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SOME CHARACTERIZATIONS OF LIGHTLIKE HYPERBOLIC QUATERNIONIC f-RECTIFYING CURVES

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Abstract. We study f-rectifying curves in hyperbolic quaternionic space $\mathbb{H}$. A curve is called f-rectifying if its f-position vector field always lie in its rectifying space. In this paper, we establish some characterizations of lightlike hyperbolic quaternionic f-rectifying curves.

1 Introduction

By a *quaternion* [1, 2], we mean an expression of the form

$$q = \sum_{i=0}^{3} q_i e_i,$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $e_0 = 1, e_1, e_2, e_3$ are unit quaternions that meet the following quaternionic product rules:

$$\begin{cases} e_i e_i = -1, \ i = 1, 2, 3; \\ e_i e_j = e_k = -e_j e_i; \\ e_i e_j e_k = -1, \end{cases}$$

where $(i\ j\ k)$ stands for an even permutation of $(1\ 2\ 3)$.

In similar way, a hyperbolic quaternion introduced by A. Macfarlane in 1902 [3] is expressed as

$$q = \sum_{i=0}^{3} q_i e_i,$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $e_0 = 1, e_1, e_2, e_3$ are called unit hyperbolic quaternions that admit the following conditions:

$$e_i e_i = 1, \quad e_i e_j = e_k = -e_j e_i, \quad e_i e_j e_k = 1,$$

where $(i\ j\ k)$ is an even permutation of $(1\ 2\ 3)$. The collection of all hyperbolic quaternions, denoted by $\mathbb{H}$, forms a Lorentzian vector space with a basis $\{1, e_1, e_2, e_3\}$. For a hyperbolic quaternion q, its conjugate denoted by q^* is defined by [4]

$$q^* = q_0 - \sum_{i=1}^{3} q_i e_i.$$

For all $q \in \mathbb{H}$, the product

$$-qq* = -q_0^2 + q_1^2 + q_2^2 + q_3^2$$

defines the quadratic form on the semi-Euclidean space $\mathbb{R}^4_1$ with signature $(-,+,+,+)$. In this sense, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4_1$.

In 2003, B. Y. Chen [5] introduced rectifying curves as smooth space curves whose position vectors lie in their rectifying planes (the planes spanned by the tangent and binormal vectors of the concerned curve). After that, the study was expanded to multiple ambient spaces. In this context, we mention that Gungor and Tosun [6] studied quaternionic rectifying curves in 2011.

In 2018, F. Hathout [7] introduced f-rectifying curves (a generalized version of rectifying curves). The same topic has been studied in various spaces, including Minkowski spaces [8, 9, 10] and Euclidean spaces [11]. Our aim in this paper is to explore characterizations of lightlike hyperbolic quaternionic f-rectifying curves.

The structure of this article is as follows: We start with a discussion on essential informations in the preliminaries section. Following that, we introduce the concept of hyperbolic quaternionic f-rectifying curves. Finally, we explore some of their characterizations.

2 Preliminaries

The scalar product $\sigma : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ on $\mathbb{H}$ is defined by [4]

$$(p,q) \mapsto \sigma(p,q) := -\frac{1}{2}\left(pq^* + qp^*\right) = -\,p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3$$

for all $p, q \in \mathbb{H}$. The norm of a hyperbolic quaternion q is denoted and defined by

$$\|q\| := \sqrt{|\sigma(q,q)|} = \sqrt{|-q_0^2 + q_1^2 + q_2^2 + q_3^2|}.$$

Since the scalar product σ is non-degenerate everywhere on $\mathbb{H}$, so we can consider a hyperbolic quaternion q as *spacelike* (when $\sigma(q,q) > 0$ or $q = 0$) or *lightlike* (when $\sigma(q,q) = 0$ but $q \neq 0$) or *timelike* (when $\sigma(q,q) < 0$)[12, 13, 14].

As in [6], a hyperbolic quaternionic curve $\alpha : J \to \mathbb{H}$ can be expressed as

$$\alpha(t) = \sum_{i=0}^{3} \alpha_i(t)e_i \;\; \forall\, t \in J,$$

where $J \subset \mathbb{R}$ is an interval and $\alpha_0, \alpha_1, \alpha_2, \alpha_3 : J \to \mathbb{R}$ are differentiable. Its derivative is provided by

$$\alpha'(t) = \sum_{i=0}^{3} \alpha_i'(t)e_i \;\; \forall\, t \in J.$$

A hyperbolic quaternionic curve α may be *spacelike* (when $\sigma\left(\alpha'(t), \alpha'(t)\right) > 0$ or $\alpha'(t) = 0$ for all $t \in J$) or *lightlike* (when $\sigma\left(\alpha'(t), \alpha'(t)\right) = 0$ or $\alpha'(t) \neq 0$ for all $t \in J$) or *timelike* (when $\sigma\left(\alpha'(t), \alpha'(t)\right) < 0$ for all $t \in J$) [12, 13, 14].

