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Stability of Set-Valued Linear Functional Equations By Fixed Point and Direct Methods

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Abstract. In this paper, using the fixed point and direct methods, we investigate the Hyers-Ulam stability for the set-valued general linear functional equation

$$G(a_1u + b_1u') = a_2G(u) \oplus b_2G(u'), \forall u, u' \in U$$

for a set-valued mapping $G : U \to \mathcal{B}_{cb}(V)$ of a real vector space U into a set of all closed convex bounded subset of Banach space V with $B \oplus B' = \overline{B + B'}$ where $B,\ B' \in \mathcal{B}_{cb}(V)$ and $a_1,\ b_1,\ a_2,\ b_2$ are fixed non zero scalars. Also, we obtain some interesting consequences.

1 Introduction

A set-valued (SV) function, also called a multi-valued function, is similar to a function but may associate several values to each input. More precisely, a multivalued function from a domain U to a codomain V associates each u in U to one or more values v in V. The theory of SV functions has been much related to mathematical economics and control theory.

Lu and Park [25] examined the Ulam stability problem for set-valued functional equations (SVFEs), thereby generalizing classical stability results from single-valued functions to set-valued mappings. Although the stability of such equations is frequently expressed using set inclusions or the Hausdorff metric, the equations themselves fundamentally involve equalities between sets; hence, the terminology set-valued functional equations is used. In their work, the authors focused on the following additive SVFEs:

$$f(a_1u + b_1u') = a_2f(u) \oplus b_2f(u'), \qquad f(u + u' + u'') = 2f\left(\frac{u' + u'}{2}\right) \oplus f(u''),$$

where $a_1, b_1 > 0$, $a_2, b_2 \in \mathbb{R}$ satisfy $a_1 + b_1 = a_2 + b_2 \neq 1$. They established the Ulam stability of these equations. Several authors have investigated Ulam stability problems for various forms of SVFE (see [23, 30, 31, 32, 38] and references cited therein). Kenary et al. [24] used the Hausdorff metric defined on closed convex subsets of a Banach space to characterize a functional inequality and investigated Ulam's stability of SVFE (a Cauchy-Jensen-type additive SVFE, a Jensen-type additive-quadratic SVFE, a generalized quadratic SVFE, and a Jensen-type cubic SVFE) by using a fixed point technique, which is used to deal with the stability of SVFE. SVFE has been thoroughly researched by a number of authors, and there are a number of intriguing findings (see [8, 11, 26, 33] and references cited therein).

In the theory of Ulam's stability, one can find an efficient tool to evaluate the errors, that is, to study the existence of an exact solution of the perturbed FE, which is not far from the given function. In 1940, the stability problem for the FE was first raised by S.M. Ulam [37].

Ulam's question. Let $(R_1, *)$ be a group and (R_2, Δ, d) be a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $f : R_1 \to R_2$ satisfies

$$d\big(f(u * v),\ f(u)\Delta, f(v)\big) < \delta \quad \text{for all } u, v \in R_1,$$

then there exists a homomorphism $F : R_1 \to R_2$ such that

$$d\big(f(u), F(u)\big) < \varepsilon \quad \text{for all } u \in R_1?$$

If the answer to this problem is affirmative, then the homomorphism functional equation

$$F(u * v) = F(u)\Delta F(v)$$

is said to be stable.

Hyers [21] give an affirmative partial answer to the problem of Ulam in Banach space. Thereafter, Aoki [3] and Rassias [34] generalized Hyers' theorem for additive and linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruta [18] generalized Rassias's theorem and discussed the stability of linear FE. Stability results of several functional equations in the literature have been studied by many authors in recent years (see [1, 2, 4, 5, 6, 7, 15, 16, 36, 39], references cited therein).

In 1968, Diaz and Margolis [14] introduced one of the fundamental results of fixed point theory in generalized metric spaces. The main distinction between a generalized metric and a usual metric is that the range of the former is allowed to include infinity. Isac and Rassias [22] were the first to show how to use the stability theory of FE to prove new FP theorems with applications in 1996. A number of authors have extensively examined the stability difficulties of numerous FE utilizing FP approaches (see [9, 10, 19, 28, 29, 35]).

