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WEAKLY S-1-ABSORBING IDEALS IN NONCOMMUTATIVE RINGS

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Abstract This paper investigates weakly S-1-absorbing prime ideals in noncommutative rings, extending the concept of weakly 1-absorbing prime ideals and related structures. A proper ideal P of a ring $\mathcal{H}$, which does not intersect a given m-system S, is called *weakly S-1-absorbing prime* if for all non-invertible elements $k, m, l \in \mathcal{H}$ satisfying $\{0\} \neq k\mathcal{H}m\mathcal{H}l \subseteq P$, there exists an element $s \in S$ such that either $km\langle s\rangle \subseteq P$ or $l\langle s\rangle \subseteq P$. We establish several characterizations of such ideals, with particular emphasis on their behavior in local and prime rings. Key results include a structural criterion for prime rings and conditions under which weakly S-1-absorbing prime ideals satisfy relations such as $J^3 = \{0\}$ in certain local rings. This study broadens the theory of 1-absorbing prime ideals and examines the properties of weakly S-1-absorbing prime ideals across various ring constructions, including homomorphic images, idealizations, direct products, and prime, local, and local prime rings.

1 Introduction

The notions of (weakly) 1-absorbing prime ideals were first introduced in [13, 18]. These concepts, developed within the framework of commutative rings with identity, provide a meaningful generalization of the classical theory of prime ideals. Specifically, a proper ideal K of a commutative ring $\mathcal{H}$ is called a (weakly) 1-absorbing prime ideal if, for any elements $k, l, t \in \mathcal{H}$, the condition $klt \in K$ (or $0 \neq klt \in K$ in the weakly case) implies that either $kl \in K$ or $t \in K$. This generalization has attracted considerable attention and has been examined extensively in recent studies such as [8, 13, 15, 16, 17, 18]. The development of weakly 1-absorbing prime ideals not only extends the classical notion of prime ideals but also plays a significant role in analyzing the internal structure of rings, thereby contributing to broader advances in ideal theory. Moreover, generalizations of absorbing ideals investigate the structural properties of various algebraic systems [19, 20, 21].

Our study also generalizes the notion of weakly right S-prime ideals introduced in [11]. In [11], an ideal K is called a weakly right S-prime ideal if, for ideals $A, B \lhd \mathcal{H}$, the condition $0 \neq AB \subseteq K$ implies the existence of an element $s \in S$ such that either $A\langle s\rangle \subseteq K$ or $B\langle s\rangle \subseteq K$. This definition serves as a natural extension of the right S-prime ideals introduced in [1]. The concept has subsequently been further investigated and developed in several studies, including [2, 3, 9]. Moreover, the case where S is a multiplicatively closed subset of $\mathcal{H}$ has been examined in works such as [6]. Collectively, these contributions highlight the significance of such generalizations in advancing the understanding of ideal structures within noncommutative ring theory.

The concept of weakly 1-absorbing ideals in the setting of noncommutative rings was first introduced in [12]. In this context, a proper ideal K of a noncommutative ring $\mathcal{H}$ is called weakly 1-absorbing if, for all nonunit elements $k, l, t \in \mathcal{H}$, the condition $0 \neq k\mathcal{H}l\mathcal{H}t \subseteq K$ implies that either $kl \in K$ or $t \in K$. Although this notion extends the idea of weakly 1-absorbing

ideals from commutative to noncommutative rings, it has not yet been extensively explored in the latter setting. Nevertheless, it offers valuable insights into the internal structure of ideals in noncommutative rings and has motivated ongoing research aimed at further generalizations and applications.

In this study, we advance the theory by introducing the notion of weakly S-1-absorbing prime ideals in noncommutative rings, incorporating an m-system S into the framework. Specifically, let K be a proper ideal of $\mathcal{H}$ such that $K \cap S = \emptyset$. Then K is called a weakly S-1-absorbing prime ideal if, for all nonunit elements $k, l, t \in \mathcal{H}$ satisfying $0 \neq k\mathcal{H}l\mathcal{H}t \subseteq K$, there exists an element $s \in S$ such that either $kl\langle s\rangle \subseteq K$ or $t\langle s\rangle \subseteq K$. This definition not only extends the concept of weakly 1-absorbing ideals to a broader setting but also provides a unified approach to studying such ideals in noncommutative rings with identity.

The main objective of this paper is to investigate the structural properties of weakly S-1-absorbing prime ideals in noncommutative rings. We aim to establish various characterizations and equivalent conditions for these ideals, with particular emphasis on their behavior in local rings, prime rings, and other significant classes of rings. Furthermore, we examine how these ideals interact with different ring constructions, such as trivial ring extensions and amalgamations along an ideal. We also present new results concerning the transferability of weakly S-1-absorbing prime ideals across such constructions, providing deeper insight into their behavior within these extended frameworks.

For example, Corollary 2.7 establishes that in a local ring $\mathcal{H}$, if an ideal J is weakly S-1-absorbing prime but not S-1-absorbing prime, then $J^3 = \{0\}$ necessarily holds. A refined characterization in the setting of prime rings is provided in Lemma 2.20, where we show that in a local prime ring with a unique maximal ideal M, an ideal K is weakly S-1-absorbing prime if and only if it is either weakly right S-prime or satisfies the condition $M^2\langle s\rangle \subseteq K \subset M$. Furthermore, Theorem 2.21 demonstrates that if K is a weakly S-1-absorbing prime ideal of a prime ring $\mathcal{H}$ which is not weakly right S-prime, then $\mathcal{H}$ must necessarily be a local ring.

In this paper, we focus on noncommutative rings and m-systems S within a ring $\mathcal{H}$. Unless otherwise specified, all ideals considered are assumed to be two-sided. The notation $\mathcal{P}(\mathcal{H})$ denotes the prime radical of $\mathcal{H}$. We conduct a detailed investigation of weakly S-1-absorbing prime ideals, examining their fundamental properties, exploring their applications, and clarifying their connections with key concepts in ring theory. This study provides a meaningful contribution to the understanding of ideal structures in noncommutative rings and supports the broader development of ideal theory in this setting.

2 Some Properties of Weakly S-1-absorbing Prime Ideals

We now proceed to formally define the notion of a weakly S-1-absorbing prime ideal. In connection with this definition, we also introduce the concept of a weak-s-triple zero, which will serve as a key tool in analyzing the relationship between such ideals and the structure of local rings.

Definition 2.1. Let $\mathcal{H}$ be a ring and let K be a proper ideal of $\mathcal{H}$ with $K \cap S = \varnothing$. The ideal K is said to be a weakly S-1-absorbing prime ideal of $\mathcal{H}$ if, for all nonunit elements $k, l, t \in \mathcal{H}$ such that $\{0\} \neq k\mathcal{H}l\mathcal{H}t \subseteq K$, there exists an element $s \in S$ such that either $kl\langle s\rangle \subseteq K$ or $t\langle s\rangle \subseteq K$.

This definition generalizes the classical concept of weakly 1-absorbing prime ideals in commutative rings by incorporating the multiplicative subset S and extending the theory to noncommutative rings. For ease of notation, throughout this paper we use the abbreviations w-S-1AP (respectively, w-1AP) ideal to refer to a weakly S-1-absorbing prime (respectively, weakly 1-absorbing prime) ideal.

