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ENUMERATION OF PSEUDO ORDERED GROUPS ON $\mathbb{Z}_n$

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Abstract Unlike finite partially ordered groups, finite pseudo ordered groups can possess non-trivial pseudo orders. In this paper, we count the number of non-isomorphic pseudo ordered groups on the cyclic group $\mathbb{Z}_n$. An elementary formula is obtained, when n is a prime. Counting the number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_n$, when n is composite, is still open. However, we obtain the result for all $n \leq 25$.

1 Introduction

A partially ordered group (po-group) is an algebraic object having the structure of both, a group and a poset, which are connected in a very natural way. These structures, which simultaneously carry the properties of a group and a partial order that is compatible with the group operation, play a fundamental role in diverse areas, including representation theory, cohomology, and the theory of valuations.

Birkhoff [3] initiated the study of partially ordered groups. This structure has been extensively developed in papers such as Paul Conrad ([5, 6, 7]) and A. M. W. Glass ([10, 11, 12]). Dropping the assumption of transitivity, Helen Skala ([13, 14]) extended partially ordered sets (partially ordered groups) to pseudo ordered sets (pseudo ordered groups). Pseudo ordered sets and trellises are further investigated in [1, 2, 9]. Maria D Cruz gave a necessary and sufficient condition for a group to admit a compatible pseudo order [8].

The trivial partial order ($a \leq b$ if and only if $a = b$) is the only partial order compatible with a finite group. The relaxation on transitivity of po-group opens the door to a more nuanced combinatorial and algebraic investigation in the context of finite groups. In fact, there are multiple distinct pseudo orders (reflexive and antisymmetric relations) compatible with a finite group.

This paper develops a systematic approach for enumerating the non-isomorphic pseudo orders on $\mathbb{Z}_n$, by defining an appropriate group action and employing Burnside's Lemma. In particular, the total number of non-isomorphic pseudo orders on $\mathbb{Z}_p$, for a prime p is given by the formula $1 + \sum_{m=2}^{\frac{p+1}{2}} \sum_{k} \phi(k) \frac{(p-(2k+1))(p-(4k+1))\cdots(p-(2(s-1)k+1))}{k^s s!}$, where the second summation is taken over all odd divisors k of $\gcd(m-1, p-1)$ and ϕ is the Euler's Totient function. This investigation not only contributes to the structural theory of generalized ordered groups but also highlights interesting combinatorial properties inherent in finite abelian groups. The enumeration of pseudo orders on arbitrary groups remains a promising direction for future research.

2 Pseudo ordered groups

A pseudo ordered group is a generalization that relaxes some of the strict requirements of ordered groups, allowing more flexibility while preserving certain order-like behaviors. The following

definitions and results are due to Skala [14] and Maria D Cruz[8].

Definition 2.1. [14] A reflexive and antisymmetric binary relation $\trianglelefteq$ on a non-empty set P is called a pseudo order. The set P together with this pseudo order $\trianglelefteq$ is called a pseudo ordered set (psoset).

We can easily construct a pseudo order by removing non-reflexive pairs from a partial order.

Definition 2.2. [14] A group $(G, *)$, together with a pseudo order $\trianglelefteq$ on G, is called a pseudo ordered group (pso-group) if it satisfies the property: $a, b \in G$ and $a \trianglelefteq b$ implies $x*a*y \trianglelefteq x*b*y$ for every $x, y \in G$.

Definition 2.3. [14] Let e be the identity element in a pso-group G. The subset P of G defined by $P = \{x \in G; e \trianglelefteq x\}$ is called the positive part.

If $P^{-1} = \{x^{-1}; x \in P\}$, then from the definition of P, it follows that $P \cap P^{-1} = \{e\}$.

Theorem 2.4. *[8] A group $(G, *)$ can be pseudo ordered with positive part P if and only if*

i $P \cap P^{-1} = \{e\}$ *and*

ii $gxg^{-1} \in P$ *for every* $g \in G$ *and* $x \in P$.