The direct method is a classical technique in Hyers–Ulam stability in which stability is proved through direct manipulations and estimates of the functional inequality. This approach originates from early foundational results and has been repeatedly used in later works (see [17] and the references cited therein).

Throughout the paper, U is a real vector space, $Q \subset U$ is a nonempty and V is a Banach space. Moreover, for a function r mapping a nonempty set Q into Q we write $r^0(u) = u$, $r^n = r \circ r^{n-1}$, and if r is invertible, $r^{-n} = r^{-1} \circ r^{-n+1}$ for $u \in Q$, $n \in \mathbb{N} \cup \{0\}$. We define the following:

2^V: The set of all subset of V;

$\mathcal{B}_b(V)$: The set of all closed bounded subset of V;

$\mathcal{B}_c(V)$: The set of all closed convex subset of V;

$\mathcal{B}_{cb}(V)$: The set of all closed convex bounded subset of V.

On 2^V, we consider the addition and the scalar multiplication as follows:

$$B + B' = \{u + u' : u \in C, u' \in B'\}, \quad \mu B = \{\mu u : u \in B\},$$

where $B,\ B' \in 2^V$ and $\mu \in \mathbb{R}$. Further, if $B,\ B' \in \mathcal{B}_c(V)$, then we denote

$$B \oplus B' = \overline{B + B'},$$

where $\overline{B + B'}$ denotes the closure of $B + B'$.

It is simple to check that

$$\mu B + \mu B' = \mu(B + B'), \quad (\mu_1 + \mu_2)B \subseteq \mu_1 B + \mu_2 B.$$

In addition, when $\mathcal{B}$ is convex, we have $(\mu_1 + \mu_2)B = \mu_1 B + \mu_2 B$ for all $\mu_1, \mu_2 \in \mathbb{R}^+$.

It is clear that if $B \in \mathcal{B}_{cb}(V)$, then for all $\mu_1, \mu_2 \in \mathbb{R}^+$, we have

$$\mu_1 B \oplus \mu_2 B = \overline{\mu_1 B + \mu_2 B} = \overline{(\mu_1 + \mu_2)B} = (\mu_1 + \mu_2)B.$$

For a given set $B \in 2^V$, the distance function $d(., B)$ and the support function $s_p(., B)$ are respectively defined by

$$d(u, B) = \inf\{\|u - v\| : v \in B\}, u \in V,$$

$$s_p(u^*, B) = \sup\{\langle u^*, u\rangle; u \in B\}, u^* \in V^*.$$

For every pair $B, B' \in \mathcal{B}_b(V)$, we define the Pompeiu-Hausdorff (PH) distance between B and B' by

$$h(B, B') = \inf\{\Lambda > 0 : B \subseteq B' + \Lambda B_V,\ B' \subseteq B + \Lambda B_V\},$$

where B_V is the closed unit ball in V.

Some properties of the PH distance are shown in the following proposition.

Proposition 1.1. *[24] The following properties are true for every $B, B', K, K' \in \mathcal{B}_{cb}(V)$ and $\Lambda > 0$,*

(a) $h(B \oplus B', K \oplus K') \leq h(B, K) + h(B', K')$;

(b) $h(\Lambda B, \Lambda K) = \Lambda h(B, K)$.

Definition 1.2. Let $G : U \to \mathcal{B}_{cb}(V)$. The general linear SVFE is defined by

$$G(a_1 u + b_1 u') = a_2 G(u) \oplus b_2 G(u'), \tag{1.1}$$

for all $u, u' \in U$ and fixed nonzero scalars a_1, b_1, a_2, b_2 such that $a_1 + b_1 > 0$ and $a_2 + b_2 > 0$.