If α is a spacelike or timelike curve in $\mathbb{H}$ and we modify the parameter t using arc length $s : J \to I$ based at $t_0 \in J$ provided by $s(t) = \int_{t_0}^{t} \|\alpha'(v)\|dv$ then we find $\sigma(\alpha'(t), \alpha'(t)) = \pm 1$ for every possible $s \in I$ respectively, in this case α is called unit-speed in $\mathbb{H}$. Again, if α is a non-linear lightlike curve in $\mathbb{H}$ and we modify the parameter t using pseudo-arc length $s : J \to I$ based at $t_0 \in I$ provided by $s(t) = \int_{t_0}^{t} \sqrt{\|\alpha''(v)\|}dv$ then we get $\sigma(\alpha''(s), \alpha''(s)) = 1$ for every possible $s \in I$, in this case α is called unit-speed in $\mathbb{H}$.

For a lightlike curve α in $\mathbb{H}$, the tangent, normal, first binormal and second binormal vectors are defined as follows:

$$T(s) = \sum_{i=0}^{3} \alpha_i'(s)e_i,$$

$$N(s) = \sum_{i=0}^{3} \alpha_i''(s)e_i,$$

$$B_1(s) = \sum_{i=0}^{3} \left(-\frac{\varepsilon_{N'}}{2}\alpha_i'(s) - \alpha_i'''(s)\right)e_i,$$

$$\begin{aligned}B_2(s) = &(\alpha_1'\alpha_2''\alpha_3''' - \alpha_1'\alpha_2'''\alpha_3'' - \alpha_1''\alpha_2'\alpha_3''' + \alpha_1'''\alpha_2'\alpha_3'' + \alpha_1''\alpha_2'''\alpha_3' - \alpha_1'''\alpha_2''\alpha_3')(s)e_0\\ &+ (\alpha_0'\alpha_2'''\alpha_3'' - \alpha_0'\alpha_2''\alpha_3''' + \alpha_0''\alpha_2'\alpha_3''' - \alpha_0'''\alpha_2'\alpha_3'' - \alpha_0''\alpha_2'''\alpha_3' + \alpha_0'''\alpha_2''\alpha_3')(s)e_1\\ &+ (\alpha_0'\alpha_1'''\alpha_3'' - \alpha_0'\alpha_1''\alpha_3''' + \alpha_0''\alpha_1'\alpha_3''' - \alpha_0'''\alpha_1'\alpha_3'' - \alpha_0''\alpha_1'''\alpha_3' + \alpha_0'''\alpha_1''\alpha_3')(s)e_2\\ &+ (\alpha_0'\alpha_1'''\alpha_2'' - \alpha_0'\alpha_1''\alpha_2''' + \alpha_0''\alpha_1'\alpha_2''' - \alpha_0'''\alpha_1'\alpha_2'' - \alpha_0''\alpha_1'''\alpha_2' + \alpha_0'''\alpha_1''\alpha_2')(s)e_3,\end{aligned}$$

respectively, where $\varepsilon_{N'} = \sigma(N', N')$. Here $B_2 = T \wedge_{\mathbb{L}} N \wedge_{\mathbb{L}} B_1$, where $\wedge_{\mathbb{L}}$ denotes the Lorentzian cross product[12] on $\mathbb{E}_1^4$. Thus $\{T(s), N(s), B_1(s), B_2(s)\}$ provides the Frenet frame for the curve α at $\alpha(s)$ and the Frenet equations are

$$\begin{cases} T' &= \nu_1 N,\\ N' &= \nu_2 T - \nu_1 B_1,\\ B_1' &= -\nu_2 N + \nu_3 B_2,\\ B_2' &= -\nu_3 T, \end{cases} \tag{2.1}$$

where ν_1, ν_2 and ν_3 represents the first, second and third curvature respectively also

$$\begin{cases} \sigma(T,T) = 0, & \sigma(T,N) = 0,\\ \sigma(T,B_1) = 1, & \sigma(T,B_2) = 0,\\ \sigma(N,N) = 1, & \sigma(N,B_1) = 0,\\ \sigma(N,B_2) = 0, & \sigma(B_1,B_1) = 0,\\ \sigma(B_1,B_2) = 0, & \sigma(B_2,B_2) = 1. \end{cases} \tag{2.2}$$

In this situation, the only possibility for the first curvature is $\nu_1 = 1$.