Definition 2.2. Let $\mathcal{H}$ be a ring and K be a proper ideal of $\mathcal{H}$ with $K \cap S = \varnothing$. We say that K is a *fixed weakly S-1-absorbing prime ideal* of $\mathcal{H}$ if for all nonunit elements $k, l, t \in \mathcal{H}$ such that $\{0\} \neq k\mathcal{H}l\mathcal{H}t \subseteq K$, there exists a fixed element $s \in S$ such that either $kl\langle s\rangle \subseteq K$ or $t\langle s\rangle \subseteq K$.

Remark 2.3. It is clear that every fixed weakly S-1-absorbing prime ideal is a weakly S-1-absorbing prime ideal. Definition 2.2 plays a key role in understanding the relationship between colon ideals and the m-system.

In order to examine the properties of w-S-1AP ideals within local rings, we introduce a new notion called the weak-s-triple zero. This concept is defined by adapting the idea of an s-triple zero, which was previously introduced in [11]. The weak-s-triple zero condition will help clarify how w-S-1AP ideals interact with the structure of local rings, providing deeper insight into their behavior in this setting. In the following sections, we investigate the relationships among w-S-1AP ideals, the weak-s-triple zero condition, and local rings, and we present new results and characterizations based on these definitions.

Definition 2.4. Let K be a w-S-1AP ideal of $\mathcal{H}$, and let $k, l, m \in \mathcal{H}$ be nonunits. We call (k, l, m) a *weak-s-triple zero* of K if $k\mathcal{H}l\mathcal{H}m = \{0\}$, $kl\langle s\rangle \nsubseteq K$, and $m\langle s\rangle \nsubseteq K$ for every $s \in S$.

It is evident that if K is a w-S-1AP ideal that is not an S-1AP ideal, then K possesses a weak-s-triple zero. Throughout the remainder of the paper, the abbreviation w-s-tz will be used to denote weak-s-triple zeros.

Lemma 2.5. *Let $\mathcal{H}$ be a local ring and K a w-S-1AP ideal of $\mathcal{H}$. Suppose that (k, l, t) is a w-s-tz of K, where $k, l, t \in \mathcal{H}$ are nonunits. Then the following holds.*

(i) $k\mathcal{H}lK = \{0\}$

(ii) $Kl\mathcal{H}t = \{0\}$

(iii) $kKt = \{0\}$

(iv) $K^2t = \{0\}$

(v) $kK^2 = \{0\}$

(vi) $KlK = \{0\}$

Proof. (i) For $k\mathcal{H}lK = \{0\}$: Suppose $k\mathcal{H}lK \neq \{0\}$. Then there exist $a \in \mathcal{H}$ and $b \in K$ with $kalb \neq 0$. We obtain

$$kal(b + t) = kalb + kalt = kalb \neq 0$$

Hence, $\{0\} \neq k\mathcal{H}l\mathcal{H}(b + t) \subseteq K$. Since $kl\langle s\rangle \nsubseteq K$, the element $b + t$ cannot be a unit; otherwise, we would have $kl \in K$, which contradicts the assumption that $kl\langle s\rangle \nsubseteq K$. Therefore,

$$(b + t)\langle s\rangle \subseteq K,$$

which implies $t\langle s\rangle \subseteq K$, a contradiction because (k, l, t) is a w-s-tz.

(ii) For $Kl\mathcal{H}t = \{0\}$: Assume $Kl\mathcal{H}t \neq \{0\}$, so there exist $a \in \mathcal{H}$ and $b \in K$ such that $blat \neq 0$. Then:

$$(k + b)lat = klat + blat = blat \neq 0$$

Hence, $\{0\} \neq (k + b)l\mathcal{H}t \subseteq K$. Since $\mathcal{H}$ is local, the set of nonunits forms an ideal, and therefore $k + b$ is a nonunit. As K is a w-S-1AP ideal, it follows that either

$$(k + b)l\langle s\rangle \subseteq K \quad \text{or} \quad t\langle s\rangle \subseteq K.$$

However, this contradicts the assumption that (k, l, t) is a w-s-tz.

(iii) For $kKt = \{0\}$: Assume that $kKt \neq \{0\}$; then one can find an element $a \in K$ for which $kat \neq 0$. We compute:

$$k(l + a)t = klat + kat = kat \neq 0$$

Thus, $\{0\} \neq k\mathcal{H}(l + a)\mathcal{H}t \subseteq K$. Since $\mathcal{H}$ is local, $l + a$ is a nonunit. By the w-S-1AP property of K, we obtain either

$$k(l + a)\langle s\rangle \subseteq K \quad \text{or} \quad t\langle s\rangle \subseteq K,$$

which contradicts the fact that (k, l, t) is a w-s-tz. Thus, $\{0\} \neq k\mathcal{H}(l + a)\mathcal{H}t \subseteq K$. Since $\mathcal{H}$ is local, $l + a$ is a nonunit. By the w-S-1AP property of K, we obtain either

$$k(l + a)\langle s\rangle \subseteq K \quad \text{or} \quad t\langle s\rangle \subseteq K,$$

which contradicts the fact that (k, l, t) is a w-s-tz.

(iv) For $K^2t = \{0\}$: Assume $K^2t \neq \{0\}$, so there exist $a, b \in K$ such that $abt \neq 0$. Then:

$$(k+a)(l+b)t = klt + kbt + alt + abt = abt \neq 0$$

Thus, $\{0\} \neq (k+a)\mathcal{H}(l+b)\mathcal{H}t \subseteq K$. Since $\mathcal{H}$ is local, both $(k+a)$ and $(l+b)$ are nonunits. By the w-S-1AP property of K, we conclude that either

$$(k+a)(l+b)\langle s\rangle \subseteq K \quad \text{or} \quad t\langle s\rangle \subseteq K,$$

which contradicts the assumption that (k, l, t) is a w-s-tz.

(v) For $kK^2 = \{0\}$: Suppose $kK^2 \neq \{0\}$, so there exist $a, b \in K$ such that $kab \neq 0$. We obtain

$$k(l+a)(t+b) = klt + klb + kat + kab = kab \neq 0$$

Hence, $\{0\} \neq k\mathcal{H}(l+a)\mathcal{H}(t+b) \subseteq K$. Since $(l+a)$ and $(t+b)$ are nonunits, the w-S-1AP property of K implies that either

$$k(l+a)\langle s\rangle \subseteq K \quad \text{or} \quad (t+b)\langle s\rangle \subseteq K,$$

which contradicts the fact that (k, l, t) is a w-s-tz.

(vi) For $KlK = \{0\}$: Assume $KlK \neq \{0\}$, so there exist $a, b \in K$ such that $alb \neq 0$. Then:

$$(k+a)l(t+b) = klt + klb + alt + alb = alb \neq 0$$

Thus, $\{0\} \neq (k+a)\mathcal{H}l\mathcal{H}(t+b) \subseteq K$. Since $(k+a)$ and $(t+b)$ are nonunits, the w-S-1AP property of K implies that either

$$(k+a)l\langle s\rangle \subseteq K \quad \text{or} \quad (t+b)\langle s\rangle \subseteq K,$$

which leads to a contradiction.

