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INDEX CHARACTERIZATION AND MONOGENEITY OF A CERTAIN CLASS OF NONIC NUMBER FIELDS DEFINED BY $x^9 + ax^2 + bx + c$

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Abstract Let $K = \mathbb{Q}(\theta)$ be a number field generated by a root of the irreducible polynomial

$$F(x) = x^9 + ax^2 + bx + c,$$

where $a, b, c \in \mathbb{Z} \setminus \{0\}$. We investigate the common index divisor $i(K)$ of K. We prove that the only possible prime divisors of $i(K)$ are 2 and 3. Moreover, using Newton polygon techniques, we establish sufficient conditions on the coefficients of F ensuring that each of these primes occurs in $i(K)$. Several examples are given to illustrate the effectiveness of the criteria.

1 Introduction

Let K be a number field of degree n and $\mathbb{Z}_K$ its ring of integers. For any primitive element α of $\mathbb{Z}_K$, the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$ is finite. This index will be denoted by $\text{ind}(\alpha)$. Therefore, $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ if and only if $\text{ind}(\alpha) = 1$. In such a case $(1, \alpha, \ldots, \alpha^{n-1})$ is a power integral basis of $\mathbb{Z}_K$ and we say that K is monogenic. Otherwise, K is said to be non-monogenic.

Let $i(K)$ denote the index of K, that is

$$i(K) = \gcd\{\text{ind}(\alpha) \mid \alpha \in \mathbb{Z}_K \text{ and } \mathbb{Q}(\alpha) = K\}.$$

A prime p dividing $i(K)$ is called a common index divisor of K. Observe that if K is monogenic, then $i(K) = 1$. Consequently, the existence of a prime p such that $\nu_p(i(K)) > 0$ implies that K is non-monogenic.

Monogeneity is a classical topic in algebraic number theory, dating back to the pioneering works of Dedekind and Hensel. A vast literature has been devoted to the study of indices and monogeneity of number fields. Nevertheless, for every degree $n \geq 4$ a complete characterization of monogenic number fields of degree n is still unknown. For instance, the monogeneity of pure quartic fields $\mathbb{Q}(\sqrt[4]{m})$ has not yet been completely characterized, even when m is square-free (see the recent 2024 paper [2] by Arnóczki and Nyul.

In [22], Győry proved that there exist only finitely many $\mathbb{Z}$-equivalence classes of generators yielding power integral bases of $\mathbb{Z}_K$, and that a complete system of representatives can be determined effectively. These results led to the first general algorithms for deciding whether a number field is monogenic and for determining all its power integral bases.

The monograph of Evertse and Győry [17] contains comprehensive surveys of discriminant and index form theory and their applications to Diophantine equations and monogeneity problems. See also [7] by Bérczes, Evertse, and Győry, and [16] by Evertse.

Problem 22 in Narkiewicz's book [21] asks whether an explicit formula for $\nu_p(i(K))$ can be obtained. Important partial answers to this problem were given by Engstrom [15] and Nart [31].

In [27], Llorente and Nart studied the indices of cubic number fields. The indices and Monogeneity of quartic number fields have been studied in several papers, see for example [1] by Akhtari, [12] by Davis and Spearman, and [33] by Pethő and Ziegler. For quartic number fields defined by the quadrinomial $x^4 + ax^3 + bx + c \in \mathbb{Z}[x]$, see [5] by Ben Yakkou. For quintic number fields, see [19] by Gaál and Győry, and for sextic number fields, see [8] by Bilu, Gaál and Győry, and for sextic reciprocal dihedral polynomials, see [25] by Jones. For pure number fields, refer to [34] by Remete, and [29] by Kchit, [13] by Didi et al.. For a certain class of number fields defined by trinomials, see [14] by Sahmoudi et al., [3, 4, 6] by Ben Yakkou, [28] by Kchit, [23] by Ibara et al., and [24, 26] by Jones et al.

In order to examine the relative Monogeneity of extensions of number fields, Sahmoudi and Charkani [35, 10] and Sahmoudi et al. [9, 11, 36, 37, 38] provided novel and practical results connecting the index-ideal in tower extensions. See the works [17] by Evertse and Gy'Hory, [18] by Ga'al, and [30] by Narkiewicz for more findings and a bibliography.

Let $F(x) = x^9 + ax^2 + bx + c \in \mathbb{Z}[x]$ be an irreducible monic quadrinomial. Let $K = \mathbb{Q}(\theta)$ be a number field generated by a complex root θ of $F(x)$. The purpose of this paper is to establish sufficient conditions on the coefficients a, b, and c for p to divide the index $i(K)$, where $p \in \{2, 3\}$. In particular, under these conditions, K is not monogenic. The main tools are Newton polygons and the factorization of prime ideals in number fields.