3 Null hyperbolic quaternionic f-rectifying curves in $\mathbb{H}$

A unit speed lightlike curve $\alpha : I \to \mathbb{H}$ is called a rectifying curve[5] in $\mathbb{H}$ if the position vector field of α lies in its rectifying space, i.e., if the following equation is satisfied by its position vector field:

$$\alpha(s) = \zeta_1(s)T(s) + \zeta_2(s)B_1(s) + \zeta_3(s)B_2(s)$$

for every $s \in I$, where $\zeta_1, \zeta_2, \zeta_3 : I \to \mathbb{R}$ are smooth functions with parameter s, and $\{T, N, B_1, B_2, \nu_1 = 1, \nu_2, \nu_3\}$ is the Frenet apparatus for α.

The f-position vector field α_f of the curve α is now defined by

$$\alpha_f(s) := \int f(s)d\alpha = \sum_{i=0}^{3} \left(\int f(s)\alpha_i'(s)ds \right) e_i$$

for every $s \in I$, where $f : I \to \mathbb{R}$ is an everywhere non-zero integrable function. A unit speed lightlike curve $\alpha : I \to \mathbb{H}$ is said to be an f-rectifying curve in $\mathbb{H}$ if the f-position vector $\alpha_f(s)$ of α is of the form

$$\alpha_f(s) = \zeta_1(s)T(s) + \zeta_2(s)B_1(s) + \zeta_3(s)B_2(s)$$

or equivalently,

$$\begin{aligned}\alpha_f(s) = &\Big\{\zeta_3\left(\alpha_1'\alpha_2''\alpha_3''' - \alpha_1'\alpha_2'''\alpha_3'' - \alpha_1''\alpha_2'\alpha_3''' + \alpha_1'''\alpha_2'\alpha_3'' + \alpha_1''\alpha_2'''\alpha_3' - \alpha_1'''\alpha_2''\alpha_3'\right)\\ &+\left(\zeta_1 - \frac{\varepsilon_{N'}}{2}\zeta_2\right)\alpha_0' - \zeta_2\alpha_0'''\Big\}(s)\, e_0\\ +&\Big\{\zeta_3\left(\alpha_0'\alpha_2'''\alpha_3'' - \alpha_0'\alpha_2''\alpha_3''' + \alpha_0''\alpha_2'\alpha_3''' - \alpha_0'''\alpha_2'\alpha_3'' - \alpha_0''\alpha_2'''\alpha_3' + \alpha_0'''\alpha_2''\alpha_3'\right)\\ &+\left(\zeta_1 - \frac{\varepsilon_{N'}}{2}\zeta_2\right)\alpha_1' - \zeta_2\alpha_1'''\Big\}(s)\, e_1\\ +&\Big\{\zeta_3\left(\alpha_0'\alpha_1'''\alpha_3'' - \alpha_0'\alpha_1''\alpha_3''' + \alpha_0''\alpha_1'\alpha_3''' - \alpha_0'''\alpha_1'\alpha_3'' - \alpha_0''\alpha_1'''\alpha_3' + \alpha_0'''\alpha_1''\alpha_3'\right)\\ &+\left(\zeta_1 + \frac{2}{\epsilon_{N'}}\zeta_2\right)\alpha_2' - \zeta_2\alpha_2'''\Big\}(s)\, e_2\\ +&\Big\{\zeta_3\left(\alpha_0'\alpha_1'''\alpha_2'' - \alpha_0'\alpha_1''\alpha_2''' + \alpha_0''\alpha_1'\alpha_2''' - \alpha_0'''\alpha_1'\alpha_2'' - \alpha_0''\alpha_1'''\alpha_2' + \alpha_0'''\alpha_1''\alpha_2'\right)\\ &+\left(\zeta_1 - \frac{\varepsilon_{N'}}{2}\zeta_2\right)\alpha_3' - \zeta_2\alpha_3'''\Big\}(s)\, e_3 \qquad (3.1)\end{aligned}$$

for every $s \in I$, where $\zeta_1, \zeta_2, \zeta_3 : I \to \mathbb{R}$ are smooth functions.

