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On the G-Edge Colored Diagram Algebras

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Abstract. In this article, we focus on the Jones result in Bloss [1] which tells that G-edge colored diagram algebra $\overrightarrow{P}_k(x,G)$ is a centralizer algebra of the group $G \wr S_n$ on $W^{\otimes k}$, where $W = \mathbb{C}^{n|G|}$. In specific, the action of $G \wr S_n$ is restricted to the diagonal action of $G \wr A_n$ and we describe a basis for the centralizer algebra in this diagonal action. Also, we prove that $\overrightarrow{P}_k(x,G)$ is isomorphic to its centralizer when $n \geq 2k+2$. Further, we provide the R-S correspondence of $\overrightarrow{P}_k(x,G)$ which gives a bijection among the edge colored diagrams of $\overrightarrow{P}_k(x,G)$ and the pair of multi-vacillating tableaux of the shape. In addition, we drive the Knuth relations for $\overrightarrow{P}_k(x,G)$.

1 Introduction

The algebra generated by the set partitions diagrams is called a partition algebra and is denoted by $P_k(x)$ which was introduced by P. Martin [11] and Jones [7] independently. A decade afterward, the representation theory of $P_k(x)$ and its structure have been studied by many more authors. The algebra $P_k(n)$ consider as the centralizer algebra of the action of S_n on the k-fold tensor space $V^{\otimes k}$, where $V = \mathbb{C}^n$by Jones in [7].

The partition algebra in which the edges are labeling using an arbitrary group elements is called a G-edge colored partition algebra and is denoted by $\overrightarrow{P}_k(x,G)$ which was studied by Bloss. He considered the algebra $\overrightarrow{P}_k(x,G)$ as the centralizer algebra of the group $G \wr S_n$ acting on $W^{\otimes k}$, where $W = \mathbb{C}^{n|G|}$ in [1] and studied the representation theory of this algebra in [3].

Bloss restricted the action of S_n on $V^{\otimes k}$ to the alternating group A_n in $P_k(x)$ and showed that $End_{A_n}(V^{\otimes k}) \cong P_k(n)$, for $n \geq 2k+2$ in [2]. The action of $S_n \times G$ is restricted to $A_n \times G$ by Kennedy *et al.* in [9] and proved the isomorphism $End_{A_n \times G}(W^{\otimes k}) \cong P_k(n,G)$ if $n \geq 2k+2$.

The finite tensor product algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$ has been introduced recently by Kennedy and Jaish in [8]. They considered the algebra $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ as a centralizer algebra of the group $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ on the tensor space $W^{\otimes k}$, where $W = C^{n_1 n_2 \cdots n_m}$. In [6], they restricted the diagonal action of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ to $A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$ and proved that for all $n_1, n_2, ..., n_m \geq 2k+2$, $End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k}) \cong P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$.

The R-S correspondence was introduced independently by Robinson [14] and Schensted [15] which states that there is a bijection between the symmetric group S_n and a pairs of standard tableaux. The R-S correspondence of $P_k(x)$ studied by Halverson and Lewandowski in [4], which provides a bijection between the set partitions and pairs of vacillating tableaux of same shape. The R-S correspondence of G-Brauer algebras was studied by Parvathi and Tamilselvi in [13] and they derived the Kunth relations of this algebras.

In this article, we describe a basis and dimension for the centralizer $End_{G \wr A_n}(W^{\otimes k})$ by restricting the action of $G \wr S_n$ to the action of $G \wr A_n$ in $\overrightarrow{P}_k(n,G)$. We show that $End_{G \wr A_n}(W^{\otimes k}) \cong \overrightarrow{P}_k(n,G)$ if $n \geq 2k+2$. Further, we provide the R-S correspondence of $\overrightarrow{P}_k(x,G)$, which is

a $1-1$ correspondence between the elements of $\overrightarrow{P}_k(n,G)$ and the pairs of multi-vacillating tableaux of same shape. We also prove the identity $n^k|G|^k = \sum_{[\lambda]\in\Gamma^G_{k,n}} f^{[\lambda]}m_k^{[\lambda]}$ by constructing a bijective map between the sequence of tuples $((a_1,h_1),(a_2,h_2),...,(a_k,h_k))$, $1 \le a_j \le n, h_i \in G$ and the pairs $(T_{[\lambda]}, P_{[\lambda]})$ of standard multi-tableax $T_{[\lambda]}$ and multi-vacillating tableau$P_{[\lambda]}$ of the same shape $[\lambda]$. In addition, we drive the Knuth relations for $\overrightarrow{P}_k(n,G)$.

2 Preliminaries

Definition 2.1. [16] A partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ of a positive integer n is a sequence that satisfying the following condition $\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_m > 0$ and $\sum \lambda_i = n$. It is denoted by $\lambda \vdash n$.

A young diagram of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \vdash n$ is an array of total n boxes with λ_i boxes in the i^{th} row $\forall i = 1, 2, ..., m$.

A young diagram filled with the numbers $1, 2, ..., n$ is called a young tableau of shape $\lambda \vdash n$ and the numbers are in the increasing order then the tableau is called a standard tableau of shape $\lambda \vdash n$.

Definition 2.2. [5] A sequence of partition $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(r)}) \vdash n$ is said to be a multi-partition of size n if $\lambda^{(1)} \vdash n_1, \lambda^{(2)} \vdash n_2, ..., \lambda^{(r)} \vdash n_r$ such that $n_1 + n_2 + \cdots + n_r = n$ and is denoted by $[\lambda] \vdash n$.

Definition 2.3. [5] Suppose $[\lambda] \vdash n$. A sequence of tableau $[t] = (t^{(1)}, t^{(2)}, ..., t^{(r)})$ is said to be a multi-tableau of shape $[\lambda]$ if the corresponding sequence of Young diagrams are filled with the numbers $\{1, 2, ..., n\}$.

The entries of each tableau $t^{(i)}$ is increasing strictly along rows and columns then the tableau $[t]$ is called a standard multi-tableau of shape $[\lambda]$.

Let $ST^m([\lambda]) = \{[t] \mid [t] \text{ is a standard multi-tableau of } [\lambda] \vdash n\}$.

Definition 2.4. [16] A two-line array π of positive integers is said to be a generalized permutation if the columns are in lexicographic order and the top entry taking precedence.

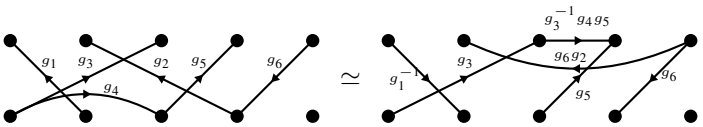
$$\pi = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

Theorem 2.5. *[10, 16] Let x, y be two generalized permutations. The permutations x and y are Knuth equivalent $(i.e.\ x \overset{K}{\sim} y)$ iff the P-tableau of x and y are same $(i.e\ P(x) = P(y))$.*

Theorem 2.6. *[10, 16] Let x, y be two generalized permutations. The permutations x and y are dual Knuth equivalent $(i.e.\ x \overset{K^*}{\sim} y)$ iff the Q-tableau of x and y are same $(i.e\ Q(x) = Q(y))$.*

2.1 The G-Edge Colored Diagram Algebras

A (G,k)-edge colored diagram is a k-partition diagram in which the edges are oriented and named by the elements of G, where G is an arbitrary group. Two (G,k)-edge colored partition diagrams are similar if the following conditions holds: (i) the fundamental partition diagrams are similar (ii) the G-diagrams are similar up to the "vector addition". Let $g_1, g_2, ..., g_6 \in G$ then:

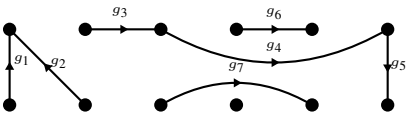


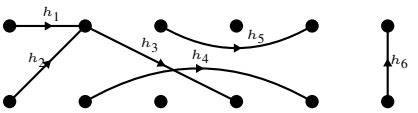
The product of two (G,k) partition diagrams d and d' defined in the following way:

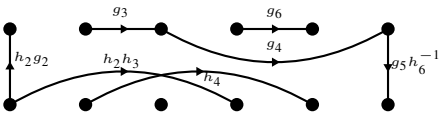
- The fundamental diagram of $d'd$ formed by multiplying the fundamental diagrams of d and d'.
- The lower edges of d and upper edges of d' are coincides with an equivalent orientation however the labeling is different, then $d'd = 0$ during the concatenation.