2 Results

Throughout this paper, K is a nonic number field generated by a root θ of a monic irreducible quadrinomial $F(x) = x^9 + ax^2 + bx + c \in \mathbb{Z}[x]$. Let $\nu_p(m)$ denote the p-adic valuation of m, and define $m_p := \frac{m}{p^{\nu_p(m)}}$ for a prime p and any rational integer m. For the rational integers a_1, a_2, a_3, b_1, b_2 and b_3, by the notation $(a_1, a_2, a_3) \equiv (b_1, b_2, b_3) (\mathrm{mod}\ p)$, we mean that $a_i \equiv b_i (\mathrm{mod}\ p)$ for $i = 1, 2, 3$. Using the above notations , our results are as follows.

Theorem 2.1. *Under the above notations, we have the following statements:*

(i) If $2 \mid a$, $2 \mid b$, $2 \mid c$, *and* $\frac{8}{7}\nu_2(a) < \nu_2(b) < \frac{\nu_2(a)+\nu_2(c)}{2}$, *then* $\nu_2(i(K)) = 1$.

(ii) If $2 \mid b$, $2 \mid c$, 2 *does not divide* a, *and* $2\nu_2(b) < \nu_2(c)$, *then* $\nu_2(i(K)) = 1$.

(iii) If $2 \mid a$, $2 \mid c$, $2 \nmid b$. *Then the following hold:*

- *a. If* $(a,\ b,\ c) \in \{(0,3,4), (0,3,12), (0,11,4), (0,11,12), (8,3,4), (8,3,12), (8,11,4), (8,11,12), (4,3,0), (4,3,8), (4,11,0), (4,11,8), (12,3,0), (12,3,8), (12,11,0), (12,11,8), (4,7,4), (12,7,12), \}(\mathrm{mod}\ 16)$, *then* $\nu_2(i(K)) \geq 1$.
- *b. If* $(a,\ b,\ c) \in \{(2,7,6), (2,15,14), (10,7,14), (10,15,6), (6,7,2), (6,15,10), (14,7,10), (14,15,2)\}(\mathrm{mod}\ 16)$, *then* $\nu_2(i(K)) = 1$.
- *c. If* $(a,\ b,\ c) \in \{(0,15,16), (0,31,0), (16,15,0), (16,31,16), (8,15,8), (8,31,24), (24,15,24), (24,31,8)\}(\mathrm{mod}\ 32)$, *then* $\nu_2(i(K)) \geq 1$.
- *d. If* $(a,\ b,\ c) \in \{(2,11,50), (2,43,18), (34,11,18), (34,43,50), (2,27,34), (2,59,2), (34,27,2), (34,59,34), (18,11,34), (18,43,2), (50,11,2), (50,43,34), (18,27,18), (18,59,50), (50,27,50), (50,59,18), (10,11,42), (10,43,10), (42,11,10), (42,43,42), (10,27,26), (10,59,58), (42,27,58), (42,59,26), (26,11,26), (26,43,58), (26,27,10), (26,59,42), (56,27,42), (6,3,54), (6,35,22), (38,3,22), (38,35,54), (6,19,38), (6,51,6), (38,19,6), (38,51,38), (22,3,38), (22,35,6), (54,3,6), (54,35,38), (22,19,22), (22,51,54), (54,19,54), (54,51,22), (14,3,46), (14,35,14), (46,3,14), (46,35,46), (14,19,30), (14,51,62), (46,19,62), (46,51,30), (30,3,30), (30,35,62), (62,3,62), (62,35,30), (30,19,14), (30,51,46), (62,19,46), (62,51,14)\}(\mathrm{mod}\ 64)$, *then* $\nu_2(i(K)) = 1$.

In particular, under any of these conditions, K *is non-monogenic.*

Theorem 2.2. *If one of the following conditions holds:*

(i) $(a, b, c) \in \{(0,72,73), (27,45,19), (54,18,46)\}(\mathrm{mod}\ 81)$, $\nu_3(b - 2a + 9) \geq 4$, $\nu_3(a + c - b - 1) \geq 7$, *and* $\nu_3(b - 2a + 9) < \frac{2+\nu_3(a+c-b-1)}{2}$.

(ii) $(a,b,c) \in \{(0,72,8),(27,18,35),(54,45,62),(9,54,17),(36,0,44),$ $(63,27,71)\}(\text{mod } 81)$, $\nu_3(b+2a+9) \geq 4$, $\nu_3(1+a+b+c) \geq 7$, *and* $\nu_3(b+2a+9) < \frac{2+\nu_3(a+b+c+1)}{2}$.

(iii) $(a,b,c) \in \{(18,36,26),(72,9,80)\}(\text{mod } 81)$, $\nu_3(b+2a+9) \geq 6$, $\nu_3(1+a+b+c) \geq 10$, *and* $\nu_3(b+2a+9) < \frac{2+\nu_3(a+b+c+1)}{2}$.