4 Some characterizations of lightlike hyperbolic quaternionic f-rectifying curves in $\mathbb{H}$

Now, we establish some characterizations of lightlike hyperbolic quaternionic f-rectifying curves in $\mathbb{H}$. In this regard, we state and prove the following theorems:

Theorem 4.1. *Let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Then, up to isometries of $\mathbb{H}$, the relation*

$$a\nu_2'(s) + \left(a\int \nu_3(s)ds - b\right)\nu_3(s) = f(s) \qquad (4.1)$$

is valid for each $s \in I$ if and only if α is congruent to an f-rectifying curve in $\mathbb{H}$, where a and b are two constants with $b \neq 0$.

Proof. At first, we assume that $\alpha : I \to \mathbb{H}$ is a lightlike hyperbolic quaternionic curve with and $f : I \to \mathbb{R}$ is an everywhere non-zero integrable function such that the relation (4.1) is satisfied. We define $\mathcal{V}$ as a vector field along α by

$$\mathcal{V}(s) = \alpha(s) - a\nu_2(s)\, T(s) - aB_1(s) - \left(-a\int \nu_3(s)ds + b\right)B_2(s)$$

for every $s \in I$. By using (2.1) and (4.1) after differentiating the last equation, we get $\mathcal{V}'(s) = 0$ for every $s \in I$. This shows that $\mathcal{V}$ is a constant vector field along α. Therefore, up to isometries on $\mathbb{H}$, the curve α is congruent to an f-rectifying curve in $\mathbb{H}$.

Conversely, let us assume that $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic f-rectifying curve, where $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Then, for some smooth functions
$\zeta_1, \zeta_2, \zeta_3 : I \to \mathbb{R}$, its f-position vector field α_f satisfies the relation (3.1).

Applying (2.1) after differentiating (3.1), we get

$$\Big(\zeta_1(s)-\nu_3(s)\zeta_3(s)-f(s)\Big)\sum_{i=0}^{3}\alpha_i'(s)e_i+\Big(\zeta_1(s)-\nu_2(s)\zeta_2(s)\Big)\sum_{i=0}^{3}\alpha_i''(s)e_i$$
$$+\zeta_2'(s)\sum_{i=0}^{3}\left(\frac{2}{\epsilon_{N'}}\alpha_i'(s)-\alpha_i'''(s)\right)e_i+\Big(\zeta_3'(s)+\nu_3(s)\zeta_2(s)\Big)$$
$$\Big\{\left(\alpha_1'\alpha_2''\alpha_3'''-\alpha_1'\alpha_2'''\alpha_3''-\alpha_1''\alpha_2'\alpha_3'''+\alpha_1'''\alpha_2'\alpha_3''+\alpha_1''\alpha_2'''\alpha_3'-\alpha_1'''\alpha_2''\alpha_3'\right)(s)e_0$$
$$+\left(\alpha_0'\alpha_2'''\alpha_3''-\alpha_0'\alpha_2''\alpha_3'''+\alpha_0''\alpha_2'\alpha_3'''-\alpha_0'''\alpha_2'\alpha_3''-\alpha_0''\alpha_2'''\alpha_3'+\alpha_0'''\alpha_2''\alpha_3'\right)(s)e_1$$
$$+\left(\alpha_0'\alpha_1'''\alpha_3''-\alpha_0'\alpha_1''\alpha_3'''+\alpha_0''\alpha_1'\alpha_3'''-\alpha_0'''\alpha_1'\alpha_3''-\alpha_0''\alpha_1'''\alpha_3'+\alpha_0'''\alpha_1''\alpha_3'\right)(s)e_2$$
$$+\left(\alpha_0'\alpha_1'''\alpha_2''-\alpha_0'\alpha_1''\alpha_2'''+\alpha_0''\alpha_1'\alpha_2'''-\alpha_0'''\alpha_1'\alpha_2''-\alpha_0''\alpha_1'''\alpha_2'+\alpha_0'''\alpha_1''\alpha_2'\right)(s)e_3\Big\}=0.$$