- Follow a path starting from d into d', implement a"vector addition" of the edge labels between d and d' along imposed connections . In this process, the d edge-labels are multiplied by d' edge-labels on the right side.
- For every middle level connected components, the factor $x|G|$ should be added in to the product.

For example, let $g_s, h_t \in G(1 \leq s \leq 7)$ and $(1 \leq t \leq 6)$,

$d =$ 

$d' =$ 

$d'd =$ $\delta^{(h_1,h_5)}_{(g_1g_2^{-1},g_7)}$ $x^2|G|^2$ 

Note that $\delta^{(h_1,h_5)}_{(g_1g_2^{-1},g_7)}$ is the Kronecker delta

$$\text{(i.e.,)} \quad \delta^{(h_1,h_5)}_{(g_1g_2^{-1},g_7)} = \begin{cases} 1 & \text{if } (g_1g_2^{-1}, g_7) = (h_1, h_5) \\ 0 & \text{if } (g_1g_2^{-1}, g_7) \neq (h_1, h_5). \end{cases}$$

Up to equivalence of such diagrams this product is well defined and associative. The algebra $\overrightarrow{P}_k(x, G)$ is defined by F-span of set of all edge colored diagrams. For a finite group G

$$dim\ \overrightarrow{P}_k(x, G) = \sum_{l=1}^{l=2k} |G|^{2k-l} S(2k, l).$$

Let $G \wr S_n$ be a group of order $|G|^n n!$ and the elements of this group are in the following form $\pi_{(g_1,g_2,...,g_n)}$, where $\pi \in S_n$ and $g_1, g_2, ..., g_n \in G$. This group is called a wreath product group of S_n with G. The product of two elements defined as follows:

$$\pi_{(g_1,g_2,...,g_n)}\pi'_{(g'_1,g'_2,...,g'_n)} = (\pi\pi')_{(g_{\pi'(1)}g'_1, g_{\pi'(2)}g'_2,..., g_{\pi'(n)}g'_n)}. \tag{2.1}$$

Let $W = \text{Span}_{\mathbb{C}}\{v_{(i,j)} \mid 1 \leq i \leq n \text{ and } j \in G\}$ and the action of $G \wr S_n$ on $W^{\otimes k}$ is defined as

$$\pi_{(g_1,g_2,...,g_n)}(v_{(i_1,j_1)} \otimes \cdots \otimes v_{(i_k,j_k)}) = v_{(\pi(i_1),g_{i_1}j_1)} \otimes \cdots \otimes v_{(\pi(i_k),g_{i_k}j_k)}. \tag{2.2}$$

Let $A \in End(W^{\otimes k})$. Define the matrix entries $A^{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k})} \in \mathbb{C}$ of A by

$$A(v_{(i_1,j_1)} \otimes v_{(i_2,j_2)} \otimes \cdots \otimes v_{(i_k,j_k)})$$

$$= \sum A^{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k})}(v_{(i_{k+1},j_{k+1})} \otimes v_{(i_{k+2},j_{k+2})} \otimes \cdots \otimes v_{(i_{2k},j_{2k})}).$$

We will write it as $A(v_I) = \sum_j A^I_J(v_J)$ where $I, J \in \mathbb{S}^k$ ($\mathbb{S} = [n] \times G$). Then the equation (2.2) may be written as $\pi_{(g_1,g_2,...,g_n)}(v_I) = v_{\pi_{(g_1,g_2,...,g_n)}(I)}$.

Lemma 2.7. *[1]. $A \in End_{G\wr S_n}(W^{\otimes k}) \iff A^I_J = A^{\pi_{(g_1,g_2,...,g_n)}(I)}_{\pi_{(g_1,g_2,...,g_n)}(J)}, \forall \pi_{(g_1,g_2,...,g_n)} \in G \wr S_n$.*

In [1], Bloss defined a homomorphism $\overrightarrow{\phi} : \overrightarrow{P}_k(n,G) \longrightarrow End(W^{\otimes k})$ on a G-edge colored partition diagram d as follows:

$$\overrightarrow{\phi}(d) = \left(\overrightarrow{\phi}(d)^{(i_1,h_1),(i_2,h_2),\ldots,(i_k,h_k)}_{(i_{k+1},h_{k+1}),(i_{k+2},h_{k+2}),\ldots,(i_{2k},h_{2k})}\right) \tag{2.3}$$

$$= \left(\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}\ \delta_{h_{l_1}g_1,h_{l_2}}\ \delta_{h_{l_3}g_3,h_{l_4}}\cdots\delta_{h_{l_m}g_m,h_{l_{m+1}}}\right), \tag{2.4}$$

where

$$\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} = \begin{cases} 1 & \text{if } i_a = i_b \text{ whenever the vertices } a \text{ and } b \text{ have connection in } d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Define an action of $\overrightarrow{P}_k(n,G)$ on the tensor space $W^{\otimes k}$ by

$$d(v_{(i_1,h_1)} \otimes \cdots \otimes v_{(i_k,h_k)}) = \overrightarrow{\phi}(d)(v_{(i_1,h_1)} \otimes \cdots \otimes v_{(i_k,h_k)}).$$

Theorem 2.8. *[1]. $\mathbb{C}[G \wr S_n]$ and $\overrightarrow{P}_k(n,G)$ generate full centralizers of each other in $End(W^{\otimes k})$. That is, for $n \geq 2k$,*

(i) $\overrightarrow{P}_k(n,G) \cong End_{G\wr S_n}(W^{\otimes k})$,

(ii) $G \wr S_n$ *generates* $End_{\overrightarrow{P}_k(n,G)}(W^{\otimes k})$.

Theorem 2.9. *[2]. $\mathbb{C}[A_n]$ and $P_k(n)$ generate full centralizers of each other in $End(V^{\otimes k})$. That is, for $n \geq 2k+2$,*

(i) $P_k(n) \cong End_{A_n}(V^{\otimes k})$,

(ii) A_n *generates* $End_{P_k(n)}(V^{\otimes k})$.

Theorem 2.10. *[3]. Let $G^{[\lambda]}$ be an irreducible $G(r,1,n)$-module and let W denote the permutation representation of $G(r,1,n)$ with respect to the subgroup $G(r,1,n-1)$. Then*

$$G^{[\lambda]} \otimes W \cong (G^{[\lambda]} \downarrow^{G(r,1,n)}_{G(r,1,n-1)}) \uparrow^{G(r,1,n)}_{G(r,1,n-1)} \cong \bigoplus_{[\mu]=([\lambda]^-)^+} G^{[\mu]},$$

where $([\lambda]^-)^+$ is an r-tuple of partitions with n boxes total that is obtained by removing a box from $[\lambda]$ and then adding a box.