(iv) $(a,b,c) \equiv (72,63,26)(\text{mod } 81)$ *and* $\nu_3(1+a+b+c) \geq 6$,

then $\nu_3(i(K)) = 1$. *In particular,* K *is not monogenic.*

Theorem 2.3. *Let* $K = \mathbb{Q}(\theta)$, *where* θ *is a root of an irreducible polynomial* $F(x) = x^9 + ax^2 + bx + c \in \mathbb{Z}[x]$. *Then the index* $i(K)$ *is not divisible by any prime* $p \geq 5$.

3 Preliminaries

We recall the basic notions on Newton polygons and their use in the factorization of primes in number fields. We also state Ore's theorem, which is a key ingredient in our proofs. For further details, see [21, 20] by Guàrdia, Montes and Nart, and [32] by Ore.

Let p be a prime and let $\phi \in \mathbb{Z}[x]$ be monic and irreducible modulo p. The ϕ-adic development of $F(x)$ is $F(x) = \sum_{i=0}^{n} a_i(x)\phi(x)^i$, $\deg a_i < \deg \phi$, and is unique. Set $u_i = \nu_p(a_i(x))$. The ϕ-Newton polygon $N_\phi(F)$ is the lower convex hull of the points (i,u_i) with $a_i(x) \neq 0$. It is the union of sides $S_1,\ldots,S_g$ with increasing slopes $\lambda_1 < \cdots < \lambda_g$, and we write $N_\phi(F) = S_1 + \cdots + S_g$. The ϕ-principal Newton polygon $N_\phi^+(F)$ is the subpolygon formed by the sides of negative slope. Its length equals $\nu_{\overline{\phi}}(\overline{F})$, the largest power of $\overline{\phi}$ dividing $\overline{F}$ in $\mathbb{F}_p[x]$.

Let $\mathbb{F}_\phi = \mathbb{Z}[x]/(p,\phi(x)) \simeq \mathbb{F}_p[x]/(\overline{\phi})$. For $0 \leq i \leq n$, define $c_i \in \mathbb{F}_\phi$ by

$$c_i = \begin{cases} 0, & \text{if } (i,u_i) \text{ lies strictly above } N_\phi^+(F), \\ \left(\frac{a_i(x)}{p^{u_i}}\right) \pmod{(p,\phi(x))}, & \text{if } (i,u_i) \text{ lies on } N_\phi^+(F). \end{cases}$$

Let S be a side of N_ϕ^+F and $\lambda = -\frac{h}{e}$ its slope, where e and h are two positive coprime integers. When S is projected to the horizontal axis, its length is marked by the symbol $l(S)$, and when it is projected to the vertical axis, it is marked by the symbol $h(S)$. S's degree is $d = d(S) = gcd(l(S),h(S))$. It is worth noting that if (s,u_s) is the starting point of S, then the points with integer coordinates lying on S are exactly (s,u_s), $(s+e,u_s-h),\cdots,(s+de,u_s-dh)$. We attach to S the residual polynomial defined by

$$R_\lambda(F)(y) = c_s + c_{s+e}y + \cdots + c_{s+(d-1)e}y^{d-1} + c_{s+de}y^d \in \mathbb{F}_\phi[y].$$

The $ind_\phi(F)$ symbol stands for the ϕ-index of $F(x)$, which is defined in [28, 14] as $\deg(\phi)$ times the number of points with natural integer coordinates that strictly lie above the horizontal axis, strictly above the vertical axis, below or on the polygon $N_\phi^+(F)$.

The polynomial $F(x)$ is said to be ϕ-regular with respect to p if, for each side S of N_ϕ^+F, the corresponding residual polynomial $R_\lambda(F)(y)$ is separable in $\mathbb{F}_\phi[y]$. Let $F(x) \equiv \prod_{i=1}^{t} \phi_i(x)^{l_i}$ be the factorization into monic irreducible polynomials in $\mathbb{F}_p[x]$ (modulo p). So, the polynomial $F(x)$ is said to be p-regular if $F(x)$ is ϕ_i-regular for $i = 1,\ldots,t$. Let $R_{\lambda_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{n_{ijs}}(y)$ be the factorization of $R_{\lambda_{ij}}(F)(y)$ in $\mathbb{F}_{\phi_i}[y]$. Then we have the following theorem of Ore (see [21, Theorem 3.9], and [32]):

Theorem 3.1. *(Ore's Theorem)*
With notation as above, If $F(x)$ *is* p*-regular, then*

$$p\mathbb{Z}_K = \prod_{i=1}^{t}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}},$$

where e_{ij} *is the ramification index of the side* S_{ij} *and* $f_{ijs} = deg(\phi_i) \times deg(\psi_{ijs})$ *is the residue degree of* $\mathfrak{p}_{ijs}$ *over* p.

The following lemma, which characterizes the prime common index divisors of K, will be needed in the proof of our results.