Moreover, if G is abelian, then G can be pseudo ordered with positive part P if and only if $P \cap P^{-1} = \{e\}$.

Thus any subset P of an abelian group G satisfying $P \cap P^{-1} = \{e\}$ uniquely defines a pseudo order on G. In fact, for any $x, y \in G$, we may define $x \trianglelefteq y$ if $x^{-1} * y \in P$.

To avoid notational complexity, a pseudo ordered group G with positive part P is denoted by (G, P), unless there is an ambiguity on the binary operation $*$ and pseudo order $\trianglelefteq$.

Definition 2.5. [8] Let (G, P) and (H, Q) be pseudo ordered groups. A map $f : G \to H$ which is both an order isomorphism and a group isomorphism is called a *pso-isomorphism* and in this case we write $(G, P) \approxeq_f (H, Q)$. Two pso-groups are isomorphic when there is a pso-isomorphism between them.

The following theorem characterizes isomorphism of pso-groups.

Theorem 2.6. *[8] Let (G, P) and (H, Q) be pseudo ordered groups. A map $f : G \to H$ is a pso-isomorphism if and only if f is a group isomorphism and $f(P) = Q$.*

3 Pseudo ordered groups on $\mathbb{Z}_n$

Throughout this paper, $\mathbb{Z}_n$ is the group $\{0, 1, 2, \cdots n-1\}$ with binary operation addition modulo n.

For any $n \in \mathbb{N}$, even though there are several pseudo ordered groups on $\mathbb{Z}_n$, many of them are isomorphic. For example, there are 3 pseudo orders that make $\mathbb{Z}_3 = \{0, 1, 2\}$ a pso-group. The corresponding positive parts for each of these pseudo orderers are $P_1 = \{0\}, P_2 = \{0, 1\}$ and $P_3 = \{0, 2\}$. However, $(\mathbb{Z}_3, P_2) \approxeq_f (\mathbb{Z}_3, P_3)$, where $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ is defined by $f(n) = 2n$. Hence there are only two pso-groups on $\mathbb{Z}_3$, up to isomorphism.

Similarly, there are 9 pseudo orders that make $\mathbb{Z}_6$ a pso-group. The positive parts corresponding to these 9 pseudo orders are $P_1 = \{0\}, P_2 = \{0, 1\}, P_3 = \{0, 2\}, P_4 = \{0, 4\}, P_5 = \{0, 5\}, P_6 = \{0, 1, 2\}, P_7 = \{0, 1, 4\}, P_8 = \{0, 2, 5\}$ and $P_9 = \{0, 4, 5\}$. However, many of them are isomorphic. For example $(\mathbb{Z}_6, P_6)$ is isomorphic to $(\mathbb{Z}_6, P_9)$. Every pseudo order set (psoset) can be represented as a directed graph, as illustrated in Figure 1. The isomorphism between the pso-groups $(\mathbb{Z}_6, P_6)$ and $(\mathbb{Z}_6, P_9)$ is also depicted therein. Eliminating such isomorphic cases, we get only 5 pso-groups on $\mathbb{Z}_6$ up to isomorphism.

In this paper, we enumerate the non-isomorphic pso-groups on $\mathbb{Z}_n$.

Let X_m be the set of all positive parts on $\mathbb{Z}_n$ with m elements. The idea is to define a suitable group action on X_m, so that any two pso-groups are pso-isomorphic if and only if the corresponding positive parts are in the same orbit. Hence counting the number of non-isomorphic