Note that Cauchy type additive SVFE $G(u + u') = G(u) \oplus G(u')$ is a special case of the general linear SVFE (1.1), if $a_1 = b_1 = a_2 = b_2 = 1$. Moreover the equation (1.1) also includes Cauchy-Jensen type additive SVFE for $a_1 = b_1 = a_2 = b_2 = \frac{1}{2}$.

In this paper, using the FP method and the direct method, we investigate the stability of the general linear SVFE (1.1) in the setting of Banach space.

2 Fixed Point Method

In this section, we use the FP method to investigate the stability of general linear SVFE (1.1) in the setting of Banach spaces.

Let $(\mathcal{B}_{cb}(V), \oplus, h)$ be endowed with the PH distance h. Since V is a Banach space, $(\mathcal{B}_{cb}(V), \oplus, h)$ is a complete metric semigroup (see [12]). Further, Debreu [13] proved that $(\mathcal{B}_{cb}(V), \oplus, h)$ is isometrically embedded in a Banach space.

The following result of Diaz and Margolis [14] will be used in this section.

Theorem 2.1. *Let (U, d) be a generalized complete metric space. Assume that $T : U \to U$ is a strictly contractive operator with Lipschitz constant $L < 1$. If there exists a non negative integer n_0 such that $d(T^{n_0+1}u, T^{n_0}u) < \infty$ for some $u \in U$, then the following statements are true:*

(i) The sequence $\{T^n u\}$ converges to a FP u^* of T;

(ii) u^* is the unique FP of T in $Z = \{z \in U, d(T^{n_0}u, z) < \infty\}$;

(iii) If $z \in Z$ then

$$d(z, u^*) \leq \frac{1}{1 - L} d(Tz, z).$$

Theorem 2.2. *Let Q be a nonempty subspace of U, $r : Q \to Q$, $\phi : Q^2 \to \mathbb{R}$, and $f : Q \to (\mathcal{B}_{cb}(V), \oplus, h)$ such that following conditions hold*

$$h(f(a_1 u + b_1 u'), a_2 f(u) \oplus b_2 f(u')) \leq \phi(u, u'), \tag{2.1}$$

for every $u, u' \in Q$, and fixed nonzero scalars a_1, b_1, a_2, b_2 such that $a_1 + b_1 > 0$ and $a_2 + b_2 > 0$. Suppose that

$$\phi(r(u), r(u')) \leq L\, \phi(u, u'), \tag{2.2}$$

where $L < 1$, and $r = q^{\epsilon}$ where $q(u) = (a_1 + b_1)u$ for $u \in \mathcal{U}$ with $a_1 + b_1 > 0$, $q^{\epsilon}(Q) \subset Q$, $\epsilon = \{1, -1\}$.

Then

$$G(u) = \lim_{n\to\infty} (\Gamma^{-\epsilon})^n f(r^n u),$$

exists for each $u \in Q$ *and define as a unique general linear SV map* $G : Q \to (\mathcal{B}_{cb}(V), h)$ *such that*

$$h(G(u), f(u)) \leq \begin{cases} \dfrac{\Gamma^{-\epsilon}}{1 - L\Gamma^{-\epsilon}}\phi(u,u);\ if\ \epsilon = 1, \Gamma \geq 1 \\ \dfrac{L}{1 - L\Gamma^{-\epsilon}}\phi(u,u);\ if\ \epsilon = -1, 0 < \Gamma < 1 \end{cases} \tag{2.3}$$

where $\Gamma = a_2 \oplus b_2$.

Moreover, if there exist positive real numbers m *and* μ *with* $\mu > 0$ *such that diam* $f(u) \leq m \|u\|^{\mu}$ *for all* $u \in Q$, *then* $G(u)$ *is a singleton.*

Proof. Take $u = u'$ in (2.1), we have

$$h(f(q(u)), \Gamma(f(u))) \leq \phi(u,u). \tag{2.4}$$

Using the properties of PH distance, equation (2.4) implies that, for each $u \in Q$ and $q^{\epsilon}(Q) \subset Q$,

$$h(\Gamma^{-\epsilon} f(r(u)), f(u)) \leq H_{\epsilon}(u), \tag{2.5}$$

where $r := q^{\epsilon}$, $H_1(u) :=\mid \Gamma \mid^{-1} \phi(u,u)$, $H_{-1}(u) := \phi(q^{-1}(u), q^{-1}(u))$. The proof of this theorem have two cases either $\epsilon = 1$ or $\epsilon = -1$.