Lemma 3.2. *([15])*
Let p be a rational prime and K a number field. For every positive integer f, let $\mathcal{P}_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p with residue degree f and $\mathcal{N}_f$ the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree f. Then p divides the index $i(K)$ if and only if $\mathcal{P}_f > \mathcal{N}_f$ for some positive integer f.

To apply the last lemma, one needs to know the number $N_p(m)$ of the monic irreducible polynomials over $\mathbb{F}_p$ of degree m. As shown by Gauss, this number is given in the following proposition (see [21, Proposition 4.35]).

Proposition 3.3. *[21]The number of monic irreducible polynomials of degree m over the field $\mathbb{F}_p[X]$ is*

$$N_p(m) = \frac{1}{m}\sum_{d|m}\mu(d)p^{\frac{m}{d}},$$

where μ is the Möbius function.

Let p be a prime and K a number field. For every positive integer f, let $L_p(f)$ denote the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p with residue degree f and $N_p(f)$ denote the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree f.

In the case of a number field of degree 9, the following lemma describes the prime index divisor of K. It is an immediate consequence of propositions 2.13, 4.36, 4.33, 4.34, and 4.35 of [30].

Lemma 3.4. *[30] Let K be a nonic number field generated by a root of a monic polynomial $F(x)$ having discriminant $\Delta(F)$. If p is a prime and $p \mid i(K)$, then $p \in \{2,3,5,7\}$, and $p^2 \mid \Delta(F)$. Moreover,*

(i) $2 \mid i(K)$ *if and only if* $L_2(1) \geq 3$ *or* $L_2(2) \geq 2$ *or* $L_2(3) = 3$.

(ii) $3 \mid i(K)$ *if and only if* $L_3(1) \geq 4$ *or* $L_3(2) \geq 4$.

(iii) $5 \mid i(K)$ *if and only if* $L_5(1) \geq 6$.

(iv) $7 \mid i(K)$ *if and only if* $L_7(1) \geq 8$.

4 Proofs of main results

Recall that, according to the factorization given in Theorem 3.1, we use triple indices in the factorization of $p\mathbb{Z}_K$. Namely,

$$p\mathbb{Z}_K = \prod_{i=1}^{r}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}}\mathfrak{p}_{ijs}^{e_{ij}}.$$

Here, e_{ij} is the ramification index of $\mathfrak{p}_{ijs}$, and

$$f_{ijs} = \deg(\phi_i)\cdot\deg(\psi_{ijs})$$

is its residue degree for every (i,j,s).

To establish our results, we proceed as follows:

- Firstly, we compute the discriminant $\Delta(\alpha)$ of the given polynomial.
- Secondly, according to the index formula (1.1), the rational primes that may divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ are those whose squares divide the discriminant $\Delta(\alpha)$.
- Using the coefficients of the polynomial and the primes identified above, we determine the prime ideal factorization of $p\mathbb{Z}_K$ in each case based on the Newton polygon's technique and Theorem 3.1. Then, Lemma 3.2 is used to check whether p divides the index $i(K)$.

- Finally, based on the previous steps, we compute the index $i(K)$ in terms of its prime divisors.

Proof of Theorem 2.1. In all cases, we prove that 2 is a common index divisor of K.

(i) If $2 \mid a$, $2 \mid b$ and $2 \mid c$, then $F(x) \equiv x^9 (\mathrm{mod}\ 2)$. Let $\phi_1 = x$, so $F(x) = \phi_1(x)^9 + a\phi_1(x)^2 + b\phi_1(x) + c$. We discuss two subcases depending on whether 7 divides $\nu_2(a)$ or not.

- If $\frac{8}{7}\nu_2(a) < \nu_2(b) < \frac{\nu_2(a)+\nu_2(c)}{2}$ and $\nu_2(a) \not\equiv 0 (\mathrm{mod}\ 7)$, then $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three distincts sides joining the points $(0, \nu_2(c))$, $(1, \nu_2(b))$, $(2, \nu_2(a))$ and $(9, 0)$ with $\deg(S_{11}) = \deg(S_{12}) = \deg(S_{13}) = 1$ (see FIGURE 1). It follows that $F(x)$ is ϕ_1-regular, and so it is 2-regular. Applying Theorem 3.1, we see that $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}^7_{131}$, where $\mathfrak{p}_{111}, \mathfrak{p}_{121}, \mathfrak{p}_{131}$ are three distinct prime ideals of $\mathbb{Z}_K$ with respective residue degrees $f_{111} = f_{121} = f_{131} = 1$. So 2 divides $i(K)$. Applying [[15], Corollary, p: 230], we get $\nu_2(i(K)) = 1$.