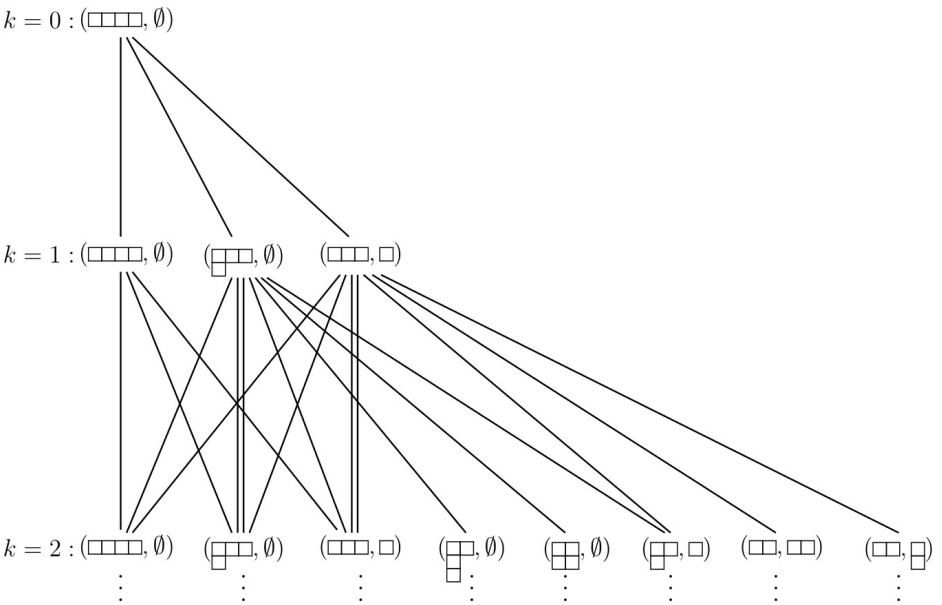


Figure 1: Bratteli diagram for $G(2,1,4)$

The set $\overrightarrow{P}_k\widehat{(n,\mathbb{Z}/r\mathbb{Z})} = \{[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(r-1)}) \mid \sum_i |\lambda^{(i)}| = n, n-k \leq |\lambda^{(0)}| \leq n, 0 \leq |\lambda^{(0)^*}| \leq k\}$ indexes all the irreducible representation of $\overrightarrow{P}_k(n, \mathbb{Z}/r\mathbb{Z})$.

The Bratteli diagram for $G(r, 1, n)$ and $\overrightarrow{P}_k(n, \mathbb{Z}/r\mathbb{Z})$ as they act on $W^{\otimes k}$, where $r = 2$ and $n = 4$ is given in Figure 1.

3 The Algebra $End_{G\wr A_n}(W^{\otimes k})$

As in §2 $G \wr S_n$ acts on the tensor space $W^{\otimes k}$ diagonally, and we restrict this diagonal action from $G \wr S_n$ to the subgroup $G \wr A_n$. Now we give a basis of the algebra $End_{G\wr A_n}(W^{\otimes k})$. Also, $End_{G\wr S_n}(W^{\otimes k}) \subseteq End_{G\wr A_n}(W^{\otimes k})$. Along with that, we prove $End_{G\wr A_n}(W^{\otimes k}) \cong \overrightarrow{P}_k(n, G)$ for $n \geq 2k+2$ and compute the dimension of $End_{G\wr A_n}(W^{\otimes k})$.

Lemma 3.1. $A \in End_{G\wr A_n}(W^{\otimes k}) \iff A^I_J = A^{\pi_{(g_1,g_2,...,g_n)}(I)}_{\pi_{(g_1,g_2,...,g_n)}(J)}, \forall \pi_{(g_1,g_2,...,g_n)} \in G \wr A_n$.

Proof. Let $A \in End_{G\wr A_n}(W^{\otimes k})$

$$\begin{aligned}
&\Leftrightarrow \pi_{(g_1,g_2,...,g_n)} A = A\pi_{(g_1,g_2,...,g_n)}, \quad \forall\ \pi_{(g_1,g_2,...,g_n)} \in G \wr A_n \\
&\Leftrightarrow \pi_{(g_1,g_2,...,g_n)} A(v_I) = A\pi_{(g_1,g_2,...,g_n)}(v_I), \forall\ v_I \\
&\Leftrightarrow \pi_{(g_1,g_2,...,g_n)} \sum_J A^I_J(v_J) = A(v_{\pi_{(g_1,g_2,...,g_n)}(I)}) \\
&\Leftrightarrow \sum_J A^I_J \pi_{(g_1,g_2,...,g_n)}(v_J) = \sum_J A^{\pi_{(g_1,g_2,...,g_n)}(I)}_J (v_J) \\
&\Leftrightarrow \sum_J A^I_J v_{\pi_{(g_1,g_2,...,g_n)(J)}} = \sum_J A^{\pi_{(g_1,g_2,...,g_n)}(I)}_{\pi_{(g_1,g_2,...,g_n)}(J)} v_{\pi_{(g_1,g_2,...,g_n)(J)}}
\end{aligned}$$

The lemma follows from linearly independence. □

Hence we can describe a basis of $End_{G\wr A_n}(W^{\otimes k})$ by describing $G \wr A_n$-orbits on

$$\Omega := (\{1, 2, ..., n\} \times G)^{2k}$$

$$= \Big\{ \begin{bmatrix} (i_1,j_1),(i_2,j_2),...,(i_k,j_k) \\ (i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k}) \end{bmatrix} \Big| \ 1 \leq i_1, i_2, ..., i_{2k} \leq n \text{ and } j_1, j_2, ..., j_{2k} \in G \Big\}.$$

The wreath product group $G \wr S_n$ act on Ω by

$$\sigma \begin{bmatrix} (i_1,j_1),(i_2,j_2),...,(i_k,j_k) \\ (i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k}) \end{bmatrix} = \begin{bmatrix} \sigma(i_1,j_1),\sigma(i_2,j_2),...,\sigma(i_k,j_k) \\ \sigma(i_{k+1},j_{k+1}),\sigma(i_{k+2},j_{k+2}),...,\sigma(i_{2k},j_{2k}) \end{bmatrix},$$

where $\sigma \in G \wr S_n$.

From, §2 we know that the dimension of $End_{G\wr S_n}(W^{\otimes k})$ is nothing but number of $G \wr S_n$-orbits on Ω. This $G \wr S_n$-orbits and the set partitions of $\{1, 2, , ..., 2k\}$ are in one-to-one correspondence. However, as we see the $G \wr A_n$-orbits with the equivalence relations on the set $\{1, 2, , ..., 2k\}$ are not necessarily in $1-1$ correspondence. Since an entire $G \wr S_n$-orbits on the set Ω is nothing but may be a disjoint union of $G \wr A_n$-orbits more than one.

The equivalence relation on the set $\{1, 2, , ..., 2k\}$ with l classes is denoted by $\sim_l$. We have $S(2k, l)$ total number of such equivalence relations and for each such relations there are $|G|^{2k-l}$ ways to pick the elements from the group G. Therefore, corresponding to the equivalence relations $\sim_l$ there are $|G|^{2k-l}S(2k, l)$ basis elements T of the algebra $End_{G\wr S_n}(W^{\otimes k})$. Let $T_{\sim_l} \in End_{G\wr S_n}(W^{\otimes k})$ denote one such basis element of type $|G|^{2k-l}S(2k, l)$, so that according to $\sim_l$ the basis element $T_{\sim_l}$ entries have indices partition. Define the following set:

$$\Omega(T_{\sim_l}) = \Big\{ \begin{bmatrix} (i_1,j_1),(i_2,j_2),...,(i_k,j_k) \\ (i_{k+1},j_{k+1}),...,(i_{2k},j_{2k}) \end{bmatrix} \Big| \ j_1, ..., j_{2k} \in G \text{ and}$$

$1 \leq i_1, ..., i_{2k} \leq n,$ where the entries $i_1, i_2, ..., i_{2k}$ are partitioned corresponding to $\sim_l$ $\Big\}$.