□

Theorem 2.6. *Let J be an ideal of a local ring $\mathcal{H}$ with $J^3 \neq \{0\}$. Then J is a w-S-1AP of $\mathcal{H}$ if and only if it is an S-1AP.*

Proof. Assume that J is a w-S-1AP ideal of $\mathcal{H}$ that is not an S-1AP ideal. Then J has a w-s-tz (k, l, t) for some nonunits $k, l, t \in \mathcal{H}$. Suppose that $abc \neq 0$ for some $a, b, c \in J$. Then, by Lemma 2.5, we obtain

$$(k+a)(l+b)(t+c) = abc \neq 0.$$

Thus,

$$\{0\} \neq (k+a)\mathcal{H}(l+b)\mathcal{H}(t+c) \subseteq J.$$

Since $\mathcal{H}$ is local, the set of nonunits of $\mathcal{H}$ forms an ideal. Because J is w-S-1AP, we get either

$$(k+a)(l+b)\langle s\rangle \subseteq J \quad \text{or} \quad (t+c)\langle s\rangle \subseteq J.$$

Hence either $kl\langle s\rangle \subseteq J$ or $t\langle s\rangle \subseteq J$, a contradiction. Therefore, J is an S-1AP ideal. The converse is clear.

□

Corollary 2.7. *If an ideal J is a w-S-1AP of a local ring $\mathcal{H}$ that is not an S-1AP, then $J^3 = \{0\}$.*

Proof. Clear from the Theorem 2.6. □

Now, we will present an important result regarding the nil radical of a ring and the relationship between the nil radical and a w-S-1AP ideal.

Corollary 2.8. *If $J \lhd \mathcal{H}$ is a w-S-1AP that is not an S-1AP, then $J \subseteq \mathcal{P}(\mathcal{H})$.*

Proof. Since $J^3 = 0 \subseteq \mathcal{P}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ is semiprime, then $J \subseteq \mathcal{P}(\mathcal{H})$.

□

In ring theory, once a property is defined for ideals, it is natural to investigate rings in which every (proper) ideal satisfies that property. In this context, the following proposition shows that in certain rings, every proper ideal is a w-S-1AP ideal.

Remark 2.9. We also note that the same conclusion holds for the special radical ϱ, known as the Kurosh–Amitsur radical, since ϱ is semiprime.

Corollary 2.10. *If $J \lhd \mathcal{H}$ is a w-S-1AP that is not an S-1AP, then $J \subseteq \varrho(\mathcal{H})$.*

Proof. This conclusion is immediate from Remark 2.9. □

Proposition 2.11. *Let $\mathcal{H}$ be a local ring with the unique maximal ideal M such that $M^3 = \{0\}$. Then each proper ideal of $\mathcal{H}$ is a w-S-1AP.*

Proof. Let J be a nonzero proper ideal of $\mathcal{H}$ that is not a w-S-1AP. Then there exist nonunits $a, b, c \in \mathcal{H}$ such that $\{0\} \neq a\mathcal{H}b\mathcal{H}c \subseteq J$ with $ab\langle s\rangle \nsubseteq J$ and $c\langle s\rangle \nsubseteq J$. Since a, b, c are nonunits, they are contained in M. Thus $a\mathcal{H}b\mathcal{H}c \subseteq M^3 = \{0\}$, a contradiction. □

Absorbing ideals possess several interesting properties related to idempotent ideals. We now examine these properties in the context of w-S-1AP ideals.

Theorem 2.12. *Let $\mathcal{H}$ be a local ring in which every ideal is a w-S-1AP. Then any idempotent ideal $J \neq 0$ is an S-1AP.*

Proof. Take $J \lhd \mathcal{H}$ such that $\{0\} \neq J = J^2$. Then $J^3 = J \neq \{0\}$. Since J is a w-S-1AP ideal and $J^3 \neq \{0\}$, Theorem 2.6 implies that J is an S-1AP ideal. □

Corollary 2.13. *[Theorem 5.18, [12]] Assume that $\mathcal{H}$ is a local ring in which each ideal is a w-1AP, then any idempotent ideal $J \neq 0$ is a 1AP.*

Proof. It follows from Theorem 2.12 by taking $S = \{1\}$. □

Remark 2.14. Recall that if the prime radical of a ring $\mathcal{H}$ is equal to the set of nilpotent elements in $\mathcal{H}$, then $\mathcal{H}$ is called 2-primal. Also if an ideal I is a 2-primal ring without identity, then it is said to be 2-primal ideal. On the other hand I is 2-primal if $\mathcal{H}/I$ is a 2-primal ring.

Proposition 2.15. *Assume that $\mathcal{H}$ is a local ring and J is a w-S-1AP that is not an S-1AP of $\mathcal{H}$. Then J is a 2-primal ideal if and only if $\mathcal{H}$ is 2-primal.*

Proof. It is obtained from Corollary 2.7 and [Proposition 2.4, [7]]. □

We provide several examples to show that a w-S-1AP ideal is a more general notion than a w-1AP ideal, an S-1AP ideal, or a weakly right S-prime (respectively, weakly right prime) ideal. In particular, Example 2.16 illustrates that

$$\text{weakly 1-absorbing prime} \;\Rightarrow\; \text{weakly } S\text{-1-absorbing prime},$$

$$\text{weakly } S\text{-1-absorbing prime} \;\not\Rightarrow\; \text{weakly 1-absorbing prime}$$

and Example 2.17 shows that

$$\text{S-1-absorbing prime} \;\Rightarrow\; \text{weakly } S\text{-1-absorbing prime},$$

$$\text{weakly } S\text{-1-absorbing prime} \;\not\Rightarrow\; \text{S-1-absorbing prime}.$$

Example 2.16. Let $\mathcal{H} = M_2(\mathbb{Z}_{30}[X])$ and $K = M_2(\langle 9X\rangle)$. Consider the set $S = \{s, s^2, s^4, s^8, \dots\}$ where $s = 3(e_{11} + e_{22})$. It is clear that S forms an m-system of $\mathcal{H}$ such that $K \cap S = \emptyset$. Now, consider the matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} M_2(\mathbb{Z}_{30}[X]) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} M_2(\mathbb{Z}_{30}[X]) \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \subseteq M_2(\langle 9X\rangle),$$

$$\text{but neither } \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \in K \text{ nor } \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \in K.$$

Thus, K is not a w-1AP ideal. Now, consider $0 \neq \mathcal{M}_1\mathcal{H}\mathcal{M}_2\mathcal{H}\mathcal{M}_3 \subseteq M_2(\langle 9X\rangle)$ for some matrices $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in \mathcal{H}$.

Then, there exist matrices $A_1, A_2, A_3 \in \mathcal{H}$ such that $\mathcal{M}_1 = xA_1$, or $\mathcal{M}_2 = xA_2$, or $\mathcal{M}_3 = xA_3$. If $\mathcal{M}_1 = xA_1$, then we have $\mathcal{M}_1\mathcal{M}_2\langle s^2\rangle \subseteq K$. If $\mathcal{M}_2 = xA_2$, then again $\mathcal{M}_1\mathcal{M}_2\langle s^2\rangle \subseteq K$. If $\mathcal{M}_3 = xA_3$, then $\mathcal{M}_3\langle s^2\rangle \subseteq K$. Hence, K is a w-S-1AP ideal, but it is not a w-1AP ideal.

Example 2.17. Let $\mathcal{H} = M_2(\mathbb{Z}_{15}[X])$ and $K = M_2(\langle 0\rangle)$. Consider the set $S = \{s, s^2, s^4, \dots\}$ where $s = 2(e_{11} + e_{22})$. It is clear that S forms an m-system of $\mathcal{H}$, and $K \cap S = \emptyset$. Therefore, K is a w-S-1AP ideal of $\mathcal{H}$. Now, consider the following matrix products:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} M_2(\mathbb{Z}_{15}[X]) \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} M_2(\mathbb{Z}_{15}[X]) \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \subseteq K,$$

$$\text{but neither } \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \langle s\rangle \subseteq K \text{ nor } \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \langle s\rangle \subseteq K.$$

Thus, K is not an S-1AP ideal, even though it is a w-S-1AP ideal.