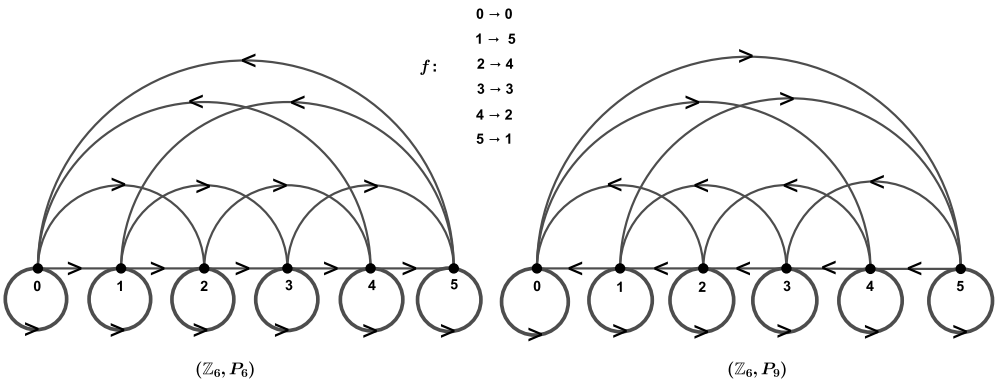


Figure 1. Isomorphic Pso-groups

pso-groups on $\mathbb{Z}_n$ reduces to counting the number of orbits, which can be done by use of Burnside's formula.

We have $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n;\ \gcd(a, n) = 1\}$ and is a group under multiplication modulo n.
Define $\psi : \mathbb{Z}_n^* \times X_m \to X_m$ by $\psi(a, P) = aP$.

Note that for any $a \in \mathbb{Z}_n^*$ and $P \in X_m$,

i $aP \cap -(aP) = \{0\}$.
If $x \in aP \cap -(aP)$, then $a^{-1}x \in P \cap -P = \{0\}$ so that $x = 0$.

ii aP contains m elements.
In fact, the map $\phi : P \to aP$, defined by $\phi(k) = ak$ is bijective and hence aP contains m elements.

Thus the map ψ is well defined.

Further, we have

i $\psi(1, P) = P$ and

ii

$$\begin{aligned}\psi(a, \psi(b, P)) &= \psi(a, bP)\\ &= a(bP)\\ &= (ab)P\\ &= \psi(ab, P) \ \text{ for each } a, b \in \mathbb{Z}_n^* \text{ and } P \in X_m.\end{aligned}$$

Thus ψ defines a group action of $\mathbb{Z}_n^*$ on X_m.

Lemma 3.1. *Two pso-groups $(\mathbb{Z}_n, P)$ and $(\mathbb{Z}_n, Q)$ are isomorphic if and only if P and Q are in the same orbit under the group action defined above.*

Proof. Suppose P and Q are in the same orbit. i.e., $Q = aP$ for some $a \in \mathbb{Z}_n^*$. Then the map $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\phi(k) = ak$ is a group isomorphism satisfying $\phi(P) = Q$. Hence by Theorem 2.6, $(\mathbb{Z}_n, P) \approx_\phi (\mathbb{Z}_n, Q)$.

Conversely, suppose that $(\mathbb{Z}_n, P) \approx_\phi (\mathbb{Z}_n, Q)$. Then by Theorem 2.6, $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ is a group isomorphism and $\phi(P) = Q$. Since ϕ is an isomorphism, $a = \phi(1) \in \mathbb{Z}_n^*$. Thus $Q = \phi(P) = aP$ and hence P and Q are in the same orbit. □

For each $a \in \mathbb{Z}_n^*$, denote $(X_m)_a = \{P \in X_m;\ aP = P\}$.

Theorem 3.2. *The number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_n$ is given by the formula* $\frac{1}{\phi(n)} \sum_{a \in \mathbb{Z}_n^*} \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} |(X_m)_a|$, *where $\lfloor . \rfloor$ is the floor function.*

Proof. By Lemma 3.1, number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_n$ with positive parts in X_m equals the number of orbits.