Case (a). Suppose that $\Gamma \geq 1$ and $\lambda := \Gamma^{-\epsilon}$. If $\epsilon = 1$ in (2.5). Then we have

$$h(f(u), \lambda f(r(u))) \leq \lambda\phi(u,u) \text{ for every } u \in Q \text{ with } q(u) \in Q. \tag{2.6}$$

Consider $Q_d = \{g : Q \to \mathcal{B}_{cb}(V), g(0) = \{0\}\}$ endowed with the norm $\|.\|$ on Q_d define as

$$\|g_1 - g_2\| = \inf\{\mu \in (0,\infty) \mid h(g_1(u), g_2(u)) \leq \mu\phi(u,u), \forall u \in Q\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to see that $(Q_d, \|.\|)$ is a complete (see [20], Theorem 2.4 and [27], Lemma 2.1).

Now, we define a linear mapping $T : Q_d \to Q_d$ by

$$Tg(u) = \lambda g(r(u)), \text{ for all } u \in Q.$$

Let $g_1, g_2 \in Q_d$ be given such that $\|g_1 - g_2\| = \varepsilon$. Then

$$h(g_1(u), g_2(u)) \leq \varepsilon\phi(u,u), \text{ for all } u \in Q.$$

Therefore, using the properties of PH distance and (2.2), we obtain

$$\begin{aligned} h(Tg_1(u), Tg_2(u)) &= h(\lambda g_1(r(u)), \lambda g_2(r(u))) \\ &= \lambda h(g_1(r(u)), g_2(r(u))) \\ &\leq \lambda\varepsilon\phi(r(u), r(u)) \\ &\leq \lambda L\varepsilon\phi(u,u). \end{aligned}$$

Suppose that $\lambda L = \alpha$, thus we have

$$h(Tg_1(u), Tg_2(u)) \leq \alpha\varepsilon\phi(u,u)$$

where $\alpha < 1$, for all $u \in Q$. Hence, $\|g_1 - g_2\| = \varepsilon$ implies that $\|Tg_1 - Tg_2\| \leq \alpha\varepsilon$. Therefore, we know that $\|Tg_1 - Tg_2\| \leq \alpha \|g_1 - g_2\|$, which means that T is a strictly contractive map with $\alpha < 1$. Moreover, we can infer from (2.6) that $\|Tf - f\| \leq \lambda$.

By the Theorem 2.1, there exists a mapping $G : Q \to V$ satisfying the following:
(i) G is a FP of T, i.e., $G(u) = \lambda G(ru)$ for all $u \in Q$. Further, G is the unique FP of T in the set $\{g \in Q_d \mid \|f - g\| < \infty\}$, which means that there exists an $\varrho \in (0, \infty)$ such that

$$h(f(u), G(u)) \leq \varrho\phi(u, u), \text{ for all } u \in Q.$$

(ii) $\|T^n f - G\| \to 0$ as $n \to \infty$. Then we get

$$\lim_{n\to\infty} \lambda^n f(r^n(u)) = G(u), \text{ for all } u \in Q.$$

(iii) $\|f - G\| \leq \dfrac{1}{1-\alpha} \|f - Tf\|$. Then we have $\|f - G\| \leq \dfrac{\lambda}{(1-\alpha)}$, which implies that (2.3) holds.