By applying (2.2) on the above equation, we get

$$\begin{cases}\zeta_1'(s)-\nu_3(s)\zeta_3(s)-f(s)=0,\\ \zeta_1(s)-\nu_2(s)\zeta_2(s)=0,\\ \zeta_2'(s)=0,\\ \zeta_3'(s)+\nu_3(s)\zeta_2(s)=0\end{cases}\tag{4.2}$$

for every $s\in I$. The final three equations of (4.2) give

$$\begin{cases}\zeta_1(s)=a\nu_2(s),\\ \zeta_2(s)=a,\\ \zeta_3(s)=-a\int\nu_3(s)ds+b\end{cases}\tag{4.3}$$

for every $s\in I$, where a and b are two constants. Now our desired equation (4.1) is obtained by replacing the first and third relations of (4.3) with the first one of (4.2). □

Remark. Since $\sigma(T,B_1)=1$, by using (3.1) and (4.3), the tangential part $\sigma(\alpha_f,T)$ of α_f is given by

$$\sigma(\alpha_f,T)=\zeta_2=a,$$

where a is a constant. This shows that $\sigma(\alpha_f,T)$ is globally vanishing if $a=0$ and $\sigma(\alpha_f,T)$ is globally non-vanishing if $a\neq 0$.

As a result, we obtain several characterizations of unit speed lightlike hyperbolic quaternionic f-rectifying curves in $\mathbb{H}$ with a globally vanishing tangential component in the subsequent theorem.

Theorem 4.2. *Let $\alpha: I\to\mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f: I\to\mathbb{R}$ be an everywhere non-zero integrable function. Then, α is an f-rectifying curve in $\mathbb{H}$ with a globally vanishing tangential part $\sigma(\alpha_f,T)$ if and only if the norm function $\|\alpha_f\|$ reduces to a positive constant.*

Proof. Let us consider that $\alpha: I\to\mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic f-rectifying curve with globally vanishing tangential part $\sigma(\alpha_f,T)$, where $f: I\to\mathbb{R}$ is an everywhere non-zero integrable function. Then, for some smooth functions $\zeta_1,\zeta_2,\zeta_3: I\to\mathbb{R}$, α_f satisfies the equation (3.1). Additionally, we get sets of equations (4.2) and (4.3), when we proved the theorem 4.1. Now $\sigma(\alpha_f,T)=0$ gives us $a=0$ and so the equations (4.3) reduce to

$$\zeta_1(s)=0,\ \zeta_2(s)=0 \text{ and } \zeta_3(s)=b\tag{4.4}$$

for every $s\in I$, where $b\neq 0$ is a constant. Then equation (3.1) becomes

$$\alpha_f(s)=b\left(\alpha_1'\alpha_2''\alpha_3'''-\alpha_1'\alpha_2'''\alpha_3''-\alpha_1''\alpha_2'\alpha_3'''+\alpha_1'''\alpha_2'\alpha_3''+\alpha_1''\alpha_2'''\alpha_3'-\alpha_1'''\alpha_2''\alpha_3'\right)(s)\,e_0$$
$$+b\left(\alpha_0'\alpha_2'''\alpha_3''-\alpha_0'\alpha_2''\alpha_3'''+\alpha_0''\alpha_2'\alpha_3'''-\alpha_0'''\alpha_2'\alpha_3''-\alpha_0''\alpha_2'''\alpha_3'+\alpha_0'''\alpha_2''\alpha_3'\right)(s)\,e_1$$
$$+b\left(\alpha_0'\alpha_1'''\alpha_3''-\alpha_0'\alpha_1''\alpha_3'''+\alpha_0''\alpha_1'\alpha_3'''-\alpha_0'''\alpha_1'\alpha_3''-\alpha_0''\alpha_1'''\alpha_3'+\alpha_0'''\alpha_1''\alpha_3'\right)(s)\,e_2$$
$$+b\left(\alpha_0'\alpha_1'''\alpha_2''-\alpha_0'\alpha_1''\alpha_2'''+\alpha_0''\alpha_1'\alpha_2'''-\alpha_0'''\alpha_1'\alpha_2''-\alpha_0''\alpha_1'''\alpha_2'+\alpha_0'''\alpha_1''\alpha_2'\right)(s)\,e_3,$$

for every $s \in I$. By using the last equation, we find

$$\sigma(\alpha_f, \alpha_f) = b^2 > 0.$$

This shows that, $\|\alpha_f\|$ is a positive constant.

Conversely, let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function such that the norm function $\|\alpha_f\|$ be a positive constant. Then a constant $b > 0$ exists in a way that

$$\sigma(\alpha_f, \alpha_f) = b.$$

Differentiating and using the everywhere non-zero property of f, we get

$$\sigma(\alpha_f, T) = 0.$$

Again, differentiating the last equation and applying (2.1) and (2.2), we get

$$\sigma(\alpha_f, N) = 0.$$

This shows that α_f is an f-rectifying curve in $\mathbb{H}$.