Using Burnside's formula [4], the number of orbits in X_m is

$$\frac{1}{|\mathbb{Z}_n^*|} \sum_{a \in \mathbb{Z}_n^*} |(X_m)_a| = \frac{1}{\phi(n)} \sum_{a \in \mathbb{Z}_n^*} |(X_m)_a|.$$

Since for any pso-group $(\mathbb{Z}_n, P)$, we have $P \cap -P = \{0\}$, P can contain at most $\left\lfloor \frac{n+1}{2} \right\rfloor$ elements.
Thus the total number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_n$ is

$$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{\phi(n)} \sum_{a \in \mathbb{Z}_n^*} |(X_m)_a| = \frac{1}{\phi(n)} \sum_{a \in \mathbb{Z}_n^*} \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} |(X_m)_a|.$$

□

4 Enumeration of pseudo ordered groups on $\mathbb{Z}_p$, p prime

In this section, we derive an elementary expression to find the total number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_p$, for any odd prime p. From Theorem 3.2, the total number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_p$ is $\frac{1}{p-1} \sum_{m=1}^{\frac{p+1}{2}} \sum_{a \in \mathbb{Z}_p^*} |(X_m)_a|$. In order to compute this quantity, we need to find $|(X_m)_a|$ for each $m = 1, 2, \cdots, \frac{p+1}{2}$ and $a = 1, 2, \cdots, p-1$. In many cases this quantity $|(X_m)_a| = 0$. The following two lemmas describe those m and a for which $|(X_m)_a| \neq 0$.

Lemma 4.1. *Let $a \in \mathbb{Z}_p^* = \{1, 2, \cdots, p-1\}$, $P \in (X_m)_a$ and $m > 1$. Then the order of a in $\mathbb{Z}_p^*$, denoted by $O(a)$ is an odd divisor of* $\gcd(m-1, p-1)$.

Proof.

$$\begin{aligned} \text{For any } n \in \mathbb{N},\ a^{2n} = 1 &\Leftrightarrow a^{2n} - 1 = 0 \\ &\Leftrightarrow (a^n - 1)(a^n + 1) = 0 \\ &\Leftrightarrow a^n - 1 = 0 \end{aligned}$$

In fact, if $a^n = -1$, then for any element $x \in P$, we get $-x = a^n x \in P$, contradicting $P \cap -P = \{0\}$.

Therefore $O(a)$ is odd, denote it by k.

Since $a \in \mathbb{Z}_p^*$, by Lagrange's theorem, $O(a) \mid O(\mathbb{Z}_p^*)$ and hence $k \mid p-1$.

The map $\sigma : P \to P$, defined by $\sigma(x) = ax$ is one-one and hence a permutation on P. The orbit of any non-zero $b \in P$ under σ is of the form $\{b, ab, a^2b, \cdots a^{k-1}b\}$ that contains k elements. If there are r orbits in the permutation σ, then $m = 1 + (r-1)k$ and hence $k \mid m-1$.

This proves the lemma. □

From the above lemma, it follows that $(X_m)_a \neq \emptyset$, only if $k = O(a)$ is an odd divisor of $\gcd(m-1, p-1)$. Thus it remains to compute $|(X_m)_a|$ when $k = O(a)$ is an odd divisor of $\gcd(m-1, p-1)$. This is done in the next lemma.

Lemma 4.2. *Let $k = O(a)$ be an odd divisor of $\gcd(m-1, p-1)$, $m > 1$ and $s = \dfrac{m-1}{k}$. Then* $| (X_m)_a | = \dfrac{(p-1)(p-(2k+1))\cdots(p-(2(s-1)k+1))}{k^s s!}$.