Let O_α denote the $G \wr A_n$-orbit of α for every α in the set $\Omega(T_{\sim_l})$. Note that the number of elements in $\Omega(T_{\sim_l})$ is $|\Omega(T_{\sim_l})| = \frac{|G|^n n!}{(n-l)!}$ and $|O_\alpha| = \frac{|G|^n n!}{2|(G \wr A_n)_\alpha|}$, where $(G \wr A_n)_\alpha \subseteq (G \wr A_n)$ is nothing but the stabilizer of α. If the disjoint union of two $G \wr A_n$-orbits is $\Omega(T_{\sim_l})$, then we could say that the element $T_{\sim_l} \in End_{G \wr S_n}(W^{\otimes k})$ splits when lifted to $End_{G \wr A_n}(W^{\otimes k})$.

Proposition 3.2. *i) If $2 < n \leq 2k$, then*

$$dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{n-2} |G|^{2k-l} S(2k, l) + 2|G|^{2k-(n-1)} S(2k, n-1) + 2|G|^{2k-n} S(2k, n).$$

ii) If $n = 2k + 1$, then

$$dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{2k-1} |G|^{2k-l} S(2k, l) + 2S(2k, 2k) = \sum_{l=1}^{2k} |G|^{2k-l} S(2k, l) + 1.$$

iii) If $n \geq 2k + 2$, then

$$dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{2k} |G|^{2k-l} S(2k, l).$$

Proof. i) We know that, $dim\ End_{G \wr S_n}(W^{\otimes k}) = \sum_{l=1}^{n} |G|^{2k-l} S(2k, l)$. Observe that

$$dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{n} c_l |G|^{2k-l} S(2k, l).$$

Here c_l is denoted by the total number of disjoint $G \wr A_n$-orbits which $\Omega(T_{\sim_l})$ comprises where $T_{\sim_l}$ is of the following type $|G|^{2k-l} S(2k, l)(c_l$ is depends only on l and independent of $\sim)$. we are going to get that $c_l = 1$ or $c_l = 2$.

Let us assume that $\alpha \in \Omega(T_{\sim_n})$ where the element $T_{\sim_n} \in End_{G \wr S_n}(W^{\otimes k})$ is of type $|G|^{2k-l} S(2k, l)$. So, the only element in $G \wr A_n$ which is the identity element fixes all of the n different entries of α. Therefore, $|O_\alpha| = |G|^n \cdot \frac{n!}{2}$ as $|\Omega(T_{\sim_n})| = |G|^n n!$, we get $c_n = 2$. Likewise, $c_{n-1} = 2$.

Now let $\alpha \in \Omega(T_{\sim_l})$ where the element $T_{\sim_n} \in End_{G \wr S_n}(W^{\otimes k})$ is of type $|G|^{2k-l} S(2k, l)$, $(1 \leq l \leq n-2)$. Observe that $(G \wr A_n)_\alpha \cong A_{n-l}$. Hence, $|O_\alpha| = (|G|^n \cdot \frac{n!}{2})/(\frac{(n-l)!}{2}) = |\Omega(T_{\sim_l})|$, so $c_l = 1, 1 \leq l \leq n-2$, and this completes the proof.

ii) We know that in this case, $dim\ End_{G \wr S_n}(W^{\otimes k}) = \sum_{l=1}^{2k} |G|^{2k-l} S(2k, l)$. We consider that $dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{2k} c_l |G|^{2k-l} S(2k, l)$, and by case (i) the only element $T_{\sim_l} \in End_{G \wr S_n}(W^{\otimes k})$ that can split into the following type $|G|^{2k-n} S(2k, n)$ or $|G|^{2k-(n-1)} S(2k, n-1)$. As the above sum will not have $c_n = c_{2k+1}$, only $T_{\sim_l} \in End_{G \wr S_n}(W^{\otimes k})$ of type $|G|^{2k-(n-1)}$ $S(2k, n-1) = S(2k, 2k)$ splits. Therefore, we have $c_{2k} = 2$ and $c_l = 1, 1 \leq l \leq n-2 = 2k-1$. Note that $S(2k, 2k) = 1$, hence $dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{2k-1} |G|^{2k-l} S(2k, l) + 2S(2k, 2k) = \sum_{l=1}^{2k} |G|^{2k-l} S(2k, l) + 1$.

iii) Likewise here also, $dim\ End_{G \wr S_n}(W^{\otimes k}) = \sum_{l=1}^{2k} |G|^{2k-l} S(2k, l)$. As $n \leq 2k + 2$, the above sum will not have c_n and c_{n-1}. So, $dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{2k} c_l |G|^{2k-l} S(2k, l)$, and therefore $c_l = 1, 1 \leq l \leq 2k(n-2)$. Hence, we get the result $dim\ End_{G \wr A_n}(W^{\otimes k}) = \sum_{l=1}^{2k} |G|^{2k-l} S(2k, l)$. □

Theorem 3.3. $\mathbb{C}[G \wr A_n]$ *and* $\overrightarrow{P}_k(n, G)$ *generate full centralizers of each other in* $End(W^{\otimes k})$. *That is, for $n \geq 2k + 2$,*

(i) $\overrightarrow{P}_k(n, G) \cong End_{G \wr A_n}(W^{\otimes k})$,

(ii) $G \wr A_n$ *generates* $End_{\overrightarrow{P}_k(n,G)}(W^{\otimes k})$.

Proof. i) We know that, $\overrightarrow{P}_k(n,G) \cong End_{G\wr S_n}(W^{\otimes k})$ for $n \geq 2k$. Then $End_{G\wr S_n}(W^{\otimes k})$ has the algebra $End_{G\wr A_n}(W^{\otimes k})$ as it's subalgebra; by (iii) of above proposition $dimEnd_{G\wr S_n}(W^{\otimes k}) = dimEnd_{G\wr A_n}(W^{\otimes k})$ when $n \geq 2k+2$. Therefore $End_{G\wr S_n}(W^{\otimes k}) = End_{G\wr A_n}(W^{\otimes k})$ when $n \geq 2k+2$. In particular, $\overrightarrow{P}_k(n,G) \cong End_{G\wr S_n}(W^{\otimes k}) = End_{G\wr A_n}(W^{\otimes k})$ when $n \geq 2k+2$.

The proof of (ii) follows from (i) and Double Centralizer Theorem. □

4 Vacillating Tableaux

Let G be a finite group and cyclic.

$$Let\ \Lambda^G_{k,n} = \{[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(r-1)}) \vdash n \mid n-k \leq |\lambda^{(0)}| \leq n, 0 \leq |\lambda^{(0)^*}| \leq k\}$$
$$and\ \ \Gamma^G_{k,n} = \{[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(r-1)}) \vdash t \mid 0 \leq t \leq k\}.$$

Let $[\lambda] \in \Lambda^G_{k,n}$. A multi-vacillating tableaux of shape $[\lambda]$ with length $2k$ is a sequence,

$$\left((n,\emptyset,...,\emptyset) = [\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)} = [\lambda]\right),$$

satisfying for every i,

(i) $[\lambda]^{(i)} \in \Lambda^G_{i,n}$ and $[\lambda]^{(i+\frac{1}{2})} \in \Lambda^G_{i,n-1}$,

(ii) $[\lambda]^{(i)} \supseteq [\lambda]^{(i+\frac{1}{2})}$ and $|[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}| = 1$,

(iii) $[\lambda]^{(i+\frac{1}{2})} \subseteq [\lambda]^{(i+1)}$ and $|[\lambda]^{(i+1)}/[\lambda]^{(i+\frac{1}{2})}| = 1$.