Since $a_1 r^n(u) + b_1 r^n(u') = r^n(a_1 u + b_1 u')$ for $n \in \mathbb{N}, u, u' \in Q$, we obtain replacing in (2.1) u by $r^n(u)$, u' by $r^n(u')$ and multiplying the both sides of $\mid \lambda \mid^n$

$$\begin{aligned} h(\lambda^n f(r^n(a_1 u + b_1 u')), a_2\lambda^n f(r^n(u)) \oplus b_2\lambda^n f(r^n(u'))) &\leq \lambda^n \ \phi(r^n(u), r^n(u')) \\ &\leq \alpha^n \ \phi(u, u'). \end{aligned}$$

Since $\alpha < 1$, the last expression tends to zero as $n \to \infty$. By (ii), we conclude that G is general linear SV mapping satisfying (1.1)

$$G(a_1 u + b_1 u') = a_2 G(u) \oplus b_2 G(u'),$$

for every $u, u' \in Q$ and $a_1, b_1, a_2, b_2 \neq 0$.

If m and μ are positive real numbers with $\mu > 0$ and diam $f(u) \leq m \|u\|^\mu$ for all $u \in U$ then diam $(\lambda^n f(k^n u)) \leq \lambda^n m \|k^n u\|^\mu$ for all $u \in U$ and therefore $G(u) = \lim_{n\to\infty} \lambda^n f(k^n u)$ is a singleton.

Case (b). Suppose that $\Gamma < 1$ and $\lambda := \Gamma^{-\epsilon}$. If $\epsilon = -1$ in (2.5). Then we have

$$h(f(u), \lambda f(r(u))) \leq \phi(r(u), r(u)) \leq L\phi(u, u), \tag{2.7}$$

for every $u \in Q$ with $u \in q^{-1}(Q)$ and $L < 1$.

The remaining part of the proof is similar to the above. □

Corollary 2.3. *Let Q be a nonempty subspace of U, $r : Q \to Q$, $\phi : Q^2 \to \mathbb{R}, q(u) = (a_1 + b_1)u$ for $u \in U$ with $a_1 + b_1 > 0$ and $f : Q \to (\mathcal{B}_{cb}(V), \oplus, h)$ such that following conditions hold*

$$h(f(a_1 u + b_1 u'), a_2 f(u) \oplus b_2 f(u')) \leq \eta(\|u\|^p + \|u'\|^p), \tag{2.8}$$

for every $u, u' \in Q$, $a_1, b_1, a_2, b_2 \neq 0$ and η, p are any constants. Suppose that

$$\phi(r(u), r(u')) \leq L \ \phi(u, u'), \tag{2.9}$$

where $L < 1$ and $r = q^\epsilon$ where $q^\epsilon(Q) \subset Q$, $\epsilon = \{1, -1\}$.

Then

$$G(u) = \lim_{n\to\infty} (\Gamma^{-\epsilon})^n f(r^n u),$$

exists for each $u \in Q$ and define as a unique general linear SV mapping $G : Q \to (\mathcal{B}_{cb}(V), h)$ such that

$$h(G(u), f(u)) \leq \begin{cases} \dfrac{\Gamma^{-\epsilon}}{1-\alpha} 2\eta \ \|u\|^p & if \quad \epsilon = 1 \\ \dfrac{L}{1-\alpha} 2\eta \ \|u\|^p & if \quad \epsilon = -1, \end{cases} \tag{2.10}$$

for all $u \in Q$, where $\alpha = L\Gamma^{-\epsilon} < 1$ and $\Gamma = a_2 + b_2 > 0$.

Moreover, if there exist positive real numbers m and μ with $\mu > 0$ such that diam $f(u) \leq m \|u\|^\mu$ for all $u \in Q$, then $G(u)$ is a singleton.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(u, u') = \eta(\|u\|^p + \|u'\|^p), \text{ for all } u, u' \in Q.$$

□

3 Stability of set-valued general linear equation functional equation

In this section, we investigate the stability of general linear SVFE (1.1) in the setting of Banach spaces using the direct method. To start with, we have the following result needed in the sequel.

