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Neighborhood Domination SDD and SO Indices of Graphs and Their Bounds

Jayjit Barman and Shibsankar Das

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Corresponding Author: Shibsankar Das

Abstract The domination degree of a vertex of a graph is the number of minimal dominating sets of the graph containing that vertex. In this research work, we define a new domination degree of each vertex of the graph, is called the neighborhood domination degree of the vertex. Using the concept of this new domination degree, first neighborhood domination Zagreb, second neighborhood domination Zagreb, modified first neighborhood domination Zagreb, forgotten neighborhood domination, hyper neighborhood domination, modified forgotten neighborhood domination, neighborhood domination symmetric division (deg) (*NDSDD*) and neighborhood domination Sombor (*NDSO*) indices are introduced. For *NDSDD* and *NDSO* indices, we compute their exact values for some standard families of graphs, and later derive their lower and upper bounds using some graph parameters and other neighborhood domination degree-based topological indices.

1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple and connected graph with non-empty vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ of cardinality n and m, respectively [1]. A subset D of $V(\Gamma)$ is a *dominating set* if every vertex in $V(\Gamma) \setminus D$ has at least one neighbor in D. The *domination degree* of a vertex $s \in V(\Gamma)$, denoted as $d_d(s)$, is the number of minimal dominating sets of Γ containing s. We use the notations $\Delta_d(\Gamma)$ and $\delta_d(\Gamma)$ to denote the maximum and minimum domination degree of vertices of a graph Γ, respectively.

Chemical graph theory is a branch of mathematical chemistry that is concerned to deal with complex molecular problems through the application of graph theory. Chemical graph theory focuses on the idea of a molecular graph, where atoms are represented as vertices and bonds between atoms are represented as edges [2]. Molecules are theoretically modeled using graph to understand the physicochemical characteristics of these chemical substances.

We use the mathematical formula called *topological index* to analyze any graph that constructs a molecular structure, which is used to assess mathematical values and investigate physicochemical properties of molecules that eliminates expenses and time-taking laboratory work. It also provides information about the physicochemical and biological properties of molecules in the quantitative structure-property relationship (*QSPR*) and quantitative structure-activity relationship (*QSAR*) studies [3, 4, 5, 6].

Definition 1.1 ([7]). The mathematical expressions symmetric division (deg) and Sombor index of graph Γ are defined as

$$SDD(\Gamma) = \sum_{st \in E(\Gamma)} \left(\frac{\min(d(s), d(t))}{\max(d(s), d(t))} + \frac{\max(d(s), d(t))}{\min(d(s), d(t))} \right)$$

and

$$SO(\Gamma) = \sum_{st \in E(\Gamma)} \sqrt{d^2(s) + d^2(t)}, \text{ respectively.}$$

The idea of domination topological indices arises depending on this minimal dominating set and domination degree [8]. For more information regarding domination in the graph, reader may follow [9, 10, 11, 12, 13, 14]. At first, the domination Zagreb indices are defined as follows [8].

$$DM_1(\Gamma) = \sum_{s \in V(\Gamma)} d_d^2(s), \qquad DM_2(\Gamma) = \sum_{st \in E(\Gamma)} d_d(s) d_d(t),$$

$$DM_1^*(\Gamma) = \sum_{st \in E(\Gamma)} (d_d(s) + d_d(t)).$$

After that, the forgotten domination, hyper domination, modified forgotten domination and domination Sombor indices of graphs are respectively defined as follows [15, 16].

$$DF(\Gamma) = \sum_{s \in V(\Gamma)} d_d^3(s), \qquad DH(\Gamma) = \sum_{st \in E(\Gamma)} (d_d(s) + d_d(t))^2,$$

$$DF^*(\Gamma) = \sum_{st \in E(\Gamma)} (d_d^2(s) + d_d^2(t)), \qquad DSO(\Gamma) = \sum_{st \in E(\Gamma)} \sqrt{d_d^2(s) + d_d^2(t)}.$$

2 Defining Neighborhood Domination Degree-based Topological Indices

From the concept of domination degree, we establish the idea of a novel domination degree, and with this new domination degree, we construct neighborhood domination topological indices, which are defined below.

Definition 2.1. Let Γ be a simple connected graph. In this research paper, we define a new domination degree of each vertex $s \in V(\Gamma)$, called the neighborhood domination degree of s and denoted by $d_{nd}(s)$, which is defined as the sum of the domination degree of neighbors of the vertex s. It is mathematically formulated as

$$d_{nd}(s) = \sum_{t \in N(s)} d_d(t).$$

Motivated by the concept of domination topological index, we introduce novel topological indices based on the neighborhood domination degree. These include the first and second neighborhood domination Zagreb indices, the modified first neighborhood domination Zagreb index, the forgotten neighborhood domination index, the hyper neighborhood domination index, the modified forgotten neighborhood domination index, the neighborhood domination symmetric division (deg) index and the neighborhood domination Sombor index, as detailed in Table 1.

Classical degree-based indices such as the Zagreb indices, Randić index and Sombor-type indices depend only on the degrees of adjacent vertices [17, 18, 19, 20, 21, 22, 23, 24, 25]. While these indices efficiently describe local connectivity and bonding patterns, they do not incorporate the wider neighborhood interactions that significantly influence the graph's structural behavior. To address this limitation, neighborhood degree-based indices were proposed [26, 27], in which each vertex's degree is replaced by the sum of the degrees of its adjacent vertices. Although these indices consider a wider local environment compared to classical degree-based measures, they still depend only on degree aggregation and fail to incorporate any notion of vertex influence or control over the graph structure. In contrast, domination-based indices are formulated using

Table 1. Formulas for neighborhood domination degree-based topological indices.

Neighborhood Domination Degree-based Indices	Notation	Formula
First Neighborhood Domination Zagreb Index	$NDM_1(\Gamma)$	$\sum_{s\in V(\Gamma)} d_{nd}^2(s)$
Second Neighborhood Domination Zagreb Index	$NDM_2(\Gamma)$	$\sum_{st\in E(\Gamma)} d_{nd}(s)d_{nd}(t)$
Modified First Neighborhood Domination Zagreb Index	$NDM_1^*(\Gamma)$	$\sum_{st\in E(\Gamma)}(d_{nd}(s)+d_{nd}(t))$
Forgotten Neighborhood Domination Index	$NDF(\Gamma)$	$\sum_{s\in V(\Gamma)} d_{nd}^3(s)$
Hyper Neighborhood Domination Index	$NDH(\Gamma)$	$\sum_{st\in E(\Gamma)} (d_{nd}(s)+d_{nd}(t))^2$
Modified Forgotten Neighborhood Domination Index	$NDF^*(\Gamma)$	$\sum_{st\in E(\Gamma)}(d_{nd}^2(s)+d_{nd}^2(t))$
Neighborhood Domination Symmetric Division (Deg) Index	$NDSDD(\Gamma)$	$\sum_{st\in E(\Gamma)}\left(\frac{\min(d_{nd}(s),d_{nd}(t))}{\max(d_{nd}(s),d_{nd}(t))}+\frac{\max(d_{nd}(s),d_{nd}(t))}{\min(d_{nd}(s),d_{nd}(t))}\right)$
Neighborhood Domination Sombor Index	$NDSO(\Gamma)$	$\sum_{st\in E(\Gamma)}\sqrt{d_{nd}^2(s)+d_{nd}^2(t)}$

domination parameters, which quantify how efficiently a selected set of vertices influences the entire graph structure [8, 15, 16].

Neighborhood domination topological indices are conceptually analogous to the classical neighborhood-based topological index. The fundamental distinction lies in the parameter used: while neighborhood indices rely on the degrees of adjacent vertices, neighborhood domination topological indices are defined using the neighborhood domination degree, which is computed as the sum of the domination degrees of all neighboring vertices of a given vertex.

3 Neighborhood Domination Symmetric Division (Deg) and Neighborhood Domination Sombor Indices of a Graph

In this section, we compute the exact values of neighborhood domination symmetric division (deg) (*NDSDD*) and neighborhood domination Sombor (*NDSO*) indices for some important families of graph such as: complete graph (K_n), star graph (S_n), complete bipartite graph ($K_{m,n}$), double star graph ($S_{m,n}$), complement of $K_{m,n}$, book graph (B_r, $r \geq 3$), friendship graph (F_n), butterfly graph ($B_{m,n}$) and windmill graph (Wd_r^s).