Let the irreducible $\mathbb{C}\overrightarrow{P}_k(n,G)$-representation indexed by $[\lambda] \in \Lambda^G_{k,n}$ is denoted by $T^{[\lambda]}_k$ and let $VT^m_k([\lambda])$ denote the set of multi-vacillating tableaux of shape $[\lambda]$ with length k then

$$m^{[\lambda]}_k = dim(T^{[\lambda]}_k) = |VT^m_k([\lambda])|.$$

For $n \geq 2k$, the sets $\Lambda^G_{k,n}$ and $\Gamma^G_{k,n}$ are bijection with each other using the following maps,

$$\Lambda^G_{k,n} \to \Gamma^G_{k,n} \qquad \Gamma^G_{k,n} \to \Lambda^G_{k,n}.$$

The example of a multi-vacillating tableaux is

$$P_{[\lambda]} = ((\square\square\square\square, \emptyset, \emptyset, \emptyset), (\square\square\square, \emptyset, \emptyset, \emptyset), (\square\square\square, \emptyset, \square, \emptyset), (\square\square, \emptyset, \square, \emptyset), (\square\square, \emptyset, \square\square, \emptyset)).$$

5 The Proof of identity $n^k|G|^k = \sum_{[\lambda]\in\Lambda^G_{k,n}} f^{[\lambda]} m^{[\lambda]}_k$

To give a combinatorial proof of

$$n^k|G|^k = \sum_{[\lambda]\in\Lambda^G_{k,n}} f^{[\lambda]} m^{[\lambda]}_k, \quad \text{for } n \geq 2k. \tag{5.1}$$

Find a bijection map of the following form

$$\{((a_1,h_1),(a_2,h_2),...,(a_k,h_k)) \mid 1 \leq a_j \leq n, h_i \in G\} \longleftrightarrow \bigsqcup_{[\lambda]\in\Lambda^G_{k,n}} ST^m([\lambda]) \times VT^m_k([\lambda]).$$

Let the jeu de taquin of removal of x from S^l when x appears in the l^{th} component is denoted by $x \xleftarrow{\text{jdt}} S$ and is defined by

(i) Let $c = S^l_{i,j}$ be the box containing x.

(ii) If the box c is not a corner box, then

a. Take $\min\{S^l_{i+1,j}, S^l_{i,j+1}\}$ is in the box c';

b. Interchange the boxes c and c'.

(iii) Remove the box c.

Example 5.1. Let $S = \left(\begin{array}{|c|c|}\hline 2&3\\\hline 6&7\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right)$ then $2 \xleftarrow{\text{jdt}} S$ as follows:

$$S = \left(\begin{array}{|c|c|}\hline 2&3\\\hline 6&7\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \Rightarrow \left(\begin{array}{|c|c|}\hline 3&2\\\hline 6&7\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \Rightarrow \left(\begin{array}{|c|c|}\hline 3&7\\\hline 6&2\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \Rightarrow \left(\begin{array}{|c|c|}\hline 3&7\\\hline 6\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right).$$

The insertion of a positive integer x that is not in S into the l^{th} component of S using R-S algorithm is denoted by $x \xrightarrow{\text{RS}} S$.

(i) Let R be the first row of l^{th} component in S

(ii) If x is smaller than some element of R, then

a. Choose a smallest element y which is larger than x in R and replace the element y by x;

b. Now, take $x := y$ and go to the next row.

(iii) Put the value x at the end of the row R.

Example 5.2. Consider $S = \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 4&5&6\\\hline 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 10\\\hline 11\\\hline\end{array}\right)$ then $2 \xrightarrow{\text{RS}} S^2$ as follows:

$$S = \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 4&5&6\\\hline 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 10\\\hline 11\\\hline\end{array}\right) \Rightarrow \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 10\\\hline 11\\\hline\end{array}\right) \Rightarrow \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 4&8\\\cline{1-2} 9\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 10\\\hline 11\\\hline\end{array}\right) \Rightarrow \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 4&8\\\cline{1-2} 7\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 10\\\hline 11\\\hline\end{array}\right)$$

$$\Rightarrow \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&5&6\\\hline 4&8\\\cline{1-2} 7\\\cline{1-1} 9\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 10\\\hline 11\\\hline\end{array}\right).$$

Similarly, the reverse process also possible using the R-S reverse insertion, $(see, [16])$.

Theorem 5.3. *The function* $((a_1,h_1),(a_2,h_2),...,(a_k,h_k)) \xrightarrow{Fd} (T_{[\lambda]}, P_{[\lambda]})$ *gives a bijection between the sequence in the set* $\{((a_1,h_1),(a_2,h_2),...,(a_k,h_k)) \mid 1 \le a_j \le n, h_i \in G\}$ *and* $\bigsqcup_{[\lambda]\in\Lambda^G_{k,n}} ST^m([\lambda]) \times VT^m_k([\lambda])$, *and thus provides a proof of (5.1).*

Proof. Given $(a_1,h_1),(a_2,h_2),...,(a_k,h_k)$ with $1 \le a_j \le n, h_i \in G$, we can produce a pair $(T_{[\lambda]}, P_{[\lambda]}), [\lambda] \in \Lambda^G_{k,n}$, where $T_{[\lambda]}$ is a standard tableau and $P_{[\lambda]}$ is a vacillating tableau. Initially we start with

$$T^{(0)} = \left(\begin{array}{|c|c|c|c|}\hline 1&2&\cdots&n\\\hline\end{array}^{(0)}, \emptyset^{(1)}, \emptyset^{(2)}, \ldots, \emptyset^{(r-1)}\right)$$

Define the following standard multi-tableau $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ for $0 \le j \le k-1$ recursively by

$$T^{(j+\frac{1}{2})} = \left(a_{j+1} \xleftarrow{jdt} T^{(j)}\right)$$

$$T^{(j+1)} = \left(a_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}\right),$$

where $a_{j+1} \xleftarrow{jdt} T^{(j)}$ is the removal of a_{j+1} using jdt from the component where a_{j+1} lies and $a_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}$ is insertion of a number a_{j+1} into the $(i-1)^{th}$ component of a tableau $T^{(j+\frac{1}{2})}$ using RS if $h_{j+1} = g_i$.

Let $[\lambda]^{(j)} \in \Lambda^G_{j,n}$ and $[\lambda]^{(j+\frac{1}{2})} \in \Lambda^G_{j,n-1}$ be the shapes of tableaux $T^{(j)}$ and $T^{(j+\frac{1}{2})}$ respectively. Then

$$T_{[\lambda]} = T^{(k)} \qquad \text{and} \qquad P_{[\lambda]} = ([\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(k)}),$$

so that the shapes of a multi-vacillating tableau $P_{[\lambda]}$ and a standard multi-tableau $T_{[\lambda]}$ will be same, $[\lambda] = [\lambda]^{(k)} \in \Lambda^G_{k,n}$. Hence $((a_1,h_1),(a_2,h_2),...,(a_k,h_k)) \xrightarrow{Fd} (T_{[\lambda]}, P_{[\lambda]})$.