Proposition 3.1. *Let Q be a nonempty subspace of U, $r : Q \to Q, \psi_\circ : Q \to \mathbb{R}, f : Q \to (\mathcal{B}_{cb}(V), h)$, and $\lambda \in \mathbb{R}_+$. Suppose that for $w \in Q$.*

$$h(\lambda f(r(w)), f(w)) \leq \psi_\circ(w) \tag{3.1}$$

and

$$H(w) = \sum_{i=0}^{\infty} \mid \lambda \mid^i \psi_\circ(r^i(w)) < \infty \text{ holds.} \tag{3.2}$$

Then for every $w \in Q$,

$$F(w) = \lim_{n\to\infty} \lambda^n f(r^n(w)) \tag{3.3}$$

exists and $F : Q \to (\mathcal{B}_{cb}(V), h)$ is a unique function with

$$h(F(w), f(w)) \leq H(w) \tag{3.4}$$

and

$$\lambda F(r(w)) = F(w), \text{ for all } w \in Q. \tag{3.5}$$

Proof. First, we will prove by induction that, for every $n \in \mathbb{N}$,

$$h(\lambda^n f(r^n(w)), \lambda^{n-1} f(r^{n-1}(w))) \leq \mid \lambda \mid^{n-1} \psi_\circ(r^{n-1}(w)), \tag{3.6}$$

for $w \in Q$. From (3.1) it follows that (3.6) holds for some $n = 1$. Assume that (3.6) holds for $n \in \mathbb{N}$ and replace w by $r(w)$, multiplying both side by $\mid \lambda \mid$ we obtain

$$h(\lambda^{n+1} f(r^{n+1}(w)), \lambda^n f(r^n(w))) \leq \mid \lambda \mid^n \psi_\circ(r^n(w)),$$

for $w \in Q$ and consequently (3.6) holds with n replace by $n + 1$. Thus, by the principle of mathematical induction, we show that (3.6) is valid for every $n \in \mathbb{N}$.

From (3.6), by the triangular inequality for every $l \in \mathbb{N} \cup \{0\}$, we get

$$h(\lambda^{l+i} f(r^{l+i}(w)), \lambda^l f(r^l(w))) \leq \sum_{t=l}^{l+i-1} \mid \lambda \mid^t \psi_\circ(r^t(w)), \text{ for } w \in Q \text{ and } i \in \mathbb{N}. \tag{3.7}$$

Taking $l \to \infty$, $\sum_{t=l}^{l+i-1} \mid \lambda \mid^t \psi_\circ(r^t(w)) \to 0$ so $h(\lambda^{l+i} f(r^{l+i}(w), \lambda^l f(r^l(w))$ is arbitrary small.

Thus, we have shown that $\{\lambda^n f(r^n(w))\}$ is a Cauchy sequence. So the limit (3.3) exists for every $w \in Q$. Now we have to prove that $F : Q \to (\mathcal{B}_{cb}(V), h)$ satisfies (3.5). Further replace in (3.1) w by $r^n(w)$ and multiply by $\mid \lambda \mid^n$ then we get

$$h(\lambda\lambda^n f(r(r^n(w))), \lambda^n f(r^n(w))) \leq \mid \lambda \mid^n \psi_\circ(r^n(w)), \text{ for } w \in Q. \tag{3.8}$$

Taking $n \to \infty$ in (3.8), and using (3.2), we get (3.5). By taking $l = 0$ and $i \to \infty$ in (3.7), we get (3.4).

To prove the uniqueness of function F, we suppose another function $F_1 : Q \to (\mathcal{B}_{cb}(V), h)$ satisfying (3.4) and (3.5). Then for $n \in \mathbb{N}$, $w \in Q$, we have

$$\begin{aligned} h(F(w), F_1(w)) &= \mid \lambda \mid^n h(F(r^n(w)), F_1(r^n(w))) \\ &\leq \mid \lambda \mid^n [h(F(r^n(w)), f(r^n(w))) + h(F_1(r^n(w)) + f(r^n(w)))] \\ &\leq 2 \mid \lambda \mid^n \sum_{i=0}^{\infty} \mid \lambda \mid^i \psi_\circ(r^{n+i}(w)) \\ &= 2 \sum_{i=n}^{\infty} \mid \lambda \mid^i \psi_\circ(r^i(w)). \end{aligned} \tag{3.9}$$