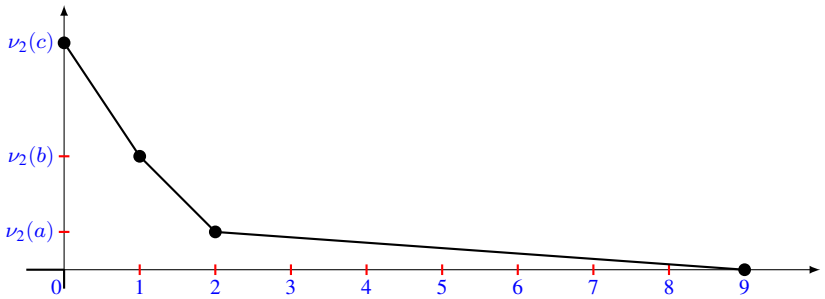


Figure 1: The ϕ_1-principal Newton polygon $N^+_{\phi_1}(F)$, with respect to ν_2 in case 1.

- If $\frac{8}{7}\nu_2(a) < \nu_2(b) < \frac{\nu_2(a)+\nu_2(c)}{2}$ and $\nu_2(a) \equiv 0 (\mathrm{mod}\ 7)$, we have $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three distinct sides joining the points $(0, \nu_2(c))$, $(1, \nu_2(b))$, $(2, \nu_2(a))$ and $(9, 0)$ with $\deg(S_{11}) = \deg(S_{12}) = 1$, and $\deg(S_{13}) = 7$. Moreover, $R_{\lambda_{131}}(F)(y) = y^7 + 1 = (y+1)(y^3+y^2+1)(y^3+y+1)$, by 3.1 so $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{132}\mathfrak{p}_{133}$ with $f_{111} = f_{121} = f_{131} = 1$ and $f_{132} = f_{133} = 3$, hence, 2 divides $i(K)$. Applying [[15], Theorem 4], we get $\nu_2(i(K)) = 1$.

(ii) If $2 \mid b$ and $2 \mid c$ and 2 does not divides a, then $F(x) \equiv x^2(x+1)(x^3+x^2+1)(x^3+x+1) (\mathrm{mod}\ 2)$. Let $\phi_1 = x$, $\phi_2 = x+1$, $\phi_3 = x^3+x^2+1$ and $\phi_4 = x^3+x+1$. As $\nu_2(b) < \frac{1}{2}\nu_2(c)$, we have $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two distinct sides, with $\deg(S_{11}) = \deg(S_{12}) = 1$. So, $F(x)$ is ϕ_1-regular. Note also that $F(x)$ is ϕ_i-regular, for $i = 1, 2, 3$. It follows that $F(x)$ is 2-regular. Using Theorem 3.1, we see that $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}\mathfrak{p}_{311}\mathfrak{p}_{411}$ with $f_{111} = f_{121} = f_{211} = 1$ and $f_{311} = f_{411} = 3$. Applying [[15], Theorem], we get 2 divides $i(K)$, then $\nu_2(i(K)) = 1$.

(iii) If $2 \mid a$ and $2 \mid c$ and 2 does not divides b, then $F(x) \equiv x(x+1)^8 (\mathrm{mod}\ 2)$. Let $\phi_1 = x$ and $\phi_2 = x+1$. We have

$$F(x) = \phi_2(x)^9 + 9\phi_2(x)^8 + 36\phi_2(x)^7 + 84\phi_2(x)^6 + 126\phi_2(x)^5 + 126\phi_2(x)^4 + 84\phi_2(x)^3 + (a+36)\phi_2(x)^2 + (2a+b+9)\phi_2(x) + (1+a+b+c).$$

a. If $(a,\ b,\ c) \in \{(0,3,4), (0,3,12), (0,11,4), (0,11,12), (8,3,4), (8,3,12), (8,11,4), (8,11,12), (4,3,0), (4,3,8), (4,11,0), (4,11,8), (12,3,0), (12,3,8), (12,11,0), (12,11,8), (4,7,4), (12,7,12)\} (\mathrm{mod}\ 16)$, then $N^+_{\phi_2}(F) = S_{21} + S_{22} + S_{23}$ has three distinct sides $\deg(S_{21}) = \deg(S_{22}) = \deg(S_{23}) = 1$, by Theorem 3.1, we have $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}^3_{221}\mathfrak{p}^4_{231}$ with $f_{111} = f_{211} = f_{221} = f_{231} = 1$. Hence, 2 divides $i(K)$, then $\nu_2(i(K)) \geq 1$.

b. If $(a,\ b,\ c) \in \{(2,7,6), (2,15,14), (10,7,14), (10,15,6), (6,7,2), (6,15,10), (14,7,10), (14,15,2)\} (\mathrm{mod}\ 16)$, then $N^+_{\phi_2}(F) = S_{21} + S_{22} + S_{23}$ has three sides

$\deg(S_{21}) = \deg(S_{22}) = \deg(S_{23}) = 1$. By 3.1, $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}^6$ with $f_{111} = f_{211} = f_{221} = f_{231} = 1$. Hence 2 divides $i(K)$, Applying [[15], Corollary, p: 230], we get $\nu_2(i(K)) = 1$.