Proposition 3.1. *If Γ be a complete graph K_n, then*

$$NDSDD(\Gamma) = n(n-1) \text{ and } NDSO(\Gamma) = \frac{n(n-1)^2}{\sqrt{2}}.$$

Proof. For every vertex $s \in V(K_n)$, $d_d(s) = 1$. So, $d_{nd}(s) = (n-1)$, for all $s \in V(K_n)$. Therefore, $NDSDD(\Gamma) = n(n-1)$ and $NDSO(\Gamma) = \frac{n(n-1)^2}{\sqrt{2}}$. □

Proposition 3.2. *If Γ be a star graph S_n, then*

$$NDSDD(\Gamma) = n^2 + 1 \text{ and } NDSO(\Gamma) = n\sqrt{n^2+1}.$$

Proof. Let Γ be a star graph S_n. Let $\{t, t_1, t_2, ..., t_n\}$ be the set of all vertices of S_n with central vertex t. The only two minimal dominating sets of S_n are $\{t\}$ and $\{t_1, t_2, ..., t_n\}$ implies $d_d(t) = 1$, for all $t \in V(\Gamma)$. Then, $d_{nd}(t) = n$ and $d_{nd}(t_i) = 1$, for all $i = 1, 2, \ldots, n$. Therefore, $NDSDD(\Gamma) = n^2 + 1$ and $NDSO(\Gamma) = n\sqrt{n^2+1}$. □

Proposition 3.3. *If Γ be a complete bipartite graph $K_{m,n}$, then*
$NDSDD(\Gamma) = m^2\left(\frac{n+1}{m+1}\right) + n^2\left(\frac{m+1}{n+1}\right)$ *and* $NDSO(\Gamma) = mn\sqrt{n^2(m+1)^2 + m^2(n+1)^2}$.

Proof. Let $\Gamma = K_{m,n}$ be a complete bipartite graph, where $m, n \geq 2$ with partite sets V_1 and V_2 of cardinality m and n respectively. Suppose $V_1 = \{s_1, s_2, ..., s_m\}$ and $V_2 = \{t_1, t_2, ..., t_n\}$. Then for any $s_i \in V_1$; $i = 1, 2, ..., m$, the minimal dominating sets are $\{s_1, s_2, ..., s_m\}$ and $\{s_i, t_j\}$, for $j = 1, 2, ..., n$. So, $d_d(s_i) = n + 1$. Similarly, for any $t_j \in V_2$; $j = 1, 2, ..., n$, the minimal dominating sets are $\{t_1, t_2, ..., t_n\}$ and $\{t_j, s_i\}$, for $i = 1, 2, ..., m$. So, $d_d(t_j) = m + 1$. Then

$$d_{nd}(s) = \begin{cases} n(m+1), & \text{for all } s \in V_1; \\ m(n+1), & \text{for all } s \in V_2. \end{cases}$$

Therefore,

$$NDSDD(\Gamma) = m^2\Big(\frac{n+1}{m+1}\Big) + n^2\Big(\frac{m+1}{n+1}\Big)$$

and

$$NDSO(\Gamma) = mn\sqrt{n^2(m+1)^2 + m^2(n+1)^2}.$$

□

Proposition 3.4. *If Γ be a double star graph $S_{m,n}$, then*

$$NDSDD(\Gamma) = \frac{m+n+1}{m+1} + \frac{m+n+1}{n+1} + m(m+1) + n(n+1)$$

and

$$NDSO(\Gamma) = m\sqrt{4(m^2+2m+2)} + n\sqrt{4(n^2+2n+2)} + 2\sqrt{(m+1)^2+(n+1)^2}.$$

Proof. Let Γ be a double star graph $S_{m,n}$. Let $\{s, s_1, s_2, ..., s_m, t, t_1, t_2, ..., t_n\}$ be the set of vertices of $S_{m,n}$. There are four minimal dominating sets $\{s,t\}$, $\{s_1, s_2, ..., s_m, t_1, t_2, ..., t_n\}$, $\{s, t_1, t_2, ..., t_n\}$ and $\{t, s_1, s_2, ..., s_n\}$. Since every vertex $v \in S_{m,n}$ contained in two minimal dominating sets, then $d_d(v) = 2$, for all $v \in V(\Gamma)$. Then,

$$d_{nd}(v) = \begin{cases} 2, & \text{for all } s_i \text{ and } t_j; \\ 2(m+1), & \text{for } s; \\ 2(n+1), & \text{for } t. \end{cases}$$

Therefore, $NDSDD(\Gamma) = \frac{m+n+1}{m+1} + \frac{m+n+1}{n+1} + m(m+1) + n(n+1)$ and

$$NDSO(\Gamma) = m\sqrt{4(m^2+2m+2)} + n\sqrt{4(n^2+2n+2)} + 2\sqrt{(m+1)^2+(n+1)^2}.$$

□

Proposition 3.5. *If Γ be the complement graph $\bar{K}_{m,n}$ of $K_{m,n}$, then* $NDSDD(\Gamma) = m^2 + n^2 - m - n$ *and* $NDSO(\Gamma) = \frac{mn}{\sqrt{2}}[(m-1)^2 + (n-1)^2]$.

Proof. Let Γ be the complement graph $\bar{K}_{m,n}$ of $K_{m,n}$. The graph $K_{m,n}$, $m, n \geq 2$ with partite sets V_1 and V_2 of cardinality m and n, respectively. Suppose $V_1 = \{s_1, s_2, ..., s_m\}$ and $V_2 = \{t_1, t_2, ..., t_n\}$. Then for any $s_i \in V_1(\Gamma)$; $i = 1, 2, ..., m$, the minimal dominating sets are $\{s_i, t_j\}$, for $j = 1, 2, ..., n$. So, $d_d(s_i) = n$. Similarly, for any $t_j \in V_2(\Gamma)$; $j = 1, 2, ..., n$, the minimal dominating sets are $\{t_j, s_i\}$, for $i = 1, 2, ..., m$. So, $d_d(t_j) = m$. Then

$$d_{nd}(v) = \begin{cases} n(m-1), & \text{for } v \in V_1; \\ m(n-1), & \text{for } v \in V_2. \end{cases}$$

Therefore, $NDSDD(\Gamma) = m^2 + n^2 - m - n$ and $NDSO(\Gamma) = \frac{mn}{\sqrt{2}}[(m-1)^2 + (n-1)^2]$. □

Lemma 3.6 ([8]). *Let $\Gamma = B_r$ be a book graph with $r \geq 3$, which is a cartesian product of star graph S_{r+1} and path graph P_2 with the total number of minimal dominating sets $2^r + 3$. Then*

$$d_d(t) = \begin{cases} 3, & t \textit{ is center vertex}; \\ 2^{r-1}+1, & \textit{otherwise}. \end{cases}$$

Theorem 3.7. *Let Γ be a book graph B_r, $r \geq 3$, then*

$$NDSDD(\Gamma) = 2 + 2r\left(\frac{2^{r-1}+4}{r(2^{r-1}+1)+3} + \frac{r(2^{r-1}+1)+3}{2^{r-1}+4} + 1\right)$$

and

$$NDSO(\Gamma) = \sqrt{2}\cdot[r(2^{r-1}+1)+3] + \sqrt{2}r\cdot(2^{r-1}+4) + 2r\sqrt{(r^2+1)\cdot 2^{2r-2} + (r^2+3r)\cdot 2^r + 2^{r+2} + r^2 + 6r + 25}.$$

Proof. Note that there are three types of edge set. These are as follows:
$E_1 = \{st \in E(\Gamma) : d_{nd}(s) = d_{nd}(t) = r(2^{r-1}+1)+3\}$,
$E_2 = \{st \in E(\Gamma) : d_{nd}(s) = 2^{r-1}+4,\ d_{nd}(t) = r(2^{r-1}+1)+3\}$,
$E_3 = \{st \in E(\Gamma) : d_{nd}(s) = d_{nd}(t) = 2^{r-1}+4\}$,
and $|E_1| = 1$, $|E_2| = 2r$, $|E_3| = r$.