Taking $n \to \infty$ in (3.9), we get $F(w) = F_1(w)$. □

Theorem 3.2. *Assume that* $\Gamma = a_2 \oplus b_2 \neq 1, Q \subset U$ *is nonempty. Let* $\Theta : Q^2 \to \mathbb{R}$, $q(u) = (a_1 + b_1)u$, *for* $u \in U$ *with* $a_1 + b_1 \neq 0$, *and* $g : Q \to (\mathcal{B}_{cb}(V), \oplus, h)$ *be a mapping satisfy*

$$h(g(a_1u + b_1u'), a_2g(u) \oplus b_2g(u')) \leq \Theta(u, u'), \tag{3.10}$$

for every $u, u' \in Q$. *Assume that* $\Gamma > 0$, $\epsilon \in \{-1, 1\}$ *such that* $q^\epsilon(Q) \subset Q$ *and*

$$\begin{gathered} H(u) := \sum_{i=0}^{\infty} \mid \Gamma \mid^{-i\epsilon} \Theta(q^{i\epsilon}(u), q^{i\epsilon}(u)) < \infty, \\ \lim_{n\to\infty} \inf \mid \Gamma^{-n\epsilon}\Theta(q^{n\epsilon}(u), q^{n\epsilon}(u)) \mid = 0. \end{gathered} \tag{3.11}$$

Then there exists a unique $G : Q \to (\mathcal{B}_{cb}(V), h)$ *such that (1.1) holds for every* $u, v \in Q$ *and*

$$h(G(u), g(u)) \leq H_\circ(u),$$

for $u \in Q$, *where*

$$H_\circ(u) = \begin{cases} \mid \Gamma \mid^{-1} H(u), & \epsilon = 1 \\ H(q^{-1}(u)), & \epsilon = -1. \end{cases} \tag{3.12}$$

Proof. Suppose that $\Gamma > 0$ and take $f = g$ in (3.10). Then we have

$$h(f(a_1u + b_1u'), a_2f(u) \oplus b_2f(u')) \leq \Theta(u, u'), \tag{3.13}$$

for every $u, u' \in Q$.
Whence $u = u'$, for each $u \in Q$,

$$h(f(q(u)), \Gamma f(u)) \leq \Theta(u, u). \tag{3.14}$$

Using the properties of PH distance, equation (3.14) implies that, for each $u \in Q$, $q^\epsilon(Q) \subset Q$,

$$h(\lambda f(r(u)), f(u)) \leq \psi_\epsilon(u),$$

where $r := q^\epsilon$, $\psi_1(u) :=\mid \Gamma \mid^{-1} \Theta(u, u)$, $\psi_{-1}(u) := \Theta(q^{-1}(u), q^{-1}(u))$ and $\lambda := \Gamma^{-\epsilon}$.

Hence by the Proposition 3.1, for every $u \in Q$, the limit $F(u)$, exists and

$$h(f(u), F(u)) \leq \sum_{i=0}^{\infty} \mid \Gamma \mid^{-i\epsilon} \psi_\epsilon(r^i(u)).$$

Since $a_1r^n(u) + b_1r^n(u') = r^n(a_1u + b_1u')$ for $n \in \mathbb{N}, u, u' \in \mathcal{U}$, we obtain replacing in (3.13) u by $r^n(u)$ and u' by $r^n(u')$ and multiplying the both sides of the inequality by λ^n

$$h(\lambda^n f(r^n(a_1u + b_1u')), a_2\lambda^n f(r^n(u)) \oplus b_2\lambda^n f(r^n(u')) \leq \lambda^n\Theta(r^nu, r^nu'), \tag{3.15}$$

for $n \in \mathbb{N}, u, u' \in Q$.