c. If $(a,\ b,\ c) \in \{(0,15,16),(0,31,0),(16,15,0),(16,31,16),(8,15,8),(8,31,24),(24,15,24),(24,31,8)\}(\text{mod } 32)$, then we have $N_{\phi_2}^+(F) = S_{21} + S_{22} + S_{23} + S_{24}$ has four distinct sides of degree 1 each. Applying Theorem 3.1, we get $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}^2\mathfrak{p}_{241}^4$ with $f_{111} = f_{211} = f_{221} = f_{231} = f_{241} = 1$. Hence 2 divides $i(K)$, then $\nu_2(i(K)) \geq 1$.

d. To be concise, consult the theorem statement for the list of triplets (a, b, c) in this case. Here, we have $N_{\phi_2}^+(F) = S_{21} + S_{22} + S_{23}$ has three sides $\deg(S_{21}) = \deg(S_{22}) = \deg(S_{23}) = 1$. Therefore, by 3.1, $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}^6$ with $f_{111} = f_{211} = f_{221} = f_{231} = 1$. Hence 2 divides $i(K)$, Applying [[15], Corollary, p: 230], we get $\nu_2(i(K)) = 1$.

□

Proof of Theorem 2.2. Let $3 \mid a$, $3 \mid b$ and 3 does not divides c,

(i) Here, we have $a \equiv b \equiv 0(\text{mod } 3)$ and $c \equiv 1(\text{mod } 3)$. Thus, $F(x) \equiv (x+1)^9(\text{mod } 3)$. Let $\phi_1 = x + 1$, so

$$F(x) = \phi_1(x)^9 - 9\phi_1(x)^8 + 36\phi_1(x)^7 - 84\phi_1(x)^6 + 126\phi_1(x)^5 - 126\phi_1(x)^4 + 84\phi_1(x)^3 + (a-36)\phi_1(x)^2 + (b-2a+9)\phi(x) + (c+a-1-b).$$

Since $(a,\ b,\ c) \in \{(0,72,73),(27,45,19),(54,18,46)\}(\text{mod } 81)$, $\nu_3(b-2a+9) \geq 4$, $\nu_3(a+c-b-1) \geq 7$ and $\nu_3(b-2a+9) < \frac{2+\nu_3(a+c-b-1)}{2}$, we have $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four sides $\deg(S_{11}) = \deg(S_{12}) = \deg(S_{13}) = \deg(S_{14}) = 1$. Then $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{141}^6$ with $f_{111} = f_{121} = f_{131} = f_{141} = 1$. Hence 3 divides $i(K)$, Applying [[15], Corollary, p: 230], we get $\nu_3(i(K)) = 1$.

(ii) In this case, we have $a \equiv b \equiv 0(\text{mod } 3)$ and $c \equiv -1(\text{mod } 3)$. It follows that $F(x) \equiv (x-1)^9(\text{mod } 3)$. Let $\phi_1 = x - 1$, so

$$F(x) = \phi_1(x)^9 + 9\phi_1(x)^8 + 36\phi_1(x)^7 + 84\phi_1(x)^6 + 126\phi_1(x)^5 + 126\phi_1(x)^4 + 84\phi_1(x)^3 + (a+36)\phi_1(x)^2 + (b+2a+9)\phi_1(x) + (1+a+b+c).$$

Since $(a,\ b,\ c) \in \{(0,72,8),(27,18,35),(54,45,62),(9,54,17),(36,0,44),(63,27,71)\}(\text{mod } 81)$, $\nu_3(b+2a+9) \geq 4$, $\nu_3(1+a+b+c) \geq 7$ and $\nu_3(b+2a+9) < \frac{2+\nu_3(1+a+b+c)}{2}$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four distinct sides, with $\deg(S_{11}) = \deg(S_{12}) = \deg(S_{13}) = \deg(S_{14}) = 1$. Then $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{141}^6$ with $f_{111} = f_{121} = f_{131} = f_{141} = 1$. Hence 3 divides $i(K)$, Applying [[15], Corollary, p: 230], we get $\nu_3(i(K)) = 1$.

(iii) $(a,\ b,\ c) \in \{(18,36,26),(72,9,80)\}(\text{mod } 81)$, $\nu_3(b+2a+9) \geq 6$, $\nu_3(1+a+b+c) \geq 10$ and $\nu_3(b+2a+9) < \frac{2+\nu_3(1+a+b+c)}{2}$. Then, for $\phi_1(x) = x - 1$, we have $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four sides $\deg(S_{11}) = \deg(S_{12}) = \deg(S_{13}) = \deg(S_{14}) = 1$. Then $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{141}^6$ with $f_{111} = f_{121} = f_{131} = f_{141} = 1$. Hence 3 divides $i(K)$, Applying [[15], Corollary, p: 230], we get $\nu_3(i(K)) = 1$.

