta!}\, dt - \mathbb{M}^w\, \rho^{-w}\, \mathrm{K}(\rho). \\ &\Longrightarrow\ \mathrm{K}(\rho)\mathbb{I} + \mathbb{M}^w\, \rho^{-w}\, \mathrm{K}(\rho) = \mathrm{K}_0\, \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \sum_{\eta=0}^{\infty} \frac{(\mathbb{U},\mathbb{R})_\eta}{\eta!} \\ &\times \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}) \int_0^\infty e^{-\rho t}\, t^\eta\, dt. \\ &\Longrightarrow\ \left(\mathbb{I} + \mathbb{M}^w\, \rho^{-w}\right) \mathrm{K}(\rho) = \mathrm{K}_0\, \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \sum_{\eta=0}^{\infty} \frac{(\mathbb{U},\mathbb{R})_\eta}{\eta!} \\ &\times \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}) \frac{\Gamma(\eta+1)}{\rho^{\eta+1}}. \\ &\Longrightarrow\ \mathrm{K}(\rho)\mathbb{I} = \mathrm{K}_0\, \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \sum_{\eta=0}^{\infty} \frac{(\mathbb{U},\mathbb{R})_\eta}{\eta!} \\ &\times \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}) \frac{\Gamma(\eta+1)}{\rho^{\eta+1}} \left(\mathbb{I} + \mathbb{M}^w\, \rho^{-w}\right)^{-1}. \\ &\Longrightarrow\ \mathrm{K}(\rho)\mathbb{I} = \mathrm{K}_0\, \Gamma(\mathbb{W})\Gamma^{-1}(\mathbb{V})\Gamma^{-1}(\mathbb{W}-\mathbb{V}) \sum_{\eta=0}^{\infty} \frac{(\mathbb{U},\mathbb{R})_\eta}{\eta!} \\ &\times \mathrm{B}_{(\alpha,\beta)}(\mathbb{V}+\tau\eta\mathbb{I}, \mathbb{W}-\mathbb{V}; \mathbb{P}, \mathbb{Q}) \frac{\Gamma(\eta+1)}{\rho^{\eta+1}} \sum_{r=0}^{\infty} \left(\mathbb{M}^w\, \rho^{-w}\right)^{-r}. \end{aligned}$$

Now, taking inverse Laplace transform both side in the above and simplifying we get the required result (4.6). □

5 Conclusion remarks

In this paper, we have introduced the generalized τ-Gauss Wright hypergeometric matrix function, a novel extension of the classical Gauss Wright hypergeometric functions, and explored its fundamental properties. By employing the Pochhammer matrix symbol and the extended beta matrix function, we derived integral representations, differential formulas, and transformation properties that characterize this function in a matrix setting. Additionally, we defined the incomplete τ -Gauss Wright hypergeometric matrix function and uncovered several important properties, including recurrence relations and asymptotic behaviors. These results offer new avenues for applying matrix-valued special functions in various areas of mathematical physics, particularly in the context of fractional dynamics. As an illustrative application, we formulated a fractional kinetic equation involving the generalized τ-Gauss Wright hypergeometric matrix function. This equation serves as a prototype for modeling non-local, memory-dependent processes in complex systems, such as anomalous diffusion and other phenomena governed by fractional differential equations. The results presented in this paper open up several possibilities for further research, particularly in exploring additional applications in statistical mechanics, quantum physics, and other fields where matrix-valued special functions and fractional calculus play an important role. Future work will focus on extending these results to more general classes of hypergeometric functions and investigating their impact on both theoretical and applied mathematics.

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References

[1] M. Abdalla and A. Bakhet, *Extension of beta matrix function*, Asian J. Math. Comput. Res., **9(3)**, 253–264, (2016). https://ikprress.org/index.php/AJOMCOR/article/view/192.

[2] M. Abdalla and A. Bakhet, *Extended Gauss hypergeometric matrix functions*, Iran. J. Sci. Technol. Trans. Sci., **42(3)**, 1465–1470, (2018).

[3] M. Abdalla, *Special matrix functions: Characteristics, achievements and future directions*, Linear Multilinear Algebra, **68(1)**, 1–28, (2020). https://doi.org/10.1080/03081087.2018.1497585.

[4] M. Ali, W. A. Khan and I. A. Khan, *Study on double integral operator associated with generalized Bessel–Maitland function*, Palestine Journal of Mathematics, **9(2)**, 991–998, (2020).

[5] A. Altin, B. Çekim and R. Sahin, *On the matrix versions of Appell hypergeometric functions*, Quaest. Math., **37(1)**, 31–38, (2014). https://doi.org/10.2989/16073606.2013.779955.

[6] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, **71**, (1999). https://doi.org/10.1017/CBO9781107325937.

[7] A. Atangana, *Fractional Operators with Constant and Variable Order with Application to Geo-hydrology*, Academic Press (Elsevier), United Kingdom, (2017).

[8] A. Bakhet, Y. Jiao and F. He, *On the Wright hypergeometric matrix functions and their fractional calculus*, Integral Transforms Spec. Funct., **30(2)**, 138–156, (2018). https://doi.org/10.1080/10652469.2018.1543669.