Proof. Let $P \in X_m$ and $aP = P$. Then

$$\begin{aligned} P = \Big\{ & 0, b_1, ab_1, a^2b_1, \cdots, a^{k-1}b_1, \\ & b_2, ab_2, a^2b_2, \cdots, a^{k-1}b_2, \\ & b_3, ab_3, a^2b_3, \cdots, a^{k-1}b_3, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \\ & b_s, ab_s, a^2b_s, \cdots, a^{k-1}b_s, \Big\} \end{aligned}$$

Starting from the beginning, b_1 has $p-1$ options and the orbit doesn't change if any of the elements of $\{b_1, ab_1, a^2b_1, \cdots a^{k-1}b_1\}$ are chosen. Having chosen b_1, we can chose b_2 such that b_2 is none of $0, b_1, ab_1, a^2b_1, \cdots a^{k-1}b_1, -b_1, -ab_1, -a^2b_1, \cdots -a^{k-1}b_1$. Hence b_2 has $p-(2k+1)$ choices. As earlier, orbit of b_2 doesn't change if any of the elements of $\{b_2, ab_2, a^2b_2, \cdots a^{k-1}b_2\}$ are chosen. Continuing in this manner, we get

$$\begin{aligned} | (X_m)_a | &= \frac{p-1}{k} \cdot \frac{p-(2k+1)}{k} \cdots \frac{p-(2(s-1)k+1)}{k} \cdot \frac{1}{s!} \\ &= \frac{(p-1)(p-(2k+1))\cdots(p-(2(s-1)k+1))}{k^s s!} \end{aligned}$$

□

We are now ready to prove the main theorem of this paper.

Theorem 4.3. *The number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_p$ is* $1 + \sum_{m=2}^{\frac{p+1}{2}} \sum_{k} \phi(k) \dfrac{(p-(2k+1))(p-(4k+1))\cdots(p-(2(s-1)k+1))}{k^s s!}$, *where the second summation is taken over all odd divisors k of $\gcd(m-1, p-1)$ and ϕ is the Euler's Totient function.*

Proof. Let N denote the total number of non-isomorphic pseudo ordered groups on $\mathbb{Z}_p$.

From Theorem 3.2, $N = \dfrac{1}{p-1} \sum_{m=1}^{\frac{p+1}{2}} \sum_{a \in \mathbb{Z}_p^*} |(X_m)_a|$.

For $m = 1$,

$$\begin{aligned} \sum_{a \in \mathbb{Z}_p^*} |(X_m)_a| &= \sum_{a \in \mathbb{Z}_p^*} |(X_1)_a| \\ &= \sum_{a \in \mathbb{Z}_p^*} 1 \\ &= |\mathbb{Z}_p^*| \\ &= p-1 \end{aligned}$$

Hence $N = 1 + \dfrac{1}{p-1} \sum_{m=2}^{\frac{p+1}{2}} \sum_{a \in \mathbb{Z}_p^*} |(X_m)_a|$.

If $k = O(\alpha^i)$, then from Lemma 4.1 and Lemma 4.2,
$|(X_m)_a| = \dfrac{(p-1)(p-(2k+1))(p-(4k+1))\cdots(p-(2(s-1)k+1))}{k^s s!}$ if k is an odd divisor of $\gcd(m-1, p-1)$, and 0 otherwise.

Since p is a prime, for any $k \mid p-1$, there are $\phi(k)$ elements of order k in $\mathbb{Z}_p^*$.

Therefore, $N = 1 + \sum_{m=2}^{\frac{p+1}{2}} \sum_{k} \phi(k) \dfrac{(p-(2k+1))(p-(4k+1))\cdots(p-(2(s-1)k+1))}{k^s s!}$, where the second summation is taken over all odd divisors k of $\gcd(m-1, p-1)$ and $s = \dfrac{m-1}{k}$. □

The following table gives the number of pso-groups on $\mathbb{Z}_p$ for small values of p.

p	2	3	5	7	11	13	17	19	23	29	31	37
N	1	2	3	6	26	63	411	1098	8054	170823	478318	10761723

5 Enumeration for an arbitrary $n \in \mathbb{N}$

We have not succeeded in reducing the equation 3.1 to an elementary expression, for composite n. In fact, determining $|(X_m)_a|$, when n is composite involves more computations. However, $|(X_m)_a| = 0$ when $a^k \equiv -1 \pmod{n}$ for some $k \in \mathbb{Z}$. This is proved in the following proposition.