(iv) Let $(a,\ b,\ c) \equiv (72,63,26)(\text{mod } 81)$ and

$\nu_3(1+a+b+c) \geq 6$, hence $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides $\deg(S_{11}) = \deg(S_{13}) = 1$ and $\deg(S_{12}) = 2$. Moreover, $R_{\lambda_2}(y) = y^2 - 1 = (y-1)(y+1)$ separable in $F_{\phi_1}[y]$. Then $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{122}\mathfrak{p}_{131}^6$ with $f_{111} = f_{121} = f_{122} = f_{131} = 1$. Hence 3 divides $i(K)$, Applying [[15], Corollary, p: 230], we get $\nu_3(i(K)) = 1$.

□

Proof of Theorem 2.3. In view of Lemma 3.4, it suffices to show that the primes 5 and 7 do not divide the index $i(K)$. We show that 5 does not divide $i(K)$, and by a similar method, we see that $i(K)$ is not divisible by 7. First, we note by Lemma 3.4 that $5 \mid i(K)$ if and only if $L_5(1) \geq 6$.

(i) If $5 \mid a$, $5 \mid b$ and $5 \mid c$, then $F(x) \equiv x^9 (\mathrm{mod}\ 5)$. Let $\phi_1 = x$, so $F(x) = \phi_1(x)^9 + a\phi_1(x)^2 + b\phi_1(x) + c$. Three distinct cases arise:

a. If $\frac{8}{7}\nu_5(a) < \nu_5(b) < \frac{\nu_5(a)+\nu_5(c)}{2}$ and $\nu_5(a) \not\equiv 0 (\mathrm{mod}\ 7)$, we have $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three distinct sides with $\deg(S_{11}) = \deg(S_{12}) = \deg(S_{13}) = 1$. Then $5\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}^7_{131}$ with $f_{111} = f_{121} = f_{131} = 1$. So, $L_5(1) = 3 < 6$. Hence, 5 does not divide $i(K)$.

b. If $\frac{8}{7}\nu_5(a) < \nu_5(b) < \frac{\nu_5(a)+\nu_5(c)}{2}$ and $\nu_5(a) \equiv 0 (\mathrm{mod}\ 7)$, then $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three distinct sides with $\deg(S_{11}) = \deg(S_{12}) = 1$ and $\deg(S_{13}) = 7$. Moreover, $R_{\lambda_{13}}(F)(y) = y^7 + a_5 = (y - c)G(y)$, where c is a non-zero element of $\mathbb{F}_5$, and $G(y)$ is an irreducible polynomial of degree 6 and $G(c) \neq 0$. Thus ϕ_1 provides three distinct prime ideals of degree 1 each, lying above 5; $L_5(1) = 3 < 6$. Hence, 5 does not divide $i(K)$.

c. If $\frac{8}{7}\nu_5(a) > \nu_5(b) > \frac{\nu_5(a)+\nu_5(c)}{2}$, then we have at most three distinct prime ideals of $\mathbb{Z}_K$ above 5 of degree 1 for each. Hence, 5 does not divide $i(K)$.

(ii) If $5 \mid a$, $5 \mid b$, and 5 does not divide c, then $F(x) \equiv \phi_1(x)\phi_2(x) (\mathrm{mod}\ 5)$ where $\deg(\phi_1) = 1$, $\deg(\phi_2) = 8$ and $\phi_2(c) \neq 0$ for every $c \in \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Then a single prime ideal of $\mathbb{Z}_K$ above 5 of degree 1. Hence 5 does not divide $i(K)$, finally $\nu_5(i(K)) = 0$.

(iii) If $5 \mid a$, $5 \mid c$ and 5 does not divide b, then we have

- If $b \equiv 2, 3 (\mathrm{mod}\ 5)$, then $F(x) \equiv \phi_1(x)\phi_2 (\mathrm{mod}\ 5)$, where $\phi_1(x) = x$ and $\phi_2(x) = x^8 + b$. By Theorem 3.1, $5\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}$, with $f_{111} = 1$ and $f_{211} = 8$.
- If $b \equiv 1 (\mathrm{mod}\ 5)$, then $F(x) \equiv \phi_1(x)\phi_2(x)\phi_3(x) (\mathrm{mod}\ 5)$, where $\phi_1(x) = x, \phi_2(x) = x^4 + 2, \phi_3 = x^4 + 3$. Therefore, $5\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{311}$, with $f_{111} = 1$ and $f_{211} = f_{311} = 4$.
- $b \equiv 4 (\mathrm{mod}\ 5)$, then $F(x) = (x-1)(x-2)(x-3)(x-4)(x^2+2)(x^2+3) (\mathrm{mod}\ 5)$. Thus, $5\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{311}\mathfrak{p}_{411}\mathfrak{p}_{511}\mathfrak{p}_{611}\mathfrak{p}_{711}$, where $f_{i11} = 1$ for $i = 1, \ldots, 5$ and $f_{i11} = 2$ for $i = 6, 7$.