Example 5.4. For $((3,g_2),(2,g_3),(5,g_2),(1,g_3),(3,g_4))$ the pair $(T_{[\lambda]},P_{[\lambda]})$ is as follows:

j	(a_j,h_j)		$T^{(j)}$
0			$\left(\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}^{(0)},\emptyset^{(1)},\emptyset^{(2)},\emptyset^{(3)}\right)$
$\frac{1}{2}$	$(3,g_2)$	$\xleftarrow{jdt}$	$\left(\boxed{1}\boxed{2}\boxed{4}\boxed{5}^{(0)},\emptyset^{(1)},\emptyset^{(2)},\emptyset^{(3)}\right)$
1	$(3,g_2)$	$\xrightarrow{RS}$	$\left(\boxed{1}\boxed{2}\boxed{4}\boxed{5}^{(0)},\boxed{3}^{(1)},\emptyset^{(2)},\emptyset^{(3)}\right)$
$1\frac{1}{2}$	$(2,g_3)$	$\xleftarrow{jdt}$	$\left(\boxed{1}\boxed{4}\boxed{5}^{(0)},\boxed{3}^{(1)},\emptyset^{(2)},\emptyset^{(3)}\right)$
2	$(2,g_3)$	$\xrightarrow{RS}$	$\left(\boxed{1}\boxed{4}\boxed{5}^{(0)},\boxed{3}^{(1)},\boxed{2}^{(2)},\emptyset^{(3)}\right)$
$2\frac{1}{2}$	$(5,g_2)$	$\xleftarrow{jdt}$	$\left(\boxed{1}\boxed{4}^{(0)},\boxed{3}^{(1)},\boxed{2}^{(2)},\emptyset^{(3)}\right)$
3	$(5,g_2)$	$\xrightarrow{RS}$	$\left(\boxed{1}\boxed{4}^{(0)},\boxed{3}\boxed{5}^{(1)},\boxed{2}^{(2)},\emptyset^{(3)}\right)$
$3\frac{1}{2}$	$(1,g_3)$	$\xleftarrow{jdt}$	$\left(\boxed{4}^{(0)},\boxed{3}\boxed{5}^{(1)},\boxed{2}^{(2)},\emptyset^{(3)}\right)$
4	$(1,g_3)$	$\xrightarrow{RS}$	$\left(\boxed{4}^{(0)},\boxed{3}\boxed{5}^{(1)},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}^{(2)},\emptyset^{(3)}\right)$
$4\frac{1}{2}$	$(3,g_4)$	$\xleftarrow{jdt}$	$\left(\boxed{4}^{(0)},\boxed{5}^{(1)},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}^{(2)},\emptyset^{(3)}\right)$
5	$(3,g_4)$	$\xrightarrow{RS}$	$\left(\boxed{4}^{(0)},\boxed{5}^{(1)},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}^{(2)},\boxed{3}^{(3)}\right)$

$T_{[\lambda]}=\left(\boxed{4},\boxed{5},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\boxed{3}\right)$

The associated multi-vacillating tableau is

$$P_{[\lambda]}=\Big((\square\square\square\square\square,\emptyset,\emptyset,\emptyset),(\square\square\square\square,\emptyset,\emptyset,\emptyset),(\square\square\square\square,\square,\emptyset,\emptyset),(\square\square\square,\square,\emptyset,\emptyset),(\square\square\square,\square,\square,\emptyset),$$
$$(\square\square,\square,\square,\emptyset),(\square\square,\square\square,\square,\emptyset),(\square,\square\square,\square,\emptyset),(\square,\square\square,\begin{array}{|c|}\hline\ \\\hline\end{array},\emptyset),(\square,\square,\begin{array}{|c|}\hline\ \\\hline\end{array},\emptyset),(\square,\square,\begin{array}{|c|}\hline\ \\\hline\end{array},\square)\Big).$$

Let us consider $[\lambda]^{(j+\frac{1}{2})}\subseteq[\lambda]^{(j+1)}$ with $[\lambda]^{(j+1)}\in\Lambda^G_{j+1,n}$, $[\lambda]^{(j+\frac{1}{2})}\in\Lambda^G_{j,n-1}$ and, also consider a standard multi-tableau $T^{(j+1)}$ has shape $[\lambda]^{(j+1)}$. We may find a_{j+1} uniquely together with a color h_{j+1} associated by a_{j+1} and a tableau $T^{(j+\frac{1}{2})}$ of shape $[\lambda]^{(j+\frac{1}{2})}$ such that $T^{(j+1)}=\left(a_{j+1}\xrightarrow{RS}T^{(j+\frac{1}{2})}\right)$. To do this, consider the box b in $[\lambda]^{(j+1)}/[\lambda]^{(j+\frac{1}{2})}$ and use RS

reverse insertion to delete the value in box b then we get a_{j+1} and $T^{(j+\frac{1}{2})}$. A color h_{j+1} is associated to a_{j+1} if the entry a_{j+1} is removed from $(i-1)^{th}$ component in $T^{(j+1)}$ then $h_{j+1} = g_i$.

Now, assume that the tableau $T^{(j+\frac{1}{2})}$ has shape $[\lambda]^{(j+\frac{1}{2})} \in \Lambda^G_{j,n-1}$ with entries from $\{1, 2, ..., n\} \setminus \{a_{j+1}\}$ and the entries of rows and columns are in increasing order and, let $[\lambda]^{(j)} \subseteq [\lambda]^{(j+\frac{1}{2})}$ with $[\lambda]^{(j)} \in \Lambda^G_{j,n}$. We may find a standard tableau $T^{(j)}$ uniquely satisfying $T^{(j+\frac{1}{2})} = \left(a_{j+1} \xleftarrow{jdt} T^{(j)}\right)$. To do this, consider the box b in $[\lambda]^{(j)}/[\lambda]^{(j+\frac{1}{2})}$ place the entry a_{j+1} in the box b of $T^{(j+\frac{1}{2})}$ and apply the reverse jdt to get $T^{(j)}$.

Given a shape $[\lambda] \in \Lambda^G_{k,n}$ and a pair $(T_{[\lambda]}, P_{[\lambda]}) \in ST^m([\lambda]) \times VT^m_k([\lambda])$ apply the above procedure in $[\lambda]^{(k-\frac{1}{2})} \subseteq [\lambda]^{(k)}$, $T^{(k)} = T_{[\lambda]}$ to produce a pair (a_k, h_k) and $T^{(k-1)}$. Continue like wise, we can get the sequence $((a_1, h_1), (a_2, h_2), ..., (a_k, h_k))$ and $T^{(k)}$, $T^{(k-1)}$, ..., $T^{(1)}$ satisfies $((a_1, h_1), (a_2, h_2), ..., (a_k, h_k)) \xrightarrow{Fd} (T_{[\lambda]}, P_{[\lambda]})$. □

6 R-S Correspondence of G-edge colored Partition Algebras

To give a combinatorial proof of

$$\sum_{l=1}^{l=2k} |G|^{2k-l} S(2k, l) = \sum_{[\lambda] \in \Gamma^G_{k,n}} (m_k^{[\lambda]})^2 \tag{6.1}$$

where G is a cyclic group of finite order, find a bijection map of the following form

$$\overrightarrow{P}_k(n, G) \longleftrightarrow \bigsqcup_{[\lambda] \in \Gamma^G_{k,n}} VT^m_k([\lambda]) \times VT^m_k([\lambda])$$

by making a function which takes a (G, k)-diagram d from $\overrightarrow{P}_k(n, G)$ and gives a pair $(P_{[\lambda]}, Q_{[\lambda]})$ of multi vacillating tableaux.

Using standard representation every diagram d of $\overrightarrow{P}_k(n, G)$ is represented as a single row with vertices of order $1, 2, ..., 2k$. Relabel the vertex j' by $2k - j + 1$.
The edges will be drawn in the following way to the one line representation of d:

Draw an edge from the vertex i to j with $i < j$ if the vertices i and j are connected by an edge in d with there is no vertex t is connected to i or j such that $i \leq t \leq j$. In a similar way connect the remaining vertices to its neighbors in the same block. Edges in a diagram $d \in \overrightarrow{P}_k(n, G)$ are labeled using groups elements of G and consider the same labeling for the edges in a single row representation.