The definition of colon ideals in noncommutative rings is slightly different from that in classical commutative rings. We now recall the following definition from Lam [14].

Definition 2.18. In noncommutative rings the colon ideals defined in two ways:
Let $\mathcal{H}$ be a ring and $I, J \lhd \mathcal{H}$.
(1) $(I : J) = \{r \in \mathcal{H} : Jr \subseteq I\}$.
(2) $(I : J)^* = \{r \in \mathcal{H} : rJ \subseteq I\}$.

Theorem 2.19. *Let $K \lhd \mathcal{H}$. The following conditions are equivalent.*

(1) K is a fixed w-S-1AP ideal of $\mathcal{H}$.
(2) There is a fixed $s \in S$ such that for all nonunits $u, v \in \mathcal{H}$ with $uv \notin (K : \langle s\rangle)^*$, $(K : u\mathcal{H}v\mathcal{H}) \subseteq (K : \langle s\rangle)^* \cup (\{0\} : u\mathcal{H}v\mathcal{H})$.
(3) There is a fixed $s \in S$ such that for all nonunits $u, v \in \mathcal{H}$ with $uv \notin (K : \langle s\rangle)^*$, $(K : u\mathcal{H}v\mathcal{H}) \subseteq (K : \langle s\rangle)^*$ or $(K : u\mathcal{H}v\mathcal{H}) \subseteq (\{0\} : u\mathcal{H}v\mathcal{H})$.
(4) For all nonunits $u, v \in \mathcal{H}$ and proper ideal I of $\mathcal{H}$ such that $\{0\} \neq u\mathcal{H}v\mathcal{H}I \subseteq K$, there is a fixed $s \in S$ such that $uv\langle s\rangle \subseteq K$ or $I\langle s\rangle \subseteq K$.
(5) For every nonunit $u \in \mathcal{H}$ and $H, L \lhd \mathcal{H}$ such that $\{0\} \neq uHL \subseteq K$, then there exists a fixed $s \in S$ such that $uH\langle s\rangle \subseteq K$ or $L\langle s\rangle \subseteq K$.
(6) For all proper ideals J, L, H of $\mathcal{H}$ such that $\{0\} \neq JLH \subseteq K$, then there is a fixed $s \in S$ such that $JL\langle s\rangle \subseteq K$ or $H\langle s\rangle \subseteq K$.

Proof. (1) $\Rightarrow$ (2) Assume that K is a fixed w-S-1AP ideal of $\mathcal{H}$. Let choose some nonunits $u, v \in \mathcal{H}$ and a fixed $s \in S$ such that $uv \notin (K : \langle s\rangle)^*$. Let $z \in (K : u\mathcal{H}v\mathcal{H})$. If $u\mathcal{H}v\mathcal{H}z = \{0\}$, then $z \in (\{0\} : u\mathcal{H}v\mathcal{H})$. Assume $0 \neq u\mathcal{H}v\mathcal{H}z \subseteq K$. Since $uv \notin (K : \langle s\rangle)^*$, then $uv\langle s\rangle \nsubseteq K$, and hence $z\langle s\rangle \subseteq K$, $z \in (K : \langle s\rangle)^*$. Hence $(K : u\mathcal{H}v\mathcal{H}) \subseteq (K : \langle s\rangle)^* \cup (\{0\} : u\mathcal{H}v\mathcal{H})$.

(2) $\Rightarrow$ (3) If a right ideal is contained by the union of two right ideals, then it should be contained by one of them. Thus the result is obtained.

(3) $\Rightarrow$ (4) Assume that $\{0\} \neq u\mathcal{H}v\mathcal{H}I \subseteq K$ for nonunits $u, v \in \mathcal{H}$ and a proper ideal I of $\mathcal{H}$. Suppose that $uv \notin (K : \langle s\rangle)^*$. Then by (3) we get $(K : u\mathcal{H}v\mathcal{H}) \subseteq (K : \langle s\rangle)^*$ or $(K : u\mathcal{H}v\mathcal{H}) \subseteq (\{0\} : u\mathcal{H}v\mathcal{H})$. Assume $(K : u\mathcal{H}v\mathcal{H}) \subseteq (\{0\} : u\mathcal{H}v\mathcal{H})$. Since $u\mathcal{H}v\mathcal{H}I \subseteq K$,

then $I \subseteq (K : u\mathcal{H}v\mathcal{H}) \subseteq (\{0\} : u\mathcal{H}v\mathcal{H})$. Hence $u\mathcal{H}v\mathcal{H}I = \{0\}$, a contradiction. Therefore $(K : u\mathcal{H}v\mathcal{H}) \subseteq (K : \langle s\rangle)^*$, so $I\langle s\rangle \subseteq K$, that completes the proof.

(4) $\Rightarrow$ (5) Assume $\{0\} \neq uHL \subseteq K$ for nonunit $u \in \mathcal{H}$ and proper ideals H, L of $\mathcal{H}$. Suppose that $uH\langle s\rangle \nsubseteq K$ and $L\langle s\rangle \nsubseteq K$. Since $uH\langle s\rangle \nsubseteq K$, there exists $a \in H$, $ua \notin (K : \langle s\rangle)^*$. Hence $u\mathcal{H}a\mathcal{H} \nsubseteq (K : \langle s\rangle)^*$. Since $\{0\} \neq uHL$, there is $b \in H$ such that $u\mathcal{H}b\mathcal{H}L \neq \{0\}$. If $u\mathcal{H}a\mathcal{H}L \neq \{0\}$, then $ua\langle s\rangle \subseteq K$ or $L\langle s\rangle \subseteq K$, a contradiction in both cases. Thus $u\mathcal{H}a\mathcal{H}L = \{0\}$. Thus $\{0\} \neq u\mathcal{H}b\mathcal{H}L = u\mathcal{H}(b+a)\mathcal{H}L \subseteq K$. Since $b+a$ is not a unit then by (4), we get $u(b+a)\langle s\rangle \subseteq K$. Also since $\{0\} \neq u\mathcal{H}b\mathcal{H}L \subseteq K$, we have $ub\langle s\rangle \subseteq K$. Hence we obtain $ua\langle s\rangle \subseteq K$, a contradiction. Thus $uH\langle s\rangle \subseteq K$ or $L\langle s\rangle \subseteq K$.

(5) $\Rightarrow$ (6) Assume that $\{0\} \neq JLH \subseteq K$ for some proper ideals J, L, H of $\mathcal{H}$. Suppose that $JL\langle s\rangle \nsubseteq K$ and $H\langle s\rangle \nsubseteq K$. Then there is $j \in J$ such that $jL\langle s\rangle \nsubseteq K$. If $jLH \neq \{0\}$, then $H\langle s\rangle \subseteq K$, a contradiction. Hence $jLH = \{0\}$. Since $\{0\} \neq JLH$, there is $a \in J$ such that $aLH \neq \{0\}$. Since $aLH \subseteq K$, then $aL\langle s\rangle \subseteq K$. Since $\{0\} \neq aLH = (a+j)LH \subseteq K$, we get $(a+j)L\langle s\rangle \subseteq K$ and so $jL\langle s\rangle \subseteq K$, that is a contradiction.