It follows that $L_5(1) \leq 4 < 6$. Consequently, 5 does not divide $i(K)$.

(iv) If $5 \mid b$, $5 \mid c$ and 5 does not divide a, then $F(x) \equiv x^2\phi_1(x)\phi_2(x) (\mathrm{mod}\ 5)$ where $\deg(\phi_1) = 1$, $\deg(\phi_2) = 6$ and $\phi_2(c) \neq 0$ for every $c \in \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Then we have at most three prime ideals of $\mathbb{Z}_K$ above 5 of residue degree 1. Hence 5 does not divide $i(K)$, finally $\nu_5(i(K)) = 0$.

(v) If $5 \mid c$ and 5 does not divide both a and b, then $F(x) \equiv x\phi_1(x)\phi_2(x) (\mathrm{mod}\ 5)$ where $\deg(\phi_1) \in \{0, 1\}$, $\deg(\phi_2) \in \{7, 8\}$, x, $\phi_1(x)$ and $\phi_2(x)$ are coprime polynomials. Further, $\phi_2(x)$ does not divide by any irreducible polynomial of degree 2. Then we have at most two prime ideals of $\mathbb{Z}_K$ above 5 of residue degree 1. Hence 5 does not divide $i(K)$, finally $\nu_5(i(K)) = 0$.

(vi) If $5 \mid b$ and 5 does not divide both a and c, then $F(x) \equiv \phi_1(x)\phi_2(x)^2\phi_3(x) (\mathrm{mod}\ 5)$ where $\deg(\phi_1), \deg(\phi_2) \in \{0, 1\}$, $\deg(\phi_3) \in \{6, 8, 9\}$, $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are coprime polynomials. Further, $\phi_3(x)$ does not divide by any irreducible polynomial of degree 2. Then we have at most three prime ideals of $\mathbb{Z}_K$ above 5 of residue degree 1. Hence 5 does not divide $i(K)$, finally $\nu_5(i(K)) = 0$.

(vii) If $5 \mid a$ and 5 does not divide both b and c, then $F(x) \equiv \phi_1(x)\phi_2(x) (\mathrm{mod}\ 5)$ where $\deg(\phi_1) \in \{0, 1, 2\}$, $\deg(\phi_2) \in \{7, 8, 9\}$, $\phi_1(x)$ and $\phi_2(x)$ are coprime polynomials. Further, $\phi_2(x)$ does not divide by any irreducible polynomial of degree 2. Then we have at most two prime ideals of $\mathbb{Z}_K$ above 5 of residue degree 1. Hence 5 does not divide $i(K)$, finally $\nu_5(i(K)) = 0$.

(viii) If 5 does not divide a, b and c, then $F(x) \equiv \phi_1(x)\phi_2(x)\phi_3(x) (\text{mod } 5)$ where $\deg(\phi_1)$, $\deg(\phi_2) \in \{0,1,2,3\}$, $\deg(\phi_3) \in \{6,7,8,9\}$, $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are coprime polynomials. Further, $\phi_3(x)$ does not divide by any irreducible polynomial of degree 2. Then we have at most four prime ideals of $\mathbb{Z}_K$ above 5 of residue degree 1. Hence 5 does not divide $i(K)$, finally $\nu_5(i(K)) = 0$.

□

5 Illustration

Let K be a number field defined by a complex root of a monic irreducible polynomial $F(x)$.

(i) We consider $F(x) = x^9 + 6x^2 + 12x + 48$. As $F(x)$ is a 3-Eisenstein polynomial, it is irreducible over $\mathbb{Q}$. So, by Theorem 2.1(1), then $\nu_2(i(K)) = 1$, hence K is not monogenic.

(ii) If $F(x) = x^9 + 7x^2 + 28x + 224$. As $F(x)$ is a 7-Eisenstein polynomial, it is irreducible over $\mathbb{Q}$. Hence, by Theorem 2.1(3), then $\nu_2(i(K)) = 1$ hence K is not monogenic.

(iii) We consider $F(x) = x^9 + 48x^2 + 3x + 12$. As $F(x)$ is a 3-Eisenstein polynomial, it is irreducible over $\mathbb{Q}$. Then, by Theorem 2.1(4), then $\nu_2(i(K)) \geq 1$, hence K is not monogenic.

(iv) We consider $F(x) = x^9 + 12x^2 + 3x + 48$. Since $F(x)$ is a 3-Eisenstein polynomial, it is irreducible over $\mathbb{Q}$. Thus, by Theorem 2.1(4), then $\nu_2(i(K)) \geq 1$, hence K is not monogenic.

Remark 5.1. We used Maple 12, which has a quick running time, for all computations of ϕ-expansions and polynomial factorization over finite fields.

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