Example 6.1. Consider the Z_4-edge colored diagram where $Z_4 = \{e = g_1, g_2, g_3, g_4\}$ as in Figure 2:

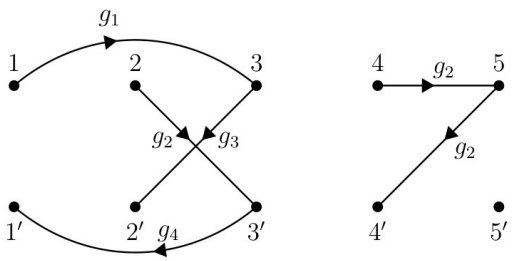


Figure 2:

A standard one line representation of the above diagram is as follows:

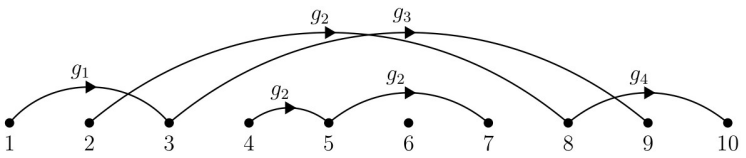


Figure 3:

We label every edge e of d with $(2k+1-v, g)$, where e is the incoming edge of the vertex v with color g as follows:

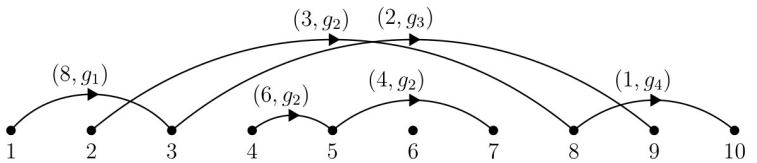


Figure 4:

The insertion sequence of $d \in \overrightarrow{P}_k(n, G)$ to be the sequence $E^{(d,k)} = \{E_j\}$ indexed by $j = \frac{1}{2}, 1, 1\frac{1}{2}, ..., 2k-1, 2k-\frac{1}{2}, 2k$ is defined as follows.

$$E_j = \begin{cases} (a, g_i), & \text{if vertex } j \text{ is the left end point of edge } (a, g_i) \\ (\emptyset, e), & \text{if vertex } j \text{ is not a left end point of edge } (a, g_i). \end{cases}$$

$$E_{j-\frac{1}{2}} = \begin{cases} (a, g_i), & \text{if vertex } j \text{ is the right end point of edge } (a, g_i) \\ (\emptyset, e), & \text{if vertex } j \text{ is not a right end point of edge } (a, g_i). \end{cases}$$

The above diagram has the following insertion sequence:

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$
E_j	$(\emptyset, e)$	$(8, g_1)$	$(\emptyset, e)$	$(3, g_2)$	$(8, g_1)$	$(2, g_3)$	$(\emptyset, e)$	$(6, g_2)$	$(6, g_2)$	$(4, g_2)$	$(\emptyset, e)$

j	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8	$8\frac{1}{2}$	9	$9\frac{1}{2}$	10
E_j	$(\emptyset, e)$	$(4, g_2)$	$(\emptyset, e)$	$(3, g_2)$	$(1, g_4)$	$(2, g_3)$	$(\emptyset, e)$	$(1, g_4)$	$(\emptyset, e)$

A pair $(P_{[\lambda]}, Q_{[\lambda]})$ will be produced for the insertion sequence $\{E_j\}$ of $d \in \overrightarrow{P}_k(n, G)$. Initially we start with the empty tableaux

$$T^{(0)} = (\emptyset, \emptyset, ..., \emptyset).$$

Then the standard tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ for $0 \leq j \leq 2k-1$ defined recursively by:

$$T^{(0)} = (\emptyset^{(1)}, \emptyset^{(2)}, ..., \emptyset^{(r)})$$

$$T^{(j+\frac{1}{2})} = \begin{cases} (a, g_i) \xleftarrow{jdt} T^{(j)} & \text{if } E_{j+\frac{1}{2}} = (a, g_i), \\ T^{(j)} & \text{if } E_{j+\frac{1}{2}} = (\emptyset, g), \text{ for any } g \in G, \end{cases}$$

where $(a, g_i) \xleftarrow{jdt} T^{(j)}$ means that delete a from the i^{th} component of $T^{(j)}$.

$$T^{(j+1)} = \begin{cases} (a, g_i) \xrightarrow{RS} T^{(j+\frac{1}{2})} & \text{if } E_{j+1} = (a, g_i)\ , \\ T^{(j+\frac{1}{2})} & \text{if } E_{j+1} = (\emptyset, g), \text{ for any } g \in G, \end{cases}$$

where $(a, g_i) \xrightarrow{RS} T^{(j+\frac{1}{2})}$ is insertion of a into i^{th} component of $T^{(j+\frac{1}{2})}$ and the remaining components of $T^{(j+\frac{1}{2})}$ are unchanged.

Let $[\lambda]^{(i)}$ and $[\lambda]^{(i+\frac{1}{2})}$ be the shapes of the tableaux $T^{(i)}$ and $T^{(i+\frac{1}{2})}$ respectively, and let $[\lambda] = [\lambda]^{(k)}$. Define

$$Q_{[\lambda]} = \left(\emptyset, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)}\right) \in VT_k^m([\lambda]) \tag{6.2}$$

$$P_{[\lambda]} = \Big([\lambda]^{(2k)}, [\lambda]^{(2k-\frac{1}{2})}, ..., [\lambda]^{(k+\frac{1}{2})}, [\lambda]^{(k)}\Big) \in VT_k^m([\lambda]). \tag{6.3}$$

We denote this process of producing a pair $(P_{[\lambda]}, Q_{[\lambda]})$ to a diagrams $d \in \overrightarrow{P}_k(n, G)$ by $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$.

For the insertion sequence above the pair of multi-vacillating tableaux is given by

j	E_j		$T^{(j)}$
0			$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
$\frac{1}{2}$	$(\emptyset, e)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
1	$(8, e)$	$\xrightarrow{RS}$	$\Big(\boxed{8}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
$1\frac{1}{2}$	$(\emptyset, e)$	$\xleftarrow{jdt}$	$\Big(\boxed{8}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
2	$(3, g_2)$	$\xrightarrow{RS}$	$\Big(\boxed{8}^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
$2\frac{1}{2}$	$(8, e)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
3	$(2, g_3)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \boxed{3}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
$3\frac{1}{2}$	$(\emptyset, e)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \boxed{3}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
4	$(6, g_2)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \boxed{3\,6}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
$4\frac{1}{2}$	$(6, g_2)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \boxed{3}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}, \Big)$
5	$(4, g_2)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \boxed{3\,4}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
$5\frac{1}{2}$	$(\emptyset, e)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \boxed{3\,4}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
6	$(\emptyset, e)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \boxed{3\,4}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
$6\frac{1}{2}$	$(4, g_2)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \boxed{3}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
7	$(\emptyset, e)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \boxed{3}^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
$7\frac{1}{2}$	$(3, g_2)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \boxed{2}^{(3)}, \emptyset^{(4)}\Big)$
8	$(1, g_4)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \boxed{2}^{(3)}, \boxed{1}^{(4)}\Big)$
$8\frac{1}{2}$	$(2, g_3)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \boxed{1}^{(4)}\Big)$
9	$(\emptyset, e)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \boxed{1}^{(4)}\Big)$
$9\frac{1}{2}$	$(1, g_4)$	$\xleftarrow{jdt}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$
10	$(\emptyset, e)$	$\xrightarrow{RS}$	$\Big(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\Big)$

$P_{[\lambda]} = \Big((\emptyset, \emptyset, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset, \square), (\emptyset, \emptyset, \emptyset, \square), (\emptyset, \emptyset, \square, \square), (\emptyset, \emptyset, \square, \emptyset),$

$$\big(\emptyset,\square,\square,\emptyset\big),\big(\emptyset,\square,\square,\emptyset\big),\big(\emptyset,\square\square,\square,\emptyset\big),\big(\emptyset,\square\square,\square,\emptyset\big),\big(\emptyset,\square\square,\square,\emptyset\big)\Big)$$

$$Q_{[\lambda]} = \Big(\big(\emptyset,\emptyset,\emptyset,\emptyset\big),\big(\emptyset,\emptyset,\emptyset,\emptyset\big),\big(\square,\emptyset,\emptyset,\emptyset\big),\big(\square,\emptyset,\emptyset,\emptyset\big),\big(\square,\square,\emptyset,\emptyset\big),\big(\emptyset,\square,\emptyset,\emptyset\big),$$

$$\big(\emptyset,\square,\square,\emptyset\big),\big(\emptyset,\square,\square,\emptyset\big),\big(\emptyset,\square\square,\square,\emptyset\big),\big(\emptyset,\square,\square,\emptyset\big),\big(\emptyset,\square\square,\square,\emptyset\big)\Big).$$