(6) $\Rightarrow$ (1) Assume that $\{0\} \neq a\mathcal{H}b\mathcal{H}c \subseteq K$ for some nonunits $a, b, c \in \mathcal{H}$. Consider $J = \mathcal{H}a\mathcal{H}, L = \mathcal{H}b\mathcal{H}, H = \mathcal{H}c\mathcal{H}$. Thus $\{0\} \neq JLH \subseteq K$. Hence from (6), we get $ab\langle s\rangle \subseteq JL\langle s\rangle \subseteq K$ or $c\langle s\rangle \subseteq H\langle s\rangle \subseteq K$. Hence K is a fixed w-S-1AP ideal.

□

Lemma 2.20. *Assume that $\mathcal{H}$ is a local prime ring with unique maximal M. Then an ideal K of $\mathcal{H}$ is a w-S-1AP ideal if and only if K is a weakly right S-prime or $M^2\langle s\rangle \subseteq K \subset M$.*

Proof. Suppose that K is a w-S-1AP. If K is not weakly right S-prime, then there exist $k\langle s\rangle, l\langle s\rangle \subseteq M - K$ for $k, l \in \mathcal{H}$ and for all $s \in S$ such that $0 \neq k\mathcal{H}l \subseteq K$. Since $\mathcal{H}$ is a local prime ring and K is a proper ideal, then we get $K \subset M$. Let $0 \neq u, v \in M$. Then $0 \neq \langle u\rangle\langle v\rangle\langle k\rangle\langle l\rangle \subseteq K$. (If u or v is zero, there is nothing to prove.) Since $l\langle s\rangle \nsubseteq K$ and K is a w-S-1AP ideal, then $\langle u\rangle\langle v\rangle\langle k\rangle\langle s\rangle \subseteq K$. Since the ring has zero product property, then we get $0 \neq \langle u\rangle\langle v\rangle\langle k\rangle\langle s\rangle \subseteq K$. As $k\langle s\rangle \nsubseteq K$ for all $s \in S$, (otherwise if $k\langle s\rangle\langle s\rangle \subseteq K$, then there exists an element $r \in \mathcal{H}$ such that $s_1 = srs \in S$, so $k\langle s_1\rangle \subseteq K$ a contradiction), and K is a w-S-1AP ideal, then $uv\langle s\rangle \subseteq K$.

Now assume that K is a right weakly S-prime ideal or $M^2\langle s\rangle \subseteq K \subset M$. If K is right weakly S-prime, then, clearly K is S-1AP. If $M^2\langle s\rangle \subseteq K \subset M$, then K is a proper ideal. Let $0 \neq k, l, t \in M$ be such that $0 \neq k\mathcal{H}l\mathcal{H}t \subseteq K$. Since $kl\langle s\rangle \subseteq M^2\langle s\rangle \subseteq K$, then we obtain that K is a w-S-1AP ideal. □

Theorem 2.21. *Assume that $\mathcal{H}$ is a prime ring. If $K \lhd \mathcal{H}$ is a w-S-1AP ideal that is not a right weakly S-prime ideal, then $\mathcal{H}$ must be a local ring.*

Proof. Let K be a w-S-1AP ideal that is not a right weakly S-prime ideal. This implies there exist nonunits $k, l \in \mathcal{H}$ such that $0 \neq k\mathcal{H}l \subseteq K$ and for all $s \in S$, we have $k\langle s\rangle, l\langle s\rangle \nsubseteq K$. Now, suppose $b \in \mathcal{H}$ is a nonzero nonunit and $v \in \mathcal{H}$ is a unit element. If we can prove that $b+v$ is a unit, then by applying Lemma 4.1 in [12], the proof will be complete. Assume, for contradiction, that $b+v$ is not a unit. Since K is a w-S-1AP ideal, and given that $l\langle s\rangle \nsubseteq K$ and $0 \neq b\mathcal{H}k\mathcal{H}l \subseteq K$, we conclude that $bk\langle s\rangle \subseteq K$. However, the assumption that $(b+v)\mathcal{H}k\mathcal{H}l \subseteq K$ implies that $(b+v)k\langle s\rangle \subseteq K$, which leads to $vk\langle s\rangle \subseteq K$. Since v is a unit, we have $k\langle s\rangle \subseteq K$, which is a contradiction. Therefore, $b+v$ must be a unit. This proves that $\mathcal{H}$ is a local prime ring. □

Proposition 2.22. *Let $\mathcal{H}$ be a prime ring and A, B be two w-S-1AP ideals of $\mathcal{H}$ that are not right weakly S-prime. Then $A \cap B$ and $A + B$ are w-S-1AP ideals.*

Proof. From Theorem 2.21 and Lemma 2.20, we obtain that $\mathcal{H}$ is a local ring with a maximal ideal M and $M^2\langle s\rangle \subseteq A \cap B$. For some nonunits $x, y, z \in \mathcal{H}$ let, $0 \neq x\mathcal{H}y\mathcal{H}z \subseteq A \cap B$. Since $x, y \in M$, then $xy\langle s\rangle \subseteq M^2\langle s\rangle \subseteq A \cap B$. Thus $A \cap B$ is a w-S-1AP. Similarly one can easily check that $A + B$ is a w-S-1AP. □

Now we present a case in which every w-S-1AP ideal of R becomes S-1AP.

Theorem 2.23. *Assume that $J \lhd \mathcal{H}$, $\mathcal{P}(\mathcal{H}) \subseteq J$ and $\mathcal{P}(\mathcal{H})$ is a right S-prime ideal of $\mathcal{H}$. Then J is a w-S-1AP ideal of $\mathcal{H}$ if and only if J is an S-1AP ideal of $\mathcal{H}$.*

Proof. Assume that J is a w-S-1AP ideal of $\mathcal{H}$ and $a\mathcal{H}b\mathcal{H}c \subseteq J$ for some nonunits $a, b, c \in \mathcal{H}$. If $\{0\} \neq a\mathcal{H}b\mathcal{H}c$, then either $ab\langle s\rangle \subseteq J$ or $c\langle s\rangle \subseteq J$. Hence assume $a\mathcal{H}b\mathcal{H}c = \{0\}$. Since $a\mathcal{H}b\mathcal{H}c = \{0\} \subseteq \mathcal{P}(\mathcal{H})$, then either $ab\langle s\rangle \subseteq \mathcal{P}(\mathcal{H}) \subseteq J$ or and $c\langle s\rangle \subseteq \mathcal{P}(\mathcal{H}) \subseteq J$. Thus, J is an S-1AP ideal of $\mathcal{H}$. The converse is clear. □

Now we will present a case in which a w-S-1AP ideal of $\mathcal{H}$ is not S-1AP.

Theorem 2.24. *Assume that K is a w-S-1AP ideal of a local ring $\mathcal{H}$ and $\{0\}$ has a w-s-tz (k, l, t) for some nonunits $k, l, t \in \mathcal{H}$ such that $kl\langle s\rangle \nsubseteq \mathcal{P}(\mathcal{H})$ and $t\langle s\rangle \nsubseteq \mathcal{P}(\mathcal{H})$. Then K is not an S-1AP of $\mathcal{H}$ if and only if $K \subseteq \mathcal{P}(\mathcal{H})$.*

Proof. Assume that K is not an S-1AP ideal of $\mathcal{H}$. Thus by Corollary 2.8 $K \subseteq \mathcal{P}(\mathcal{H})$. Conversely, assume that $K \subseteq \mathcal{P}(\mathcal{H})$ and $\{0\}$ has a w-s-tz (k, l, t) for some nonunits $k, l, t \in \mathcal{H}$ such that $kl\langle s\rangle \nsubseteq \mathcal{P}(\mathcal{H})$ and $t\langle s\rangle \nsubseteq \mathcal{P}(\mathcal{H})$. Then $kl\langle s\rangle \nsubseteq K$ and $t\langle s\rangle \nsubseteq K$. Thus (k, l, t) is a w-s-tz of K. So K is not an S-1AP of $\mathcal{H}$. □

3 S-1-absorbing ideals in related rings

3.1 Homomorphic Image

Note that if S is an m-system of $\mathcal{H}_1$, then $f(S)$ is an m-system of $\mathcal{H}_2$, where $f : \mathcal{H}_1 \to \mathcal{H}_2$ is a ring epimorphism.