$$\begin{aligned}
NDSDD(\Gamma) &= \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\
&= \sum_{st\in E_1(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\
&\quad + \sum_{st\in E_2(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\
&\quad + \sum_{st\in E_3(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\
&= 1 \cdot \left(\frac{r(2^{r-1}+1)+3}{r(2^{r-1}+1)+3} + \frac{r(2^{r-1}+1)+3}{r(2^{r-1}+1)+3}\right) + 2r \cdot \left(\frac{2^{r-1}+4}{r(2^{r-1}+1)+3}\right. \\
&\quad \left. + \frac{r(2^{r-1}+1)+3}{2^{r-1}+4}\right) + r \cdot \left(\frac{2^{r-1}+4}{2^{r-1}+4} + \frac{2^{r-1}+4}{2^{r-1}+4}\right) \\
&= 2 + 2r\left(\frac{2^{r-1}+4}{r(2^{r-1}+1)+3} + \frac{r(2^{r-1}+1)+3}{2^{r-1}+4} + 1\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&NDSO(\Gamma) \\
&= \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\
&= \sum_{st\in E_1(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} + \sum_{st\in E_2(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} + \sum_{st\in E_3(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\
&= 1 \cdot \sqrt{2\left(r(2^{r-1}+1)+3\right)^2} + 2r \cdot \sqrt{\left(2^{r-1}+4\right)^2 + \left(r(2^{r-1}+1)+3\right)^2} \\
&\quad + r \cdot \sqrt{2\left(2^{r-1}+4\right)^2} \\
&= \sqrt{2} \cdot [r(2^{r-1}+1)+3] + \sqrt{2}r \cdot (2^{r-1}+4) \\
&\quad + 2r\sqrt{(r^2+1) \cdot 2^{2r-2} + (r^2+3r) \cdot 2^r + 2^{r+2} + r^2 + 6r + 25}.
\end{aligned}$$

□

Lemma 3.8 ([28]). *Let $\Gamma = F_n$ be a friendship graph with $2n+1$ vertices that consists of n triangles sharing a common vertex. Then,*

$$d_d(t) = \begin{cases} 1, & t \text{ is center vertex}; \\ 2^{n-1}, & t \text{ is in triangle}. \end{cases}$$

Theorem 3.9. *Let Γ be a friendship graph F_n, then*

$$NDSDD(\Gamma) = 2n\left(\frac{n \cdot 2^n}{2^{n-1}+1} + \frac{2^{n-1}+1}{n \cdot 2^n} + 1\right)$$

and

$$NDSO(\Gamma) = 2n\sqrt{n^2 \cdot 2^{2n} + 2^{2n-2} + 2^n + 1} + \sqrt{2}n(2^{n-1}+1).$$

Proof. Here we calculate the neighborhood domination degree as

$$d_{nd}(t) = \begin{cases} n \cdot 2^n, & t \text{ is center vertex;} \\ 2^{n-1}+1, & t \text{ is in triangle.} \end{cases}$$

$$\begin{aligned} NDSDD(\Gamma) &= \sum_{st \in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))} \right) \\ &= 2n\left(\frac{n \cdot 2^n}{2^{n-1}+1} + \frac{2^{n-1}+1}{n \cdot 2^n} \right) + n\left(\frac{2^{n-1}+1}{2^{n-1}+1} + \frac{2^{n-1}+1}{2^{n-1}+1} \right) \\ &= 2n\left(\frac{n \cdot 2^n}{2^{n-1}+1} + \frac{2^{n-1}+1}{n \cdot 2^n} \right) + 2n \\ &= 2n\left(\frac{n \cdot 2^n}{2^{n-1}+1} + \frac{2^{n-1}+1}{n \cdot 2^n} + 1 \right). \end{aligned}$$

Similarly,

$$\begin{aligned} NDSO(\Gamma) &= \sum_{st \in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\ &= 2n\sqrt{(n \cdot 2^n)^2 + (2^{n-1}+1)^2} + n\sqrt{2(2^{n-1}+1)^2} \\ &= 2n\sqrt{n^2 \cdot 2^{2n} + 2^{2n-2} + 2^n + 1} + \sqrt{2}n(2^{n-1}+1). \end{aligned}$$

□

Lemma 3.10 ([28]). *let Γ be a butterfly graph $B_{m,n}$ with $2m+n+1$ vertices that consists of m triangles and n pendent edges sharing a common vertex. Then,*

$$d_d(t) = \begin{cases} 1, & t \textit{ is center vertex;} \\ 2^m, & t \textit{ is in pendent edge;} \\ 2^{m-1}, & t \textit{ is in triangle.} \end{cases}$$

Theorem 3.11. *Let Γ be a butterfly graph $B_{m,n}$, then*

$$NDSDD(\Gamma) = n\left(\frac{1}{(m+n) \cdot 2^m} + (m+n) \cdot 2^m \right) + 2m\left(\frac{1+2^{m-1}}{(m+n) \cdot 2^m} + \frac{(m+n) \cdot 2^m}{1+2^{m-1}} + 1 \right)$$

and

$$\begin{aligned} NDSO(\Gamma) &= n\sqrt{1 + (m^2+2mn+n^2) \cdot 2^{2m}} \\ &\quad + 2m\sqrt{(m^2+2mn+n^2) \cdot 2^{2m} + 2^{2m-1} + 2^m + 1} \\ &\quad + m\sqrt{2(2^m + 2^{2m-2} + 1)} \end{aligned}$$

Proof. Here we find the neighborhood domination degree as

$$d_{nd}(t) = \begin{cases} (m+n) \cdot 2^m, & t \text{ is center vertex;} \\ 1, & t \text{ is in pendent edge;} \\ 1+2^{m-1}, & t \text{ is in triangle.} \end{cases}$$

There are three types of edge set which are given below:
$E_1 = \{st \in E(\Gamma) : d_{nd}(s) = (m+n) \cdot 2^m, d_{nd}(t) = 1\}$,
$E_2 = \{st \in E(\Gamma) : d_{nd}(s) = (m+n) \cdot 2^m, d_{nd}(t) = 1 + 2^{(m-1)}\}$,

$E_3 = \{st \in E(\Gamma) : d_{nd}(s) = d_{nd}(t) = 1 + 2^{(m-1)}\}$,
and $|E_1| = n$, $|E_2| = 2m$, $|E_3| = m$.

$$\begin{aligned} NDSDD(\Gamma) &= \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))} \right) \\ &= n\left(\frac{1}{(m+n)\cdot 2^m} + (m+n)\cdot 2^m \right) + 2m\left(\frac{1+2^{m-1}}{(m+n)\cdot 2^m} + \frac{(m+n)\cdot 2^m}{1+2^{m-1}} \right) \\ &\quad + m\left(\frac{1+2^{m-1}}{1+2^{m-1}} + \frac{1+2^{m-1}}{1+2^{m-1}} \right) \\ &= n\left(\frac{1}{(m+n)\cdot 2^m} + (m+n)\cdot 2^m \right) + 2m\left(\frac{1+2^{m-1}}{(m+n)\cdot 2^m} + \frac{(m+n)\cdot 2^m}{1+2^{m-1}} + 1 \right). \end{aligned}$$

Similarly,

$$\begin{aligned} &NDSO(\Gamma) \\ &= \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\ &= n\sqrt{1 + \left((m+n)\cdot 2^m\right)^2} + 2m\sqrt{\left(1+2^{m-1}\right)^2 + \left((m+n)\cdot 2^m\right)^2} \\ &\quad + m\sqrt{\left(1+2^{m-1}\right)^2 + \left(1+2^{m-1}\right)^2} \\ &= n\sqrt{1 + (m^2+2mn+n^2)\cdot 2^{2m}} + 2m\sqrt{(m^2+2mn+n^2)\cdot 2^{2m} + 2^{2m-1} + 2^m + 1} \\ &\quad + m\sqrt{2(2^m + 2^{2m-2} + 1)}. \end{aligned}$$