[9] B. Çekim, *Generalized Euler's beta matrix function and related functions*, AIP Conf. Proc., **1558**, 1132–1135, (2013). http://dx.doi.org/10.1063/1.4825707.

[10] M. A. Chaudhry and S. M. Zubair, *Generalized incomplete gamma function with application*, J. Comput. Appl. Math., **55(1)**, 99–124, (1994). https://doi.org/10.1016/0377-0427(94)90187-2.

[11] A. G. Constantine and R. J. Mairhead, *Partial differential equations for hypergeometric function of two argument matrix*, J. Multivar. Anal., **2(3)**, 332–338, (1972). https://doi.org/10.1016/0047-259X(72)90020-6.

[12] A. Dahab, M. A. Bakhet and A. K., *A certain generalized gamma matrix functions and their properties*, J. Anal. Num. Theor., **3(1)**, 63–68, (2015). https://dx.doi.org/10.12785/jant/030110.

[13] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg–Dordrecht–London–New York, (2010).

[14] R. Garrappa and M. Popolizio, *Computing the matrix Mittag-Leffler function with applications to fractional calculus*, J. Sci. Comput., **77(1)**, 129–153, (2018). https://doi.org/10.48550/arXiv.1804.04883.

[15] H. Gezer and C. Kaanoglu, *On the extended Wright hypergeometric matrix function and its properties*, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat., **72(3)**, 606–617, (2023).

[16] J. S. Geronimo, *Scattering theory and matrix orthogonal polynomials on the real line*, Circuits Syst. Signal Process., **1(3)**, 471–495, (1982). https://doi.org/10.1007/BF01599024.

[17] M. Ghayasuddin, N. U. Khan and M. Ali, *A study of the extended beta, Gauss and confluent hypergeometric functions*, Int. J. Appl. Math., **33(10)**, 1–13, (2020). http://dx.doi.org/10.12732/ijam.v33i1.1

[18] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York, USA, (1982).

[19] H. J. Haubold and A. M. Mathai, *The fractional kinetic equation and thermonuclear functions*, Astrophys. Space Sci., **273(4)**, 53–63, (2000).

[20] A. T. James, *Special functions of matrix and single argument in statistics*, in Theory and Application of Special Functions, R. A. Askey (Ed.), Academic Press, New York, USA, (1975). https://doi.org/10.1016/B978-0-12-064850-4.50016-1.

[21] L. Jódar and J. C. Cortés, *Some properties of gamma and beta matrix functions*, Appl. Math. Lett., **11(1)**, 89–93, (1998). https://doi.org/10.1016/S0893-9659(97)00139-0.

[22] L. Jódar and J. C. Cortés, *On the hypergeometric matrix function*, J. Comput. Appl. Math., **99(1–2)**, 205–217, (1998). https://doi.org/10.1016/S0377-0427(98)00158-7.

[23] N. U. Khan, T. Usman and M. Aman, *Generalized Wright function and its properties using extended beta function*, Tamkang J. Math., **51(4)**, 349–363, (2020). https://doi.org/10.5556/j.tkjm.51.2020.3087.

[24] N. U. Khan, T. Usman and M. Aman, *Some properties concerning the analysis of generalized Wright function*, J. Comput. Appl. Math., **376**, 112840, (2020). https://doi.org/10.1016/j.cam.2020.112840.

[25] N. U. Khan and R. Sk, *Analysis of beta-logarithmic function and its properties*, Analysis, **45(1)**, 73–83, (2025).

[26] N. U. Khan, R. Sk and S. Husain, *Generalized k-Gamma and k-Beta functions of matrix arguments*, in Recent Advances in Special Function and their Applications, University of Kerala, India, 45–60, (2024). ISBN: 978-93-340-9964-5.

[27] N. U. Khan, R. Sk and S. Husain, *Certain results on extended beta and related functions using matrix arguments*, J. New Theory, **49**, 16–29, (2024). https://doi.org/10.53570/jnt.1534850.

[28] W. Miller, *Lie Theory and Special Functions*, Academic Press, New York, USA, (1968).

[29] E. D. Rainville, *Special Functions*, Macmillan Company, New York, (1960).

[30] M. Niyaz, A. H. Soliman and A. Bakhet, *Investigation of fractional calculus for extended Wright hypergeometric matrix functions*, Abstr. Appl. Anal., **2023(1)**, Article ID 9505980, (2023).

[31] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Ltd.), John Wiley & Sons, New York, (1984). https://doi.org/10.1137/1028045.

[32] H. M. Srivastava, P. Agarwal and S. Jain, *Generating functions for the generalized Gauss hypergeometric functions*, Appl. Math. Comput., **247**, 348–352, (2014). https://doi.org/10.1016/j.amc.2014.08.105.

[33] T. Usman, N. U. Khan, O. Khan, and D. A. Juraev, *Fractional operators and solution of fractional kinetic equations involving generalized hypergeometric function*, Palestine Journal of Mathematics,**13(3)**,711–721, (2024).

[34] A. Wiman, *Über den Fundamentalsatz in der Theorie der Funktionen $E_a(x)$*, Acta Math., **29**, 191–201, (1905).

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