Theorem 3.1. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a ring epimorphism and S be an m-system of $\mathcal{H}_1$. If K is a w-S-1AP ideal of $\mathcal{H}_1$ with $Ker(f) \subseteq K$, then $f(K)$ is a w-$f(S)$-1AP ideal of $\mathcal{H}_2$.*

Proof. Assume that $f(K) \cap f(S) \neq \phi$. Then there exists an element $a \in f(K) \cap f(S)$, which implies $a = f(b) = f(s)$ for some $b \in K$ and $s \in S$. Thus, $b - s \in Ker(f) \subseteq K$, and since $s \in S$, we have $s \in K$, which is a contradiction. Therefore, $f(K) \cap f(S) = \phi$. Let $k, l, t \in \mathcal{H}_2$ and assume $0_{\mathcal{H}_2} \neq k\mathcal{H}_2l\mathcal{H}_2t \subseteq f(K)$. Since f is an epimorphism, we can write $k = f(u)$, $l = f(v)$, $t = f(t)$ for some $u, v, t \in \mathcal{H}_1$. Then:

$$0_{\mathcal{H}_2} \neq k\mathcal{H}_2l\mathcal{H}_2t = f(u)f(\mathcal{H}_1)f(v)f(\mathcal{H}_1)f(t) = f(u\mathcal{H}_1v\mathcal{H}_1t) \subseteq f(K).$$

Since $Ker(f) \subseteq K$, it follows that $0_{\mathcal{H}_1} \neq u\mathcal{H}_1v\mathcal{H}_1t \subseteq K$. Because K is a w-S-1AP ideal, we conclude that either $uv\langle s\rangle \subseteq K$ or $t\langle s\rangle \subseteq K$. Applying the epimorphism f, we get:

$$kl\langle f(s)\rangle \subseteq f(K) \quad \text{or} \quad t\langle f(s)\rangle \subseteq f(K).$$

Hence, $f(K)$ is a w-$f(S)$-1AP ideal of $\mathcal{H}_2$. □

Theorem 3.2. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a ring epimorphism such that $f(1_{\mathcal{H}_1}) = 1_{\mathcal{H}_2}$ and $f(b)$ be a nonunit element of $\mathcal{H}_2$ for each nonunit element $b \in \mathcal{H}_1$. Let K be an ideal of $\mathcal{H}_1$. For an m-system S of $\mathcal{H}_1$, if $f(K)$ is a w-$f(S)$-1AP ideal of $\mathcal{H}_2$, then K is a w-S-1AP ideal of $\mathcal{H}_1$.*

Proof. (1) Assume $f(K)$ is a w-$f(S)$-1AP ideal and $0_{\mathcal{H}_1} \neq k\mathcal{H}_1l\mathcal{H}_1t \subseteq K$ for nonunit elements $k, l, t \in \mathcal{H}_1$. Then $0_{\mathcal{H}_2} \neq f(k\mathcal{H}_1l\mathcal{H}_1t) = f(k)\mathcal{H}_2f(l)\mathcal{H}_2f(t) \subseteq f(K)$. Since $f(K)$ is a w-$f(S)$-1AP ideal of $\mathcal{H}_2$, then $f(k)f(l)\langle f(s)\rangle \subseteq f(K)$ or $f(t)\langle f(s)\rangle \subseteq f(K)$. Thus, $kl\langle s\rangle \subseteq K$ or $t\langle s\rangle \subseteq K$. □

Recall that the set $\overline{S} = \{s + K : s \in S\}$ is an m-system of $\mathcal{H}/K$ where S is an m-system of $\mathcal{H}$. It is easy to see that $(J/K) \cap \overline{S} = \emptyset$, if and only if $J \cap S = \emptyset$. Now we will present the w-S-1AP property for quotient rings.

Proposition 3.3. *Let K be an ideal of $\mathcal{H}$, and J be an ideal of $\mathcal{H}$ that contains K. If J is a w-S-1AP ideal of $\mathcal{H}$, then JK is a w-$\overline{S}$-1AP ideal of $\mathcal{H}/K$. The converse is true if K is a w-S-1AP ideal of $\mathcal{H}$.*

Proof. Suppose J is a w-S-1AP ideal of $\mathcal{H}$, then $J \cap S = \emptyset$. Let $\overline{x}, \overline{y}, \overline{z}$ be nonunits of $\mathcal{H}/K$ with $\{\overline{0}\} \neq \overline{x}(\mathcal{H}/K)\overline{y}(\mathcal{H}/K)\overline{z} \subseteq J/K$. In this case x, y, z are nonunits of $\mathcal{H}$. Hence $\{0\} \neq x\mathcal{H}y\mathcal{H}z \subseteq J$. Thus there is $s \in S$ such that $xy\langle s\rangle \subseteq J$ or $z\langle s\rangle \subseteq J$. So $\overline{xy}\langle\overline{s}\rangle \subseteq J/K$ or $\overline{z}\langle\overline{s}\rangle \subseteq J/K$. Conversely suppose that K is a w-S-1AP ideal. Let $\{0\} \neq x\mathcal{H}y\mathcal{H}z \subseteq J$ with nonunits $x, y, z \in \mathcal{H}$. If $\{0\} \neq x\mathcal{H}y\mathcal{H}z \subseteq K$, then there exists $s \in S$ such that $xy\langle s\rangle \subseteq K \subseteq J$ or $z\langle s\rangle \subseteq K \subseteq J$. Suppose $\{0\} \neq x\mathcal{H}y\mathcal{H}z \nsubseteq K$, then $\{\overline{0}\} \neq \overline{x}\mathcal{H}\overline{y}\mathcal{H}\overline{z} \subseteq J/K$. Hence there exists $\overline{s} \in \overline{S}$ such that $\overline{xy}\langle\overline{s}\rangle \subseteq J/K$ or $\overline{z}\langle\overline{s}\rangle \subseteq J/K$. Hence $xy\langle s\rangle \subseteq J$ or $z\langle s\rangle \subseteq J$. □

3.2 Product Rings

It is well established that if the rings $\mathcal{H}_1$ and $\mathcal{H}_2$ have identities, then any ideal of $\mathcal{H}_1 \times \mathcal{H}_2$ can be expressed as $I \times J$, where I and J are ideals of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Furthermore, $\mathcal{P}(\mathcal{H}_1 \times \mathcal{H}_2) = \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$, as shown in Lemma 3 of [4]. Additionally, if S_1 is an m-system in $\mathcal{H}_1$ and S_2 is an m-system in $\mathcal{H}_2$, then $S_1 \times S_2$ forms an m-system in $\mathcal{H}_1 \times \mathcal{H}_2$. In addition, for an ideal I of $\mathcal{H}_1$, the condition $(I \times \mathcal{H}_2) \cap (S_1 \times S_2) = \emptyset$ holds if and only if $I \cap S_1 = \emptyset$.