□

Lemma 3.12 ([8]). *Let $\Gamma = Wd_p^q$ be the windmill graph formed by q copies of the complete graph K_p at a common universal vertex, where $p \geq 2$ and $q \geq 2$. Then,*

$$d_d(t) = \begin{cases} 1, & \text{if } t \text{ is in center;} \\ (p-1)^{(q-1)}, & \text{otherwise.} \end{cases}$$

Theorem 3.13. *Let Γ be the windmill graph Wd_p^q, then*

$$NDSDD(\Gamma) = q(p-1)\left(\frac{q(p-1)^q}{(p-2)(p-1)^{(q-1)}+1} + \frac{(p-2)(p-1)^{(q-1)}+1}{q(p-1)^q} \right) + q(p-1)(p-2)$$

and

$$\begin{aligned} NDSO(\Gamma) &= q(p-1)\sqrt{\left(q(p-1)^q\right)^2 + \left((p-2)(p-1)^{(q-1)}+1\right)^2} \\ &\quad + \frac{q(p-1)(p-2)}{\sqrt{2}}\left((p-2)(p-1)^{(q-1)}+1\right). \end{aligned}$$

Proof. Here we find the neighborhood domination degree as

$$d_{nd}(t) = \begin{cases} q(p-1)^q, & t \text{ is center vertex;} \\ (p-2)(p-1)^{(q-1)}+1, & \text{otherwise.} \end{cases}$$

$NDSDD(\Gamma)$

$$= \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))} \right)$$

$$= q(p-1)\left(\frac{q(p-1)^q}{(p-2)(p-1)^{(q-1)}+1} + \frac{(p-2)(p-1)^{(q-1)}+1}{q(p-1)^q} \right) + \left(\frac{pq(p-1)}{2} - q(p-1) \right) \cdot \left(\frac{(p-2)(p-1)^{(q-1)}+1}{(p-2)(p-1)^{(q-1)}+1} + \frac{(p-2)(p-1)^{(q-1)}+1}{(p-2)(p-1)^{(q-1)}+1} \right)$$

$$= q(p-1)\left(\frac{q(p-1)^q}{(p-2)(p-1)^{(q-1)}+1} + \frac{(p-2)(p-1)^{(q-1)}+1}{q(p-1)^q} \right) + q(p-1)(p-2).$$

Similarly,

$$NDSO(\Gamma) = \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}$$

$$= q(p-1)\sqrt{\left(q(p-1)^q\right)^2 + \left((p-2)(p-1)^{(q-1)}+1\right)^2} + \left(\frac{pq(p-1)}{2} - q(p-1) \right) \cdot \sqrt{\left((p-2)(p-1)^{(q-1)}+1\right)^2 + \left((p-2)(p-1)^{(q-1)}+1\right)^2}$$

$$= q(p-1)\sqrt{\left(q(p-1)^q\right)^2 + \left((p-2)(p-1)^{(q-1)}+1\right)^2} + \frac{q(p-1)(p-2)}{\sqrt{2}}\left((p-2)(p-1)^{(q-1)}+1\right).$$

□

4 Bounds of the *NDSDD* and *NDSO* Indices

In this section, we establish the bounds for the neighborhood domination symmetric division (deg) (*NDSDD*) and neighborhood domination Sombor (*NDSO*) indices of a graph Γ in terms of order n, size m, maximum domination degree $\Delta_d(\Gamma)$, minimum domination degree $\delta_d(\Gamma)$ and other considered neighborhood domination degree-based topological indices. Moreover, we also produce some mathematical relationships between these two indices (i.e. *NDSDD* and *NDSO* indices).

4.1 Bounds in Terms of Order, Size, Maximum and Minimum Domination Degrees

Theorem 4.1. *Let Γ be any graph of order n and size m with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$m \cdot \left(\frac{\delta_d}{(n-1)\Delta_d} + 1 \right) \leq NDSDD(\Gamma) \leq m \cdot \left(\frac{(n-1)\Delta_d}{\delta_d} + \frac{\delta_d}{(n-1)\Delta_d} \right)$$

with equality if and only if Γ is a $2k$-domination 1-regular graph.

Proof. Here $\Delta_d = \max\{d_d(t) : t \in V(\Gamma)\}$ and $\delta_d = \min\{d_d(t) : t \in V(\Gamma)\}$ and suppose that $1 \leq \delta_d \leq d_{nd}(s) \leq d_{nd}(t) \leq (n-1)\Delta_d$, we have

$$NDSDD(\Gamma) = \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))} \right)$$

$$\geq \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + 1 \right)$$

$$\geq \left(\frac{m\delta_d}{(n-1)\Delta_d} + m\right) = m \cdot \left(\frac{\delta_d}{(n-1)\Delta_d} + 1\right).$$

Again,

$$\begin{aligned} NDSDD(\Gamma) &= \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\ &\leq \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) \\ &\leq \sum_{st\in E(\Gamma)} \left(\frac{(n-1)\Delta_d}{\delta_d} + \frac{\delta_d}{(n-1)\Delta_d}\right) = m \cdot \left(\frac{(n-1)\Delta_d}{\delta_d} + \frac{\delta_d}{(n-1)\Delta_d}\right). \end{aligned}$$

$$\therefore\ m \cdot \left(\frac{\delta_d}{(n-1)\Delta_d} + 1\right) \leq NDSDD(\Gamma) \leq m \cdot \left(\frac{(n-1)\Delta_d}{\delta_d} + \frac{\delta_d}{(n-1)\Delta_d}\right).$$

□

Theorem 4.2. *Let Γ be any graph having n vertices and m edges with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$\frac{2m}{(n-1)} \cdot \frac{\delta_d}{\Delta_d} \leq NDSDD(\Gamma) \leq 2m(n-1) \cdot \frac{\Delta_d}{\delta_d}$$

with equality if and only if Γ is a $2k$-domination 1-regular graph.

Proof. Here $\Delta_d = \max\{d_d(t) : t \in V(\Gamma)\}$ and $\delta_d = \min\{d_d(t) : t \in V(\Gamma)\}$ and suppose that $1 \leq \delta_d \leq d_{nd}(s) \leq d_{nd}(t) \leq (n-1)\Delta_d$. Since, $\frac{1}{(n-1)\Delta_d} \leq \frac{1}{d_{nd}(s)}, \frac{1}{d_{nd}(t)} \leq \frac{1}{\Delta_d}$, then

$$\begin{aligned} &\frac{2}{(n-1)\Delta_d} \leq \left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) \leq \frac{2}{\delta_d} \\ \implies &\sum_{st\in E(\Gamma)} \frac{2}{(n-1)\Delta_d} \leq \sum_{st\in E(\Gamma)} \left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) \leq \sum_{st\in E(\Gamma)} \frac{2}{\delta_d} \\ \implies &\frac{2m}{(n-1)\Delta_d} \leq \sum_{st\in E(\Gamma)} \left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) \leq \frac{2m}{\delta_d}. \end{aligned}$$

Therefore,

$$\begin{aligned} &NDSDD(\Gamma) \\ &= \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\ &= \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) \\ &= \sum_{st\in E(\Gamma)} \left(\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) \cdot \Big(d_{nd}(s) + d_{nd}(t)\Big) - 2\right) \\ &= \sum_{st\in E(\Gamma)} \left(d_{nd}(s)\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) + d_{nd}(t)\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) - 2\right) \\ &= \sum_{st\in E(\Gamma)} \left(\min\{d_{nd}(s), d_{nd}(t)\}\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right)\right. \\ &\quad \left. + \max\{d_{nd}(s), d_{nd}(t)\}\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) - 2\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{st\in E(\Gamma)} \left(\min\{d_{nd}(s), d_{nd}(t)\}\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) + \frac{\max\{d_{nd}(s), d_{nd}(t)\}}{d_{nd}(s)} \right.\\
&\quad \left. + \frac{\max\{d_{nd}(s), d_{nd}(t)\}}{d_{nd}(t)} - 2\right)\\
&= \sum_{st\in E(\Gamma)} \left(\min\{d_{nd}(s), d_{nd}(t)\}\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right) + \frac{\max\{d_{nd}(s), d_{nd}(t)\}}{\min\{d_{nd}(s), d_{nd}(t)\}} + 1 - 2\right)\\
&\geq \sum_{st\in E(\Gamma)} \left(\min\{d_{nd}(s), d_{nd}(t)\}\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right)\right)\\
&\geq \sum_{st\in E(\Gamma)} \delta_d\left(\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}\right)\\
&\geq \delta_d \cdot \frac{2m}{(n-1)\Delta_d} = \frac{2m}{(n-1)} \cdot \frac{\delta_d}{\Delta_d}.
\end{aligned}
$$

Similarly, one can show that $NDSDD(\Gamma) \leq 2m(n-1) \cdot \frac{\Delta_d}{\delta_d}$.

$$\therefore \ \frac{2m}{(n-1)} \cdot \frac{\delta_d}{\Delta_d} \leq NDSDD(\Gamma) \leq 2m(n-1) \cdot \frac{\Delta_d}{\delta_d}.$$

□

Theorem 4.3. *Let Γ be any graph having n number of vertices and m number of edges with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$m\sqrt{2}\delta_d \leq NDSO(\Gamma) \leq m\sqrt{2}(n-1)\Delta_d.$$

In addition, equality is attained on the left-hand side when Γ is $2k$-domination 1-regular graph, while equality on the right-hand side occurs when Γ is a complete graph.