□

In the next theorem, we establish a characterization of a lightlike hyperbolic quaternionic f-rectifying curve α with respect to first binormal part and second binormal part of α_f.

Theorem 4.3. *Let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Then, α is an f-rectifying curve in $\mathbb{H}$ with a globally vanishing tangential part $\sigma(\alpha_f, T)$ if and only if the first binormal part $\sigma(\alpha_f, B_1)$ of α_f vanishes everywhere and the second binormal part $\sigma(\alpha_f, B_2)$ of α_f is provided by*

$$\sigma\left(\alpha_f(s), B_2(s)\right) = b$$

for every $s \in I$, with $b \neq 0$ is a constant.

Proof. Let us consider that $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic f-rectifying curve with globally vanishing tangential part $\sigma(\alpha_f, T)$, where $f : I \to \mathbb{R}$ is an everywhere non-zero integrable function. Then, for some smooth functions $\zeta_1, \zeta_2, \zeta_3 : I \to \mathbb{R}$, α_f satisfies the equation (3.1).Then we get equation (4.4) when we proved theorem (4.2). By using the equations (2.2), (3.1) and (4.4), we obtain the first binormal part as

$$\sigma(\alpha_f(s), B_1(s)) = \zeta_1(s) = 0$$

for every $s \in I$, and the second binormal part as

$$\sigma(\alpha_f(s), B_2(s)) = \zeta_3(s) = b$$

for every $s \in I$, where $b \neq 0$ is a constant.

Conversely, let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Also let the first binormal part and the second binormal part of α_f are respectively given by

$$\sigma(\alpha_f(s), B_1(s)) = 0$$

and

$$\sigma(\alpha_f(s), B_2(s)) = b \tag{4.5}$$

for every $s \in I$, where $b \neq 0$ is a constant.

Differentiating (4.5) and by using (2.1) and (2.2), we get

$$\sigma(\alpha_f, T) = 0.$$

Again differentiating the last equation and applying (2.1) and (2.2) we get

$$\sigma(\alpha_f, N) = 0.$$

This shows that α_f is an f-rectifying curve in $\mathbb{H}$ with globally vanishing tangential part. □

The following theorem provides multiple characterizations of a unit speed lightlike hyperbolic quaternionic f-rectifying curves in $\mathbb{H}$ with everywhere non-vanishing tangential part.

Theorem 4.4. *Let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Then, α is an f-rectifying curve in $\mathbb{H}$ with a globally non-vanishing tangential part $\sigma(\alpha_f, T)$ if and only if the norm function $\|\alpha_f\|$ satisfies*

$$\|\alpha_f(s)\| = \sqrt{|2a\mathcal{F}(s)|} \tag{4.6}$$

for every $s \in I$, where $a \neq 0$ is a constant and $\mathcal{F}$ is a primitive of f.

Proof. Let us consider that $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic f-rectifying curve with globally vanishing tangential part $\sigma(\alpha_f, T)$, where $f : I \to \mathbb{R}$ is an everywhere non-zero integrable function. Then, for some smooth functions $\zeta_1, \zeta_2, \zeta_3 : I \to \mathbb{R}$, α_f satisfies equation (3.1). Additionally, we get sets of equations (4.2) and (4.3), when we proved the theorem 4.1. Now $\sigma(\alpha_f, T) \neq 0$ gives us $a \neq 0$. Also by applying (4.3) in the first and last equations of (4.2) we get

$$a\zeta_1'(s) + \zeta_3(s)\zeta_3'(s) = af(s)$$

for every $s \in I$. Taking integration on both sides with respect to s, we get

$$2a\zeta_1(s) + \zeta_3^2(s) = 2a\mathcal{F}(s) \tag{4.7}$$

for every $s \in I$. By using (2.2), (3.1), (4.3) and (4.7), we get

$$\|\alpha_f(s)\| = \sqrt{|2a\zeta_1(s) + \zeta_3^2(s)|} = \sqrt{|2a\mathcal{F}(s)|}$$

for every $s \in I$, where $a \neq 0$ is a constant.