Theorem 3.4. *Assume that $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ and J is a proper ideal of $\mathcal{H}_1$. Let $S = S_1 \times S_2$ be an m-system of $\mathcal{H}$. Then the following are equivalent:*

(1) *J is an S-1AP of $\mathcal{H}_1$.*
(2) *$J \times \mathcal{H}_2$ is an S-1AP of $\mathcal{H}$.*
(3) *$J \times \mathcal{H}_2$ is a w-S-1AP of $\mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that J is a w-S_1-1AP ideal of $\mathcal{H}_1$ and let

$$(0,0) \neq (a_1,b_1)\mathcal{H}(a_2,b_2)\mathcal{H}(a_3,b_3) = (a_1\mathcal{H}_1a_2\mathcal{H}_1a_3, b_1\mathcal{H}_2b_2\mathcal{H}_2b_3) \subseteq J \times \mathcal{H}_2,$$

for some $a_1, a_2, a_3 \in \mathcal{H}_1$ and $b_1, b_2, b_3 \in \mathcal{H}_2$. Since $a_1\mathcal{H}_1a_2\mathcal{H}_1a_3 \subseteq J$ and J is a w-S_1-1AP ideal of $\mathcal{H}_1$, there exists $s_1 \in S_1$ such that $a_1a_2\langle s_1\rangle \subseteq J$ or $a_3\langle s_1\rangle \subseteq J$. Hence

$$(a_1,b_1)(a_2,b_2)\langle(s_1,s_2)\rangle \subseteq J \times \mathcal{H}_2 \quad \text{or} \quad (a_3,b_3)\langle(s_1,s_2)\rangle \subseteq J \times \mathcal{H}_2,$$

for $(s_1, s_2) \in S_1 \times S_2$. Thus, $J \times \mathcal{H}_2$ is a w-S-1AP ideal of $\mathcal{H}_1 \times \mathcal{H}_2$.

Conversely, suppose that $J \times \mathcal{H}_2$ is a w-S-1AP ideal of $\mathcal{H}_1 \times \mathcal{H}_2$, and let $0 \neq a_1\mathcal{H}_1a_2\mathcal{H}_1a_3 \subseteq J$ for some $a_1, a_2, a_3 \in \mathcal{H}_1$. Since $(0,0) \neq (a_1,1)\mathcal{H}(a_2,1)\mathcal{H}(a_3,1) = (a_1\mathcal{H}_1a_2\mathcal{H}_1a_3, \mathcal{H}_2) \subseteq J \times \mathcal{H}_2$ and since $J \times \mathcal{H}_2$ is an S-1AP ideal of $\mathcal{H}_1 \times \mathcal{H}_2$, there exists $(s_1, s_2) \in S_1 \times S_2$ such that

$$(a_1,1)(a_2,1)\langle(s_1,s_2)\rangle \subseteq J \times \mathcal{H}_2 \quad \text{or} \quad (a_3,1)\langle(s_1,s_2)\rangle \subseteq J \times \mathcal{H}_2.$$

Hence $a_1a_2\langle s_1\rangle \subseteq J$ or $a_3\langle s_1\rangle \subseteq J$. Thus, J is a w-S_1-1AP ideal of $\mathcal{H}_1$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $J \times \mathcal{H}_2$ be a w-S-1AP of $\mathcal{H}$ and $k\mathcal{H}_1l\mathcal{H}_1t \subseteq J$ for some nonunits $k, l, t \in \mathcal{H}_1$. Assume $0 \neq a \in \mathcal{H}_2$. Thus $\{(0,0)\} \neq (k,1)\mathcal{H}(l,1)\mathcal{H}(t,a) = (k\mathcal{H}_1l\mathcal{H}_1t, 1\mathcal{H}_2 1\mathcal{H}_2 a) \subseteq J \times \mathcal{H}_2$. Therefore, either $(k,1)(l,1)\langle(s_1,s_2)\rangle \subseteq J \times \mathcal{H}_2$ or $(t,a)\langle(s_1,s_2)\rangle \subseteq J \times \mathcal{H}_2$. Hence, either $kl\langle s_1\rangle \subseteq J$ or $t\langle s_1\rangle \subseteq J$. So, J is an S-1AP ideal of $\mathcal{H}_1$.

□

3.3 Idealization

Let $\mathcal{H}$ be a ring, and let M be a unitary $\mathcal{H}$-$\mathcal{H}$-bimodule. The idealization of M in $\mathcal{H}$ is the ring

$$\mathcal{H} \boxplus M = \{(r,m) : r \in \mathcal{H}, m \in M\},$$

equipped with the usual addition and a multiplication defined by

$$(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1),$$

for all $(r_1,m_1),(r_2,m_2) \in \mathcal{H} \boxplus M$. It follows that if S is an m-system of $\mathcal{H}$, then $S_M = S \boxplus M$ forms an m-system of $\mathcal{H} \boxplus M$. Moreover, it is well known that if K is an ideal of $\mathcal{H}$ and N is an $\mathcal{H}$-$\mathcal{H}$-bisubmodule of M, then $K \boxplus N$ is an ideal of $\mathcal{H} \boxplus M$ if and only if $KM + MK \subseteq N$. Furthermore, as stated in Remark 3.1 in [12], an element (x,n) is a nonunit in $\mathcal{H} \boxplus M$ if and only if x is a nonunit in $\mathcal{H}$.

With this information, we can now examine the following theorem.

Theorem 3.5. *Let $\mathcal{H}$ be a ring with identity and M be an $\mathcal{H}$-$\mathcal{H}$-bimodule. For a proper ideal K of $\mathcal{H}$, $K \boxplus M$ is a w-S_M-1AP ideal of $\mathcal{H} \boxplus M$ which is not S_M-1AP, if and only if K is a w-S-1AP which is not S-1AP, and for any w-s-tz (k,l,t) of K, $k\mathcal{H}l\mathcal{H}M = M\mathcal{H}l\mathcal{H}t = kMt = \{0\}$.*

Proof. Assume that $K \boxplus M$ is a w-S_M-1AP ideal of $\mathcal{H} \boxplus M$. Let $\{0\} \neq k\mathcal{H}l\mathcal{H}t \subseteq K$ for some nonunits $k, l, t \in \mathcal{H}$. For the elements $(k,0), (l,0), (t,0) \in \mathcal{H} \boxplus M$, we have

$$\{(0,0)\} \neq (k,0)\mathcal{H} \boxplus M(l,0)\mathcal{H} \boxplus M(t,0) \subseteq K \boxplus M.$$

Since $K \boxplus M$ is w-S_M-1AP, there exists $(s_1, m) \in S_M$ such that either

$$(k,0)(l,0)\langle(s_1,m)\rangle = (kl,0)\langle(s_1,m)\rangle \subseteq K \boxplus M,$$

or

$$(t,0)\langle(s_1,m)\rangle \subseteq K \boxplus M.$$

This implies $kl\langle s_1\rangle \subseteq K$ or $t\langle s_1\rangle \subseteq K$. Hence, K is a w-S-1AP ideal of $\mathcal{H}$. Next, suppose (k,l,t) is a w-s-tz of K. Assume $k\mathcal{H}lM \neq \{0\}$. Then, there exist $m_1 \in M$ and $a, b \in \mathcal{H}$ such that $kabm_1 \neq 0$. This gives

$$(0,0) \neq (kabt, kabm_1) = (k,0)(a,0)(l,0)(b,0)(t,m_1) \in (k,0)\mathcal{H} \boxplus M(l,0)\mathcal{H} \boxplus M(t,m_1).$$