Proof. Here $\Delta_d = \max\{d_d(t) : t \in V(\Gamma)\}$ and $\delta_d = \min\{d_d(t) : t \in V(\Gamma)\}$ and suppose that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Now,

$$
\begin{aligned}
&2\delta_d^2 \leq d_{nd}^2(s) + d_{nd}^2(t) \leq (n-1)^2\Delta_d^2\\
&\implies \sqrt{2}\delta_d \leq \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \leq \sqrt{2}(n-1)\Delta_d\\
&\implies \sum_{st\in E(\Gamma)} \sqrt{2}\delta_d \leq \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \leq \sum_{st\in E(\Gamma)} \sqrt{2}(n-1)\Delta_d\\
&\implies m\sqrt{2}\delta_d \leq NDSO(\Gamma) \leq m\sqrt{2}\Delta_d.
\end{aligned}
$$

Therefore,

$$m\sqrt{2}\delta_d \leq NDSO(\Gamma) \leq m\sqrt{2}(n-1)\Delta_d.$$

□

4.2 Bounds in Terms of Modified First Neighborhood Domination Zagreb and Second Neighborhood Domination Zagreb Indices

Theorem 4.4. *Let Γ represents a connected graph having n vertices and m edges. Then*

$$NDSDD(\Gamma) \geq \frac{\left(NDM_1^*(\Gamma)\right)^2}{NDM_2(\Gamma)} - 2m.$$

Moreover, the equality occurs when Γ is a complete graph.

Proof. Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, ..., n$, be two positive real number sequences. Then by using Radon's inequality [29], for all $r \geq 0$, we get

$$\sum_{i=1}^{n} \frac{{x_i}^{r+1}}{{a_i}^{r}} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^{r}}.$$

$$\begin{aligned}
\therefore NDSDD(\Gamma) &= \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right) \\
&= \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) \\
&= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s) + d_{nd}(t))^2 - 2d_{nd}(s)d_{nd}(t)}{d_{nd}(s)d_{nd}(t)}\right) \\
&= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s) + d_{nd}(t))^2}{d_{nd}(s)d_{nd}(t)} - 2\right) \\
&= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s) + d_{nd}(t))^2}{d_{nd}(s)d_{nd}(t)}\right) - 2m \\
&\geq \frac{\left(\sum_{st\in E(\Gamma)}(d_{nd}(s) + d_{nd}(t))\right)^2}{\sum_{st\in E(\Gamma)} d_{nd}(s)d_{nd}(t)} - 2m = \frac{\left(NDM_1^*(\Gamma)\right)^2}{NDM_2(\Gamma)} - 2m.
\end{aligned}$$

$$\therefore\ NDSDD(\Gamma) \geq \frac{\left(NDM_1^*(\Gamma)\right)^2}{NDM_2(\Gamma)} - 2m.$$

□

Theorem 4.5. *Let Γ be a graph of order n and size m with maximum and minimum domination degree Δ_d and δ_d, respectively. Then*

$$\frac{1}{\sqrt{2}} NDM_1^*(\Gamma) \leq NDSO(\Gamma) \leq NDM_1^*(\Gamma).$$

Moreover, equality is attained on the left-hand side when Γ is k-domination r-regular graph or complete graph, while equality on the right-hand side occurs when Γ is a complement of a complete graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then by the inequality between quadratic and arithmetic means for the positive numbers $d_{nd}(s)$ and $d_{nd}(t)$, we have

$$\begin{aligned}
&\sqrt{\frac{d_{nd}^2(s) + d_{nd}^2(t)}{2}} \geq \frac{d_{nd}(s) + d_{nd}(t)}{2} \\
&\implies \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \geq \frac{1}{\sqrt{2}}(d_{nd}(s) + d_{nd}(t)).
\end{aligned}$$

$$\begin{aligned}
\text{Now,}\ \ NDSO(\Gamma) &= \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\
&\geq \sum_{st\in E(\Gamma)} \frac{1}{\sqrt{2}}(d_{nd}(s) + d_{nd}(t)) = \frac{1}{\sqrt{2}} NDM_1^*(\Gamma).
\end{aligned}$$

$$\text{Again, } NDSO(\Gamma) = \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}$$

$$= \sum_{st\in E(\Gamma)} \sqrt{(d_{nd}(s) + d_{nd}(t))^2 - 2d_{nd}(s)d_{nd}(t)}$$

$$\leq \sum_{st\in E(\Gamma)} \sqrt{(d_{nd}(s) + d_{nd}(t))^2} = \sum_{st\in E(\Gamma)} (d_{nd}(s) + d_{nd}(t)) = NDM_1^*(\Gamma).$$

$$\therefore \frac{1}{\sqrt{2}} NDM_1^*(\Gamma) \leq NDSO(\Gamma) \leq NDM_1^*(\Gamma).$$

□

Theorem 4.6. *Let Γ be a graph of order n and size m with maximum and minimum domination degree Δ_d and δ_d, respectively. Then*

$$NDSO(\Gamma) \geq \frac{\sqrt{2}}{(n-1)\Delta_d} \cdot NDM_2(\Gamma).$$

Moreover, the equality holds if and only if Γ is a complete graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then by inequality between quadratic and harmonic means for the positives numbers $d_{nd}(s)$ and $d_{nd}(t)$, we have

$$\sqrt{\frac{d_{nd}^2(s) + d_{nd}^2(t)}{2}} \geq \frac{2}{\frac{1}{d_{nd}(s)} + \frac{1}{d_{nd}(t)}}$$

$$= \frac{2d_{nd}(s)d_{nd}(t)}{d_{nd}(s) + d_{nd}(t)}$$

$$\geq \frac{2d_{nd}(s)d_{nd}(t)}{2(n-1)\Delta_d} = \frac{d_{nd}(s)d_{nd}(t)}{(n-1)\Delta_d}$$

$$\implies \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \geq \frac{\sqrt{2}}{(n-1)\Delta_d} \cdot d_{nd}(s)d_{nd}(t).$$

Now,

$$NDSO(\Gamma) = \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}$$

$$\geq \sum_{st\in E(\Gamma)} \frac{\sqrt{2}}{(n-1)\Delta_d} d_{nd}(s)d_{nd}(t) = \frac{\sqrt{2}}{(n-1)\Delta_d} \cdot NDM_2(\Gamma).$$

Therefore, $NDSO(\Gamma) \geq \frac{\sqrt{2}}{(n-1)\Delta_d} \cdot NDM_2(\Gamma)$. □

Theorem 4.7. *Let Γ be a graph of order n and size m with maximum and minimum domination degree Δ_d and δ_d, respectively. Then*

$$NDSO(\Gamma) \leq \frac{\sqrt{(n-1)^2\Delta_d^2 + \delta_d^2}}{(n-1)\Delta_d + \delta_d} \cdot NDM_1^*(\Gamma).$$

Moreover, the equality holds on both sides if and only if $\{d_{nd}(s), d_{nd}(t)\} = \{\delta_d, (n-1)\Delta_d\}$.