Taking $n \to \infty$ in (3.15) and using (3.11), we get

$$F(a_1u + b_1u') = a_2F(u) \oplus b_2F(u'), \text{ for } u, u' \in Q.$$

Take $G = F$. Then it is easier to check that (1.1) holds for $u, u' \in Q$ and $h(G(u), g(u)) \leq H_\circ(u)$ for $u \in Q$.
Since G is unique, so is F. □

Corollary 3.3. *Assume that* $\Gamma = a_2 \oplus b_2 \neq 1, Q \subset U$ *is nonempty. Let* $\Theta : Q^2 \to \mathbb{R}, q(u) = (a_1 + b_1)u$, *for* $u \in U$ *with* $a_1 + b_1 \neq 0$, *and* $g : Q \to (\mathcal{B}_{cb}(V), \oplus, h)$ *be a mapping satisfy*

$$h(g(a_1u + b_1u'), a_2g(u) \oplus b_2g(u')) \leq \varpi(\|u\|^p + \|u'\|^p), \tag{3.16}$$

for every $u, u' \in Q$, where ϖ, p are any constant. Assume that $\Gamma > 0$, $\epsilon \in \{-1, 1\}$ such that $q^{\epsilon}(Q) \subset Q$ and

$$H(u) = \sum_{i=0}^{\infty} |\Gamma|^{-i\epsilon} \varpi(\left\|q^{i\epsilon}u\right\|^p + \left\|q^{i\epsilon}u'\right\|^p) < \infty,$$
$$\lim_{n\to\infty} \inf |\Gamma^{-n\epsilon}\varpi(\left\|q^{i\epsilon}u\right\|^p + \left\|q^{i\epsilon}u'\right\|^p)| = 0. \tag{3.17}$$

Then there exists a unique $G : Q \to (\mathcal{B}_{cb}(V), h)$ such that (1.1) holds for every $u, u' \in Q$ and

$$h(G(u), g(u)) \leq H_{\circ}(u),$$

for $u \in Q$, where

$$H_{\circ} = \begin{cases} |\Gamma|^{-1} H(u), & \epsilon = 1 \\ H(q^{-1}(u)), & \epsilon = -1. \end{cases} \tag{3.18}$$

Proof. The proof follows from Theorem 3.2 by taking

$$\Theta(u, u') = \varpi(\|u\|^p + \|u'\|^p), \text{ for all } u, u' \in Q.$$

□

Remark 3.4. If $\Gamma = 0, \epsilon = -1$ and $q^{\epsilon}(Q) \subset Q$ in the Theorem 3.2 and Corollary 3.3, the proof is trivial.

Remark 3.5. Assume $a_1 + b_1 \neq 1$ and q, ϵ are the same as in Theorem 3.2. It's then simple to verify that $q^{n\epsilon}(u) = (a_1 + b_1)^{n\epsilon}(u)$ for $n \in \mathbb{N}, u \in U$.

Corollary 3.6. *Assume that $\psi : U \to [0, \infty]$, $\lim_{n\to\infty} |\psi(a_1 + b_1)^{n\epsilon}(u)| \leq \infty$, for $\epsilon \in \{-1, 1\}, u \in U$, $\Gamma := |a_2 \oplus b_2| \neq 1$, and $g : U \to (\mathcal{B}_{cb}(V), h)$ satisfy (3.10) for all $u, u' \in U$ with $\Theta(u, u') := \psi(u - u')$. Then there is a unique mapping $G : U \to (\mathcal{B}_{cb}(V), h)$ such that $h(G(u), g(u)) \leq \frac{\psi(0)}{|1-\Gamma|}$ and (1.1) holds for all $u, u' \in U$.*

Proof. If $\Gamma > 1$ ($\Gamma < 1$, respectively), we obtain the statement from Remark 3.5 and Theorem 3.2 with $\epsilon = 1 (\epsilon = -1$ resp.). □

Conclusion

In this paper, we investigate the Hyers-Ulam-Rassias stability for the set-valued general linear functional equation using the fixed point approach and direct method in a Banach space. Also, we obtain some interesting consequences.

Now, we have an open question: Is it possible to prove Theorem 2.2 and Corollary 2.3 in the other combinations of Γ, ϵ, like $\Gamma \geq 1, \epsilon = -1$, and $\Gamma \leq 1, \epsilon = 1$?

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