Conversely, let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Also let the norm function $\|\alpha_f\|$ satisfies

$$\|\alpha_f(s)\| = \sqrt{|2a\mathcal{F}(s)|} \tag{4.8}$$

or equivalently,

$$\sigma\left(\alpha_f(s), \alpha_f(s)\right) = 2a\mathcal{F}(s)$$

for every $s \in I$, where $a \neq 0$ is a constant and $\mathcal{F}$ is the primitive function of f. By using everywhere non-zero property of f, we get after differentiation

$$\sigma\left(\alpha_f(s), T(s)\right) = a \neq 0$$

for every $s \in I$, where $a \neq 0$ is a constant. Differentiating the final equation and then using (2.1) we get

$$\sigma\left(\alpha_f(s), N(s)\right) = 0$$

for every $s \in I$. Hence α_f is an f-rectifying curve in $\mathbb{H}$ with globally non-vanishing tangential part. □

In the next theorem we establish a characterization of a lightlike hyperbolic quaternionic f-rectifying curve α with globally non-vanishing tangential part $\sigma(\alpha_f, T)$ with respect to the first and second binormal part of α_f.

Theorem 4.5. *Let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Then, α is an f-rectifying curve in $\mathbb{H}$ with a globally non-vanishing tangential part $\sigma(\alpha_f, T)$ if and only if the first binormal part $\sigma(\alpha_f, B_1)$ and the second binormal part $\sigma(\alpha_f, B_2)$ satisfies*

$$\begin{cases} \sigma\left(\alpha_f(s), B_1(s)\right) = a\nu_2(s), \\ \sigma\left(\alpha_f(s), B_2(s)\right) = -\left(a\int \nu_3(s)ds + b\right) \end{cases} \tag{4.9}$$

for every $s \in I$ respectively, where a and b are two non-zero constants.

Proof. Let us consider that $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic f-rectifying curve with globally vanishing tangential part $\sigma(\alpha_f, T)$, where $f : I \to \mathbb{R}$ is an everywhere non-zero integrable function. Then, for some smooth functions $\zeta_1, \zeta_2, \zeta_3 : I \to \mathbb{R}$, α_f satisfies equation (3.1). Additionally, we get sets of equations (4.2) and (4.3).

By using (2.2), (3.1) and (4.3), we get the first binormal part as

$$\sigma\left(\alpha_f(s), B_1(s)\right) = a\nu_2(s)$$

for every $s \in I$, where $a \neq 0$ is a constant.

Now, from the first equation of (4.2) by replacing the first two relations of (4.3), we get

$$\zeta_3(s) = \frac{a\nu_2'(s) - f(s)}{\nu_3(s)}$$

for every $s \in I$, where $a \neq 0$ is a constant. Applying (4.1) in the last equation, we get

$$\zeta_3(s) = -\left(a \int \nu_3(s)ds + b\right) \tag{4.10}$$

for every $s \in I$, where a and b are two non-zero constants.

By using (2.2), (3.1) and (4.10), we get the second binormal part is

$$\sigma\left(\alpha_f(s), B_2(s)\right) = -\left(a \int \nu_3(s)ds + b\right)$$

for every $s \in I$, where a and b are two non-zero constants.

Conversely, Let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Also let the first binormal part $\sigma(\alpha_f, B_1)$ and the second binormal part $\sigma(\alpha_f, B_2)$ satisfies

$$\begin{aligned} \sigma\left(\alpha_f(s), B_1(s)\right) &= a\nu_2(s), \\ \sigma\left(\alpha_f(s), B_2(s)\right) &= -\left(a \int \nu_3(s)ds + b\right) \end{aligned} \tag{4.11}$$

for every $s \in I$ respectively, where a and b are two non-zero constants.

Now differentiating the last equation and applying (2.1) we get

$$\sigma\left(\alpha_f(s), T(s)\right) = a$$

for every $s \in I$, where $a \neq 0$ is a constant. Differentiating the final equation and then using (2.1) we get

$$\sigma\left(\alpha_f(s), N(s)\right) = 0$$

for every $s \in I$. Hence α_f is an f-rectifying curve in $\mathbb{H}$ with non-vanishing tangential part everywhere. □

In the next theorem, we find an explicit form of α_f of an f-rectifying curve α in $\mathbb{H}$ with globally non-vanishing tangential part $\sigma(\alpha_f, T)$.