Proof. In general $\delta_d < (n-1)\Delta_d$. Then for every $d_{nd}(s), d_{nd}(t) \in [\delta_d, (n-1)\Delta_d]$,

$$\frac{2(n-1)\delta_d\Delta_d}{(n-1)^2\Delta_d^2 + \delta_d^2} \leq \frac{2d_{nd}(s)d_{nd}(t)}{d_{nd}^2(s) + d_{nd}^2(t)} \text{ holds.}$$

Now,

$$\begin{aligned}
\frac{\Big((n-1)\Delta_d+\delta_d\Big)^2}{(n-1)^2\Delta_d^2+\delta_d^2} &= \frac{(n-1)^2\Delta_d^2+\delta_d^2+2(n-1)\Delta_d\delta_d}{(n-1)^2\Delta_d^2+\delta_d^2} \\
&= 1+\frac{2(n-1)\Delta_d\delta_d}{(n-1)^2\Delta_d^2+\delta_d^2} \\
&\leq 1+\frac{2d_{nd}(s)d_{nd}(t)}{d_{nd}^2(s)+d_{nd}^2(t)} \\
&= \frac{(d_{nd}(s)+d_{nd}(t))^2}{d_{nd}^2(s)+d_{nd}^2(t)} \\
\implies \frac{(n-1)\Delta_d+\delta_d}{\sqrt{(n-1)^2\Delta_d^2+\delta_d^2}} &\leq \frac{d_{nd}(s)+d_{nd}(t)}{\sqrt{d_{nd}^2(s)+d_{nd}^2(t)}} \\
\implies \sqrt{d_{nd}^2(s)+d_{nd}^2(t)} &\leq \frac{\sqrt{(n-1)^2\Delta_d^2+\delta_d^2}}{(n-1)\Delta_d+\delta_d}\cdot(d_{nd}(s)+d_{nd}(t)) \\
\implies \sum_{st\in E(\Gamma)}\sqrt{d_{nd}^2(s)+d_{nd}^2(t)} &\leq \frac{\sqrt{(n-1)^2\Delta_d^2+\delta_d^2}}{(n-1)\Delta_d+\delta_d}\cdot\sum_{st\in E(\Gamma)}(d_{nd}(s)+d_{nd}(t)) \\
\implies NDSO(\Gamma) &\leq \frac{\sqrt{(n-1)^2\Delta_d^2+\delta_d^2}}{(n-1)\Delta_d+\delta_d}\cdot NDM_1^*(\Gamma).
\end{aligned}$$

Therefore,

$$NDSO(\Gamma) \leq \frac{\sqrt{(n-1)^2\Delta_d^2+\delta_d^2}}{(n-1)\Delta_d+\delta_d}\cdot NDM_1^*(\Gamma).$$

□

4.3 Bounds in Terms of Hyper Neighborhood Domination and Second Neighborhood Domination Zagreb Indices

Theorem 4.8. *Let Γ be a graph of size m with maximum and minimum domination degree Δ_d and δ_d respectively. Then*

$$\frac{NDH(\Gamma)}{(n-1)^2\Delta_d^2}-2m \leq NDSDD(\Gamma) \leq \frac{NDH(\Gamma)}{\delta_d^2}-2m.$$

Moreover, equality on the left hand side attains when Γ is complete graph and equality on the right hand side occurs when Γ is a $2k$-domination 1-regular graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then

$$\begin{aligned}
NDSDD(\Gamma) &= \sum_{st\in E(\Gamma)}\left(\frac{\min(d_{nd}(s),d_{nd}(t))}{\max(d_{nd}(s),d_{nd}(t))}+\frac{\max(d_{nd}(s),d_{nd}(t))}{\min(d_{nd}(s),d_{nd}(t))}\right) \\
&= \sum_{st\in E(\Gamma)}\left(\frac{d_{nd}(s)}{d_{nd}(t)}+\frac{d_{nd}(t)}{d_{nd}(s)}\right) \\
&= \sum_{st\in E(\Gamma)}\left(\frac{d_{nd}^2(s)+d_{nd}^2(t)}{d_{nd}(s)d_{nd}(t)}\right) \\
&= \sum_{st\in E(\Gamma)}\left(\frac{(d_{nd}(s)+d_{nd}(t))^2-2d_{nd}(s)d_{nd}(t)}{d_{nd}(s)d_{nd}(t)}\right)
\end{aligned}$$

$$= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s)+d_{nd}(t))^2}{d_{nd}(s)d_{nd}(t)} - 2 \right)$$

$$= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s)+d_{nd}(t))^2}{d_{nd}(s)d_{nd}(t)} \right) - 2m$$

$$\leq \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s)+d_{nd}(t))^2}{\delta_d^2} \right) - 2m = \frac{NDH(\Gamma)}{\delta_d^2} - 2m.$$

Similarly, one can show that $NDSDD(\Gamma) \geq \frac{NDH(\Gamma)}{(n-1)^2\Delta_d^2} - 2m$. Therefore,

$$\frac{NDH(\Gamma)}{(n-1)^2\Delta_d^2} - 2m \leq NDSDD(\Gamma) \leq \frac{NDH(\Gamma)}{\delta_d^2} - 2m.$$

□

Theorem 4.9. *Let Γ be a graph of order n with maximum and minimum domination degree Δ_d and δ_d respectively. Then*

$$\frac{NDH(\Gamma) - 2NDM_2(\Gamma)}{\sqrt{2}(n-1)\Delta_d} \leq NDSO(\Gamma) \leq \frac{NDH(\Gamma) - 2NDM_2(\Gamma)}{\sqrt{2}\delta_d}.$$

Moreover, equality on the left hand side attains when Γ is complete graph and equality on the right hand side occurs when Γ is a $2k$-domination 1-regular graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then

$$NDSO(\Gamma) = \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}$$

$$= \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}^2(s) + d_{nd}^2(t)}{\sqrt{d_{nd}^2(s) + d_{nd}^2(t)}} \right)$$

$$= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s)+d_{nd}(t))^2 - 2d_{nd}(s)d_{nd}(t)}{\sqrt{d_{nd}^2(s) + d_{nd}^2(t)}} \right)$$

$$= \sum_{st\in E(\Gamma)} \left(\frac{(d_{nd}(s)+d_{nd}(t))^2 - 2d_{nd}(s)d_{nd}(t)}{\sqrt{2}\delta_d} \right)$$

$$= \frac{NDH(\Gamma) - 2NDM_2(\Gamma)}{\sqrt{2}\delta_d}.$$

Similarly, one can show that $NDSO(\Gamma) \geq \frac{NDH(\Gamma)-2NDM_2(\Gamma)}{\sqrt{2}(n-1)\Delta_d}$. Therefore,

$$\frac{NDH(\Gamma) - 2NDM_2(\Gamma)}{\sqrt{2}(n-1)\Delta_d} \leq NDSO(\Gamma) \leq \frac{NDH(\Gamma) - 2NDM_2(\Gamma)}{\sqrt{2}\delta_d}.$$

□

4.4 Bounds in Terms of Modified Forgotten Neighborhood Domination Index

Theorem 4.10. *Let Γ be a graph of order n and size m with maximum and minimum domination degree Δ_d and δ_d respectively. Then*

$$\frac{NDF^*(\Gamma)}{(n-1)^2\Delta_d^2} \leq NDSDD(\Gamma) \leq \frac{NDF^*(\Gamma)}{\delta_d^2}.$$

Moreover, the equality on left hand side occurs if and only if Γ is a complete graph and the equality on the right hand side occurs if and only if Γ is a $2k$-domination 1-regular graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then

$$\begin{aligned} NDSDD(\Gamma) &= \sum_{st \in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))} \right) \\ &= \sum_{st \in E(\Gamma)} \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)} \right) \\ &= \sum_{st \in E(\Gamma)} \left(\frac{d_{nd}^2(s) + d_{nd}^2(t)}{d_{nd}(s) d_{nd}(t)} \right) \\ &\geq \frac{1}{(n-1)^2 \Delta_d^2} NDF^*(\Gamma). \end{aligned}$$