Since $K \boxplus M$ is w-S_M-1AP, we get either

$$(k,0)(l,0)\langle(s_1,m)\rangle \subseteq K \boxplus M \quad \text{or} \quad (t,m_1)\langle(s_1,m)\rangle \subseteq K \boxplus M.$$

Thus, $kl\langle s_1\rangle \subseteq K$ or $t\langle s_1\rangle \subseteq K$, which contradicts the assumption. Therefore, $k\mathcal{H}lM = \{0\}$. Similarly, assume $M\mathcal{H}l\mathcal{H}t \neq \{0\}$. Then, there exist $a \in M$ and $b, c \in \mathcal{H}$ such that $abct \neq 0$. This implies

$$(0,0) \neq (kbct, abct) = (k,a)(b,0)(l,0)(c,0)(t,0) \in (k,a)\mathcal{H} \boxplus M(l,0)\mathcal{H} \boxplus M(t,0).$$

Again, by the weakly S_M-1AP property of $K \boxplus M$, we deduce a contradiction. Hence, $M\mathcal{H}l\mathcal{H}t = \{0\}$. Finally, suppose $kMt \neq \{0\}$. Then, there exists $m \in M$ such that $kmt \neq 0$. This leads to

$$(0,0) \neq (klt, kmt) = (k,0)(1,0)(l,m)(1,0)(t,0) \in (k,0)\mathcal{H} \boxplus M(l,m)\mathcal{H} \boxplus M(t,0).$$

Once more, using the weakly S_M-1AP property, we reach a contradiction. Thus, $kMt = \{0\}$.

Conversely: Let K be a w-S-1AP ideal that is not an S-1AP ideal. Then, for every w-s-tz (k,l,t) of K, the conditions $k\mathcal{H}lM = M\mathcal{H}l\mathcal{H}t = kMt = \{0\}$ are satisfied. Assume

$$\{(0,0)\} \neq (k,a)\mathcal{H} \boxplus M(l,b)\mathcal{H} \boxplus M(t,c) \subseteq K \boxplus M,$$

for some nonunits $k, l, t \in \mathcal{H}$ and $a, b, c \in M$. Then $k\mathcal{H}l\mathcal{H}t \subseteq K$.

Case 1: If $\{0\} \neq k\mathcal{H}l\mathcal{H}t \subseteq K$, then since K is w-S-1AP, we have

$$kl\langle s_1\rangle \subseteq K \quad \text{or} \quad t\langle s_1\rangle \subseteq K.$$

Thus,

$$(k,a)(l,b)\langle(s_1,m)\rangle \subseteq K \boxplus M \quad \text{or} \quad (t,c)\langle(s_1,m)\rangle \subseteq K \boxplus M.$$

Case 2: If $\{0\} = k\mathcal{H}l\mathcal{H}t \subseteq K$, then (k,l,t) is a w-s-tz of K. From the assumption, $k\mathcal{H}lM = M\mathcal{H}l\mathcal{H}t = kMt = \{0\}$. Hence,

$$(k,a)\mathcal{H} \boxplus M(l,b)\mathcal{H} \boxplus M(t,c) \subseteq \{(0,0)\},$$

which is a contradiction. It is clear that since K has a w-s-tz, then $K \boxplus M$ has an w-(s,m)-tz. Thus, $K \boxplus M$ is not an S_M-1AP. □

Theorem 3.6. *Let $\mathcal{H}$ be a local, semiprime ring with identity that is not prime and M be an $\mathcal{H}$-$\mathcal{H}$-bimodule. Then the unique w-S-1AP ideal that is not an S-1AP of $\mathcal{H} \boxplus M$ is $\{0\} \boxplus M$.*

Proof. Since $\mathcal{H}$ is a local ring, it follows that $\mathcal{H} \boxplus M$ is also local. Let $J \lhd \mathcal{H}$ and $K = J \boxplus M$, where K is a w-S_M-1AP ideal but not an S_M-1AP ideal. Consequently, J is a w-S-1AP ideal but not an S-1AP ideal. By Corollary 2.8, we have $J \subseteq \mathcal{P}(\mathcal{H}) = \{0\}$. Thus, $J \boxplus M = \{0\} \boxplus M$. The zero ideal $\{0\}$ is a w-S-1AP ideal. Let (k, l, t) be a w-s-tz of $\{0\}$. Then

$$k\mathcal{H}l\mathcal{H}t = \{0\}, \quad \text{with } kl\langle s\rangle \neq \langle 0\rangle \text{ and } t\langle s\rangle \neq \langle 0\rangle.$$

We claim that

$$k\mathcal{H}l\mathcal{H}M = M\mathcal{H}l\mathcal{H}t = kMt = \{0\}.$$

Assume, for the sake of contradiction, that $k\mathcal{H}l\mathcal{H}M \neq \{0\}$. Then there exist $m \in M$ and $a, b \in \mathcal{H}$ such that $kabm \neq 0$. This implies

$$(0,0) \neq (0, kabm) = (k,0)(a,0)(l,0)(b,0)(t,m) \in (k,0)\mathcal{H} \boxplus M(l,0)\mathcal{H} \boxplus M(t,m).$$

Hence,

$$(k,0)\mathcal{H} \boxplus M(l,0)\mathcal{H} \boxplus M(t,m) \subseteq k\mathcal{H}l\mathcal{H}t \boxplus M = \{0\} \boxplus M = K.$$

Since K is a w-S_M-1AP ideal, we must have either

$$(k,0)(l,0)\langle(s_1,m)\rangle \subseteq K \quad \text{or} \quad (t,m)\langle(s_1,m)\rangle \subseteq K.$$

However, neither condition holds, leading to a contradiction. Thus, $k\mathcal{H}l\mathcal{H}M = \{0\}$.

Similarly, by Theorem 3.5, we can show that $M\mathcal{H}l\mathcal{H}t = \{0\}$ and $kMt = \{0\}$ using the same reasoning. On the other hand, by Corollary 2.7, since $K^3 = \{0\}$, it follows that K is not an S_M-1AP ideal of $\mathcal{H} \boxplus M$.

□

Corollary 3.7. *[Theorem 5.6, [12]] Let $\mathcal{H}$ be a ring with identity and M be an $\mathcal{H}$-$\mathcal{H}$-bimodule. For $J \lhd \mathcal{H}$, $J \boxplus M$ is a w-1AP ideal of $\mathcal{H} \boxplus M$ if and only if J is a w-1AP of $\mathcal{H}$ and for any triple zero (k, l, t) of J we get $k\mathcal{H}l\mathcal{H}M = M\mathcal{H}l\mathcal{H}t = kMt = \{0\}$.*

Proof. It follows from Theorem 3.5 by taking $S = \{1\}$. □

Corollary 3.8. *[Corollary 6.6, [12]] Let $\mathcal{H}$ be a local, semiprime ring with identity that is not prime, and let M be an $\mathcal{H}$-$\mathcal{H}$-bimodule. Then, the unique w-1AP ideal of $\mathcal{H} \boxplus M$ that is not a 1AP is $\{0\} \boxplus M$.*

Proof. By taking $S = \{1\}$ in Corollary 3.6, we get the result. □

Conflict of interest

The author declare no conflicts of interest.

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