Theorem 4.6. *Let $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve. Also, let $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function. Then α is an f-rectifying curve in $\mathbb{H}$ with $\sigma\left(\alpha_f, \alpha_f\right) \neq 0$ and globally non-vanishing tangential part $\sigma(\alpha_f, T)$ if and only if α_f satisfies the equation*

$$\alpha_f(x) = \sqrt{2a\mathcal{F}(s_0)}\, e^x \alpha(x) \tag{4.12}$$

for every $x \in J$, where a is a non-zero constant, $\mathcal{F}$ is the primitive of f and $s_0 \in I$ for which $a\mathcal{F}(s_0) > 0$ and $\alpha : J \to \mathbb{H}$ is a timelike unit-speed curve with $\|\alpha\| = 1$ and $x : I \to J$ is an arc-length function.

Proof. Let us assume that $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic f-rectifying curve such that $\sigma(\alpha_f(s), \alpha_f(s)) > 0$ and $\sigma(\alpha_f(s), T(s)) \neq 0$ for every $s \in I$, where $f : I \to \mathbb{R}$ is an everywhere non-zero integrable function. Then by using theorem (4.4), we get

$$\sigma(\alpha_f(s), \alpha_f(s)) = 2a\mathcal{F}(s) > 0$$

for every $s \in I$, where a is a non-zero constant. Next, we construct a curve α as

$$\alpha(s) := \frac{1}{\sqrt{2a\mathcal{F}(s)}}\alpha_f(s)$$

for every $s \in I$. Then we get

$$\alpha_f(s) = \sqrt{2a\mathcal{F}(s)}\alpha(s) \tag{4.13}$$

for every $s \in I$. Now, from the equation (4.13) we get

$$\sigma(\alpha(s), \alpha(s)) = 1 \tag{4.14}$$

for every $s \in I$. This shows that α is a curve in $\mathbb{H}$ with $\|\alpha\| = 1$. Differentiating the last equation, we get

$$\sigma(\alpha(s), \alpha'(s)) = 0 \tag{4.15}$$

for every $s \in I$. By applying (4.14) and (4.15) after differentiating (4.13), we get

$$\sigma(\alpha'(s), \alpha'(s)) = -\frac{f^2(s)}{4\mathcal{F}^2(s)}$$

for every $s \in I$. This shows that α is a timelike curve in $\mathbb{H}$ and

$$\|\alpha'(s)\| = \frac{f(s)}{2\mathcal{F}(s)}$$

for every $s \in I$. Now the arc length function $x : I \to J$ of α starting at $s_0 \in I$ is given by

$$x = \int_{s_0}^{s} \|\alpha'(v)\| dv.$$

This gives us

$$\mathcal{F}(s) = e^{2x}\mathcal{F}(s_0). \tag{4.16}$$

Therefore, $\alpha : J \to \mathbb{H}$ is a timelike unit speed curve with arc length function x and also it is clear that $a\mathcal{F}(s_0) > 0$. By applying (4.16) in the equation (4.13), we get (4.12). Similar argument can be derived for $\sigma(\alpha_f(s), \alpha_f(s)) < 0$.

Conversely, let us assume that $\alpha : I \to \mathbb{H}$ be a unit-speed lightlike hyperbolic quaternionic curve and $f : I \to \mathbb{R}$ be an everywhere non-zero integrable function and α_f satisfies the equation (4.12) with $a\mathcal{F}(s_0) > 0$, where $s_0 \in I$ and a is a non-zero constant. We also get

$$\sigma(\alpha(x), \alpha(x)) = 1, \ \ \sigma(\alpha(x), \alpha'(x)) = 0 \ \text{ and } \ \sigma(\alpha'(x), \alpha'(x)) = -1$$

for all $x \in J$. Hence by using (4.12), we get

$$\sigma(\alpha_f(x), \alpha_f(x)) = 2a\mathcal{F}(s_0)e^{2x} \tag{4.17}$$

for all $x \in J$. This shows that $\sigma(\alpha_f(x), \alpha_f(x)) > 0$ for every $x \in J$. Now α can be reparametrized by

$$e^{2x} = \frac{\mathcal{F}(s)}{\mathcal{F}(s_0)}.$$

Then s represents pseudo-arc length function of α. Hence, (4.17) reduces to

$$\sigma(\alpha_f(s), \alpha_f(s)) = 2a\mathcal{F}(s)$$

for every $s \in I$ and therefore we get

$$\|\alpha_f(s)\| = \sqrt{|\sigma(\alpha_f(s), \alpha_f(s))|} = \sqrt{|2a\mathcal{F}(s)|}$$

for every $s \in I$, where $a \neq 0$ is a constant. So, by using theorem (4.4), we can conclude that α is an f-rectifying curve in $\mathbb{H}$ with everywhere non-vanishing tangential part.

□

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