Similarly, one can show that $NDSDD(\Gamma) \leq \frac{NDF^*(\Gamma)}{\delta_d^2}$. Therefore,

$$\frac{NDF^*(\Gamma)}{(n-1)^2 \Delta_d^2} \leq NDSDD(\Gamma) \leq \frac{NDF^*(\Gamma)}{\delta_d^2}.$$

□

Theorem 4.11. *Let Γ be a graph of order n and size m with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$\frac{NDF^*(\Gamma)}{\sqrt{2}(n-1)\Delta_d} \leq NDSO(\Gamma) \leq \frac{NDF^*(\Gamma)}{\sqrt{2}\delta_d}.$$

Moreover, the equality on left hand side occurs if and only if Γ is a complete graph and the equality on the right hand side occurs if and only if Γ is a $2k$-domination 1-regular graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then

$$\begin{aligned} &\delta_d^2 \leq d_{nd}^2(s), d_{nd}^2(t) \leq (n-1)^2 \Delta_d^2 \\ &\implies 2\delta_d^2 \leq d_{nd}^2(s) + d_{nd}^2(t) \leq \sqrt{2}(n-1)^2 \Delta_d^2 \\ &\implies \sqrt{2}\delta_d \leq \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \leq \sqrt{2}(n-1)\Delta_d \\ &\implies \frac{1}{\sqrt{2}(n-1)\Delta_d} \leq \frac{1}{\sqrt{d_{nd}^2(s) + d_{nd}^2(t)}} \leq \frac{1}{\sqrt{2}\delta_d}. \end{aligned}$$

Now,

$$\begin{aligned} NDSO(\Gamma) &= \sum_{st \in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\ &= \sum_{st \in E(\Gamma)} \left(\frac{d_{nd}^2(s) + d_{nd}^2(t)}{\sqrt{d_{nd}^2(s) + d_{nd}^2(t)}} \right) \\ &\leq \sum_{st \in E(\Gamma)} \left(\frac{d_{nd}^2(s) + d_{nd}^2(t)}{\sqrt{2}\delta_d} \right) = \frac{NDF^*(\Gamma)}{\sqrt{2}\delta_d}. \end{aligned}$$

Similarly, one can show that $NDSO(\Gamma) \geq \frac{NDF^*(\Gamma)}{\sqrt{2}(n-1)\Delta_d}$. Therefore,

$$\frac{NDF^*(\Gamma)}{\sqrt{2}(n-1)\Delta_d} \leq NDSO(\Gamma) \leq \frac{NDF^*(\Gamma)}{\sqrt{2}\delta_d}.$$

□

Lemma 4.12 ([30]). *If $c_j, d_j \geq 0$ and $\varphi d_j \leq c_j \leq \mho d_j$ for $1 \leq j \leq n$, then*

$$\left(\sum_{j=1}^{n} c_j^2\right)\left(\sum_{j=1}^{n} d_j^2\right) \leq \frac{(\mho+\varphi)^2}{4\mho\varphi}\left(\sum_{j=1}^{n} c_j d_j\right)^2.$$

If $c_j > 0$ for some $1 \leq j \leq n$, then the equality occurs if and only if $\varphi = \mho$ and $c_j = \varphi d_j$ for every $1 \leq j \leq n$.

Theorem 4.13. *Let Γ represent any graph of size m and order n having maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$\frac{2\sqrt{m(n-1)\delta_d\Delta_d}}{\delta_d+(n-1)\Delta_d} \cdot NDF^*(\Gamma) \leq NDSO(\Gamma) \leq \sqrt{m \cdot NDF^*(\Gamma)}.$$

Moreover, equality is attained on the left-hand side when Γ is $2k$-domination 1-regular graph, while equality on the right-hand side occurs when Γ is a k-domination r-regular.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Let $a_i = \sqrt{d_{nd}^2(s)+d_{nd}^2(t)}$ and $b_i = 1$. Now,

$$\begin{aligned} &2\delta_d^2 \leq d_{nd}^2(s)+d_{nd}^2(t) \leq 2(n-1)^2\Delta_d^2 \\ &\implies \sqrt{2}\delta_d \leq \sqrt{d_{nd}^2(s)+d_{nd}^2(t)} \leq \sqrt{2}(n-1)\Delta_d. \end{aligned}$$

Here we take $\varphi = \sqrt{2}\delta_d$ and $\mho = \sqrt{2}(n-1)\Delta_d$. From Lemma 4.12, we get

$$\begin{aligned} &\sum_{st\in E(\Gamma)}\left(\sqrt{d_{nd}^2(s)+d_{nd}^2(t)}\right)^2 \cdot \sum_{st\in E(\Gamma)}(1)^2 \leq \frac{\left(\sqrt{2}\delta_d+\sqrt{2}(n-1)\Delta_d\right)^2}{8(n-1)\delta_d\Delta_d} \cdot \\ &\qquad\qquad \left(\sum_{st\in E(\Gamma)}\sqrt{d_{nd}^2(s)+d_{nd}^2(t)}\right)^2 \\ &\implies m \cdot NDF^*(\Gamma) \leq \frac{\left(\delta_d+(n-1)\Delta_d\right)^2}{4(n-1)\delta_d\Delta_d} \cdot \left(NDSO(\Gamma)\right)^2 \\ &\implies \frac{4m(n-1)\delta_d\Delta_d}{\left(\delta_d+(n-1)\Delta_d\right)^2} \cdot NDF^*(\Gamma) \leq \left(NDSO(\Gamma)\right)^2 \\ &\implies \frac{2\sqrt{m(n-1)\delta_d\Delta_d}}{\delta_d+(n-1)\Delta_d} \cdot \sqrt{NDF^*(\Gamma)} \leq NDSO(\Gamma). \end{aligned}$$

Again, by Cauchy-Bunyakovsky-Schwarz's inequality [31], it holds

$$\begin{aligned} &\left(\sum_{st\in E(\Gamma)}\sqrt{d_{nd}^2(s)+d_{nd}^2(t)}\right)^2 \leq \sum_{st\in E(\Gamma)}(1)^2 \cdot \sum_{st\in E(\Gamma)}(d_{nd}^2(s)+d_{nd}^2(t)) \\ &\implies \left(NDSO(\Gamma)\right)^2 \leq m \cdot NDF^*(\Gamma) \\ &\implies NDSO(\Gamma) \leq \sqrt{m \cdot NDF^*(\Gamma)}. \end{aligned}$$

Therefore,

$$\frac{2\sqrt{m(n-1)\delta_d\Delta_d}}{\delta_d+(n-1)\Delta_d} \cdot NDF^*(\Gamma) \leq NDSO(\Gamma) \leq \sqrt{m \cdot NDF^*(\Gamma)}.$$

□

Theorem 4.14. *Let Γ be any graph of order n and size m with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$NDSO(\Gamma) \geq \frac{2m(n-1)\delta_d\Delta_d + NDF^*(\Gamma)}{\sqrt{2}\Big((n-1)\Delta_d + \delta_d\Big)}.$$

Moreover, the equality holds if and only if Γ is a k-domination r-regular graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$. Then

$$\begin{aligned}
&\delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d \\
&\implies \delta_d^2 \leq d_{nd}^2(s), d_{nd}^2(t) \leq (n-1)^2\Delta_d^2 \\
&\implies 2\delta_d^2 \leq d_{nd}^2(s) + d_{nd}^2(t) \leq 2(n-1)^2\Delta_d^2 \\
&\implies \sqrt{2}\delta_d \leq \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \leq \sqrt{2}(n-1)\Delta_d \\
&\implies \Big(\sqrt{d_{nd}^2(s) + d_{nd}^2(t)} - \sqrt{2}\delta_d\Big) \cdot \Big(\sqrt{2}(n-1)\Delta_d - \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}\Big) \geq 0 \\
&\implies \sqrt{2}(n-1)\Delta_d \cdot \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} - \Big(d_{nd}^2(s) + d_{nd}^2(t)\Big) - 2(n-1)\delta_d\Delta_d \\
&\quad + \sqrt{2}\delta_d \cdot \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \geq 0 \\
&\implies \sqrt{2}\Big((n-1)\Delta_d + \delta_d\Big) \cdot \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \geq 2(n-1)\delta_d\Delta_d + \Big(d_{nd}^2(s) + d_{nd}^2(t)\Big) \\
&\implies \sqrt{2}\Big((n-1)\Delta_d + \delta_d\Big) \cdot \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \geq \sum_{st\in E(\Gamma)} 2(n-1)\delta_d\Delta_d \\
&\quad + \sum_{st\in E(\Gamma)} \Big(d_{nd}^2(s) + d_{nd}^2(t)\Big) \\
&\implies \sqrt{2}\Big((n-1)\Delta_d + \delta_d\Big) \cdot NDSO(\Gamma) \geq 2m(n-1)\delta_d\Delta_d + NDF^*(\Gamma).
\end{aligned}$$