Theorem 6.2. *The mapping $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$ gives a bijection between the diagrams of $\overrightarrow{P}_k(n,G)$ and the pair of multi-vacillating tableaux in the set $\bigsqcup_{[\lambda]\in\Gamma^G_{k,n}} VT^m_k([\lambda]) \times VT^m_k([\lambda])$, where G is finite order cyclic group and this gives a proof of (6.1).*

Proof. We construct the inverse of the map $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$ to prove the converse part. We produce the sequence $[\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(2k-\frac{1}{2})}, [\lambda]^{(2k)}$ using $Q_{[\lambda]}$ followed by $P_{[\lambda]}$ in the this revert process. Let us consider the initial tableau $T^{(2k)} = (\emptyset, \emptyset, ..., \emptyset)$.
Now, we present the process to construct the tableaux $T^{(i+\frac{1}{2})}$ and the entry of E_{i+1} so that $T^{(i+1)} = (E_{i+1} \xrightarrow{RS} T^{(i+\frac{1}{2})})$. If $[\lambda]^{(i+\frac{1}{2})} = [\lambda]^{(i+1)}$ then take $T^{(i+\frac{1}{2})} = T^{(i+1)}$ and $E_{i+1} = (\emptyset, e)$. Otherwise, we have $[\lambda]^{(i+1)}/[\lambda]^{(i+\frac{1}{2})}$ is a box that containing the entry a and use uninsertion on a in the box to produce a tableau $T^{(i+\frac{1}{2})}$ and E_{i+1} such that $T^{(i+1)} = (E_{i+1} \xrightarrow{RS} T^{(i+\frac{1}{2})})$. If box containing a in the i^{th} component then $E_{i+1} = (a, g_i)$.

Now, we discuss the method to construct a tableau $T^{(i)}$ and the entry of $E_{i+\frac{1}{2}}$ so that $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{jdt} T^{(i)})$. If $[\lambda]^{(i)} = [\lambda]^{(i+\frac{1}{2})}$ then let $T^{(i)} = T^{(i+\frac{1}{2})}$ and $E_{i+\frac{1}{2}} = (\emptyset, e)$. Otherwise, we have $[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}$ is a box containing a. Let the tableaux $T^{(i)}$ and $T^{(i+\frac{1}{2})}$ having the same entries and the entry $2k-i$ is in the box. Let $E_{i+\frac{1}{2}} = (2k-i, g_i)$. At any given step i, $2k-i$ is the highest value added to the tableau thus far, so that the tableau $T^{(i)}$ is standard. Further more, $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{jdt} T^{(i)})$, since $a = 2k-i$ is already in a corner box and thus jdt simply delete it.

The sequence $E_{2k}, E_{2k-1}, ..., E_{\frac{1}{2}}$ will be produced by the above iterative process, which determines the diagram d completely. Thus, we get $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. □

6.1 The Knuth relations

We give the knuth relations of G-edge colored diagrams in this section. Given $d \in \overrightarrow{P}(n,G)$ we may construct a insertion sequence $E^{(d,k)}$.

Definition 6.3. Given a insertion sequence $E^{(d,k)}$ of the G-edge colored diagram d, we may associate $[d_1, d_2, w_1, w_2, s_1, s_2]$ such that

$$d_1 = \{(x_i, g_i) \mid x_i = x_j, \forall\, 0 \le i, j \le k\}$$

$$d_2 = \{(x_i, g_i) \mid x_i = x_j, \forall\, k \le i, j \le 2k\}$$

$$w_1 = \{(x_i, g_i) \mid x_i \neq x_j, \forall\, 0 \le i, j \le k\}$$

$$w_2 = \{(x_i, g_i) \mid x_i \neq x_j, \forall\, k \le i, j \le 2k\}$$

$$s_1 = \{(\emptyset, e) \mid e = g_i, \forall\, 0 \le i \le k\}$$

$$s_2 = \{(\emptyset, e) \mid e = g_i, \forall\, k \le i \le 2k\}$$

Furthermore, the pairs in $[d_1, d_2, w_1, w_2, s_1, s_2]$ are still arranged in the same order as in the insertion sequence $E^{(d,k)}$.

Remark 6.4. In the above definition

(1) $|w_1| = t$, where the number of vertical edges are denoted by t.

(2) $w_2 = \{(x_1, g_1), (x_2, g_2), ..., (x_t, g_t)\}$ and $x_1 > x_2 > ... > x_t$.

Definition 6.5. Let $d, d' \in \overrightarrow{P}(n, G)$. The diagram d is Knuth equivalent to d' if the following conditions holds.

(i) $s_2 = s'_2$,

(ii) $d_2 = d'_2$,

(iii) $w_1 \overset{K}{\sim} w'_1$.

The Knuth equivalent of d to d' is denoted by $d \overset{K_{\overrightarrow{P}(n,G)}}{\sim} d'$.

Definition 6.6. Suppose $d, d' \in \overrightarrow{P}(n, G)$. The diagram d is dual Knuth equivalent to d' if the following conditions holds.

(i) $s_1 = s'_1$,

(ii) $d_1 = d'_1$,

(iii) $w_1 \overset{K^*}{\sim} w'_1$.

The dual Knuth equivalent of d to d' is denoted by $d \overset{K^*_{\overrightarrow{P}(n,G)}}{\sim} d'$.

Lemma 6.7. *Let $w_1 \in E^{(d,k)}$ and $w'_1 \in E^{(d',k)}$ such that $|w_1| = |w'_1| = t$. Then $w_1 \overset{K}{\sim} w'_1$ or $w_1 \overset{K^*}{\sim} w'_1 \iff w_2 = w'_2$.*

Proof. It follows from the Remark (6.4) and Theorems: (2.5) and (2.6). □

Proposition 6.8. *Suppose d and $d' \in \overrightarrow{P}(n, G)$. Then $d \overset{K_{\overrightarrow{P}(n,G)}}{\sim} d'$ if and only if $P_{[\lambda]} = P'_{[\lambda]}$ where $P_{[\lambda]}, P'_{[\lambda]}$ are defined as in Eqn.(6.3), i.e. $d \overset{K_{\overrightarrow{P}(n,G)}}{\sim} d'$ if and only if d and d' have same path from $2k$ to k.*

Proof. Let d is Knuth equivalent to d' then by the Definitions (6.3), (6.5) and Lemma (6.7), we have $E^{(d,k)} = E^{(d',k)}$, where $k + 1 \leq j \leq 2k$. Thus, we get $P_{[\lambda]} = P'_{[\lambda]}$. By retracing the steps we can prove the converse part. □

Proposition 6.9. *Let d and $d' \in \overrightarrow{P}(n, G)$. Then $d \overset{K^*_{\overrightarrow{P}(n,G)}}{\sim} d'$ if and only if $Q_{[\lambda]} = Q'_{[\lambda]}$ where $Q_{[\lambda]}, Q'_{[\lambda]}$ are as in Eqn.(6.2), i.e. $d \overset{K^*_{\overrightarrow{P}(n,G)}}{\sim} d'$ if and only if d and d' have same path from 0 to k.*

Proof. The proof is similar to the above Proposition (6.8). □

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