Therefore,

$$NDSO(\Gamma) \geq \frac{2m(n-1)\delta_d\Delta_d + NDF^*(\Gamma)}{\sqrt{2}\Big((n-1)\Delta_d + \delta_d\Big)}.$$

□

4.5 Relationship Between *NDSDD* and *NDSO* Indices

Theorem 4.15. *Let Γ be any graph of order n and size m with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$\frac{\sqrt{2}\cdot\delta_d}{(n-1)^2\cdot\Delta_d^2} \cdot NDSO(\Gamma) \leq NDSDD(\Gamma) \leq \frac{\sqrt{2}\cdot(n-1)\Delta_d}{\delta_d^2} \cdot NDSO(\Gamma).$$

Moreover, the equality occurs on both sides if and only if Γ is a $2k$-domination 1-regular graph.

Proof. We know that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$.

$$NDSDD(\Gamma) = \sum_{st\in E(\Gamma)} \left(\frac{\min(d_{nd}(s), d_{nd}(t))}{\max(d_{nd}(s), d_{nd}(t))} + \frac{\max(d_{nd}(s), d_{nd}(t))}{\min(d_{nd}(s), d_{nd}(t))}\right)$$

$$
\begin{aligned}
&= \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) \\
&= \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}^2(s) + d_{nd}^2(t)}{d_{nd}(s)d_{nd}(t)}\right) \\
&= \sum_{st\in E(\Gamma)} \left(\frac{\sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \cdot \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}}{d_{nd}(s)d_{nd}(t)}\right) \\
&\leq \left(\frac{\sqrt{2}\cdot(n-1)\Delta_d}{\delta_d^2} \cdot \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}\right) = \frac{\sqrt{2}\cdot(n-1)\Delta_d}{\delta_d^2} \cdot NDSO(\Gamma).
\end{aligned}
$$

Similarly, one can show that $NDSDD(\Gamma) \geq \frac{\sqrt{2}\cdot\delta_d}{(n-1)^2\cdot\Delta_d^2} \cdot NDSO(\Gamma)$. Therefore,

$$
\frac{\sqrt{2}\cdot\delta_d}{(n-1)^2\cdot\Delta_d^2} \cdot NDSO(\Gamma) \leq NDSDD(\Gamma) \leq \frac{\sqrt{2}\cdot(n-1)\Delta_d}{\delta_d^2} \cdot NDSO(\Gamma).
$$

□

Theorem 4.16. *Let Γ be any graph of order n and size m with maximum and minimum domination degrees Δ_d and δ_d, respectively. Then*

$$
\frac{\delta_d}{\sqrt{2}} \cdot NDSDD(\Gamma) \leq NDSO(\Gamma) \leq \frac{(n-1)\Delta_d}{\sqrt{2}} \cdot NDSDD(\Gamma).
$$

Moreover, equality is attained on the left-hand side when Γ is $2k$-domination 1-regular graph, while equality on the right-hand side occurs when Γ is a k-domination r-regular.

Proof. Suppose that $1 \leq \delta_d \leq d_{nd}(s), d_{nd}(t) \leq (n-1)\Delta_d$.

$$
\begin{aligned}
NDSO(\Gamma) &= \sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)} \\
&= \sum_{st\in E(\Gamma)} \sqrt{\frac{(d_{nd}^2(s) + d_{nd}^2(t))^2}{d_{nd}^2(s) + d_{nd}^2(t)}} \\
&= \sum_{st\in E(\Gamma)} \sqrt{\frac{\left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right)^2 \cdot d_{nd}^2(s)d_{nd}^2(t)}{d_{nd}^2(s) + d_{nd}^2(t)}} \\
&= \sum_{st\in E(\Gamma)} \sqrt{\frac{1}{\frac{1}{d_{nd}^2(s)} + \frac{1}{d_{nd}^2(t)}}} \cdot \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) \\
&\leq \sum_{st\in E(\Gamma)} \sqrt{\frac{1}{\frac{1}{(n-1)^2\Delta_d^2} + \frac{1}{(n-1)^2\Delta_d^2}}} \cdot \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) = \frac{(n-1)\Delta_d}{\sqrt{2}} \cdot NDSDD(\Gamma).
\end{aligned}
$$

Similarly, $NDSO(\Gamma) \geq \frac{\delta_d}{\sqrt{2}} \cdot NDSDD(\Gamma)$. Therefore,

$$
\frac{\delta_d}{\sqrt{2}} \cdot NDSDD(\Gamma) \leq NDSO(\Gamma) \leq \frac{(n-1)\Delta_d}{\sqrt{2}} \cdot NDSDD(\Gamma).
$$

□

Theorem 4.17. *Let Γ be a connected graph of order n and size m. Then*

$$
NDSO(\Gamma) \leq \sqrt{NDSDD(\Gamma) \cdot NDM_2(\Gamma)}.
$$

Moreover, the equality occurs if and only if Γ is a complete graph.

Proof. Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, ..., n$, be two positive real number sequences. Then using Radon's inequality [29], for all $r \geq 0$, we get

$$\sum_{i=1}^{m} \frac{{x_i}^{r+1}}{{a_i}^{r}} \geq \frac{\left(\sum_{i=1}^{m} x_i\right)^{r+1}}{\left(\sum_{i=1}^{m} a_i\right)^{r}}.$$

Now,

$$\sum_{st\in E(\Gamma)} \frac{\left(\sqrt{d_{nd}^2(s) + d_{nd}^2(t)}\right)^2}{d_{nd}(s)d_{nd}(t)} \geq \frac{\left(\sum_{st\in E(\Gamma)} \sqrt{d_{nd}^2(s) + d_{nd}^2(t)}\right)^2}{\left(\sum_{st\in E(\Gamma)} d_{nd}(s)d_{nd}(t)\right)}$$

$$\implies \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}^2(s) + d_{nd}^2(t)}{d_{nd}(s)d_{nd}(t)}\right) \geq \frac{\left(NDSO(\Gamma)\right)^2}{NDM_2(\Gamma)}$$

$$\implies \sum_{st\in E(\Gamma)} \left(\frac{d_{nd}(s)}{d_{nd}(t)} + \frac{d_{nd}(t)}{d_{nd}(s)}\right) \geq \frac{\left(NDSO(\Gamma)\right)^2}{NDM_2(\Gamma)}$$

$$\implies NDSDD(\Gamma) \geq \frac{\left(NDSO(\Gamma)\right)^2}{NDM_2(\Gamma)}$$

$$\implies \left(NDSO(\Gamma)\right)^2 \leq NDSDD(\Gamma) \cdot NDM_2(\Gamma).$$

Therefore,

$$NDSO(\Gamma) \leq \sqrt{NDSDD(\Gamma) \cdot NDM_2(\Gamma)}.$$

□

5 Conclusion

In this article, we defined the neighborhood domination degree of a graph and then established neighborhood domination topological indices of the graph using the same. Here, we calculated the exact values of the *NDSDD* and *NDSO* indices of some families of graphs. In addition, we presented several bounds on the *NDSDD* and *NDSO* indices in terms of the order, size, maximum and minimum domination degree and other neighborhood domination degree-based topological indices. Moreover, we obtained some relationships between the *NDSDD* and *NDSO* indices of the graph.

Author Contributions

All the authors contributed equally.

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My manuscript has no associated data.

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Author information

Jayjit Barman, Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, Uttar Pradesh, India.
E-mail: jayjit44@bhu.ac.in

Shibsankar Das, Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, Uttar Pradesh, India.
E-mail: shibsankar@bhu.ac.in, shib.cgt@gmail.com

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