Proposition 5.1. *Let $a \in \mathbb{Z}_n^*$. Then $(X_m)_a = \emptyset$ for all $m > 1$ if and only if $a^k \equiv -1 \pmod{n}$ for some $k \in \mathbb{Z}$.*

Proof. Let $(X_m)_a = \emptyset$ for all $m > 1$. If $a^k \not\equiv -1 \pmod{n}$ for any $k \in \mathbb{Z}$, then the set $P = \{0, a, a^2, \cdots\}$ is a positive part satisfying $aP = P$. Taking $m = |P|$, we get $P \in (X_m)_a$, a contradiction. Thus $a^k \equiv -1 \pmod{n}$ for some $k \in \mathbb{Z}$.

Conversely, let $a^k \equiv -1 \pmod{n}$ for some $k \in \mathbb{Z}$. If $(X_m)_a \neq \emptyset$ for some $m > 1$, choose $P \in (X_m)_a$ and a non-zero element $x \in P$. Since $aP = P$, we get $a^k x \in P$, which gives $-a \in P$, a contradiction as $P \cap -P = \{0\}$. Thus $(X_m)_a = \emptyset$ for all $m > 1$. □

Corollary 5.2. $(X_m)_{n-1} = \emptyset$ *for all* $m > 1$

Proof. This follows as $a = n-1 \in \mathbb{Z}_n^*$ and $a^1 = n-1 \equiv -1 \pmod{n}$. □

Proposition 5.3. $\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} |(X_m)_1| = 3^{\lfloor \frac{n-1}{2} \rfloor}$

Proof. $1P = P$ for any P and hence $(X_m)_1 = X_m$ for all $m \geq 1$. Denote $P_k = \{k, n-k\}$ for $k = 1, 2, \cdots \lfloor \frac{n-1}{2} \rfloor$. For any $P \in X_m$, we have $0 \in P$ and $\dfrac{n}{2} \notin P$, and $P - \{0\}$ contains at most one element from each P_k. We can choose $m-1$ P_k's in $\binom{\lfloor \frac{n-1}{2} \rfloor}{m-1}$ ways. Since each P_k contains 2 elements, we get $|X_m| = \binom{\lfloor \frac{n-1}{2} \rfloor}{m-1} 2^{m-1}$. Thus

$$\begin{aligned}\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}|(X_m)_1| &= \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}|X_m| \\ &= \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{\lfloor\frac{n-1}{2}\rfloor}{m-1}2^{m-1} \\ &= \sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{\lfloor\frac{n-1}{2}\rfloor}{m}2^{m} \\ &= 3^{\lfloor\frac{n-1}{2}\rfloor}\end{aligned}$$

□

Proposition 5.4. *Let $a \in \mathbb{Z}_n^*$ and $P \in X_m$. Then $P \in (X_m)_a$ if and only if P is a union of orbits of the homomorphism $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$, given by $\phi(1) = a$ (Orbits are defined as this map ϕ is a permutation).*

Proof. Let k denote the order of a in $\mathbb{Z}_n^*$.

Suppose $P \in (X_m)_a$. Then $aP = P$ and hence for any $x \in P$, we must have $\{x, \phi(x), \phi^2(x), \cdots\} = \{x, ax, a^2x, \cdots, a^{k-1}x\} \subseteq P$. Thus P is a union of orbits of the permutation ϕ.

On the other hand, if P is a union of orbits of the homomorphism ϕ, then for each $x \in P$, we have $ax = \phi(x) \in P$ so that $aP \subseteq P$. This would imply $a^2P \subseteq aP, a^3P \subseteq a^2P, \cdots, P = a^kP \subseteq a^{k-1}P$, which gives $aP = P$ and hence $P \in (X_m)_a$. □

Computing $|(X_2)_a|$, when $n = p^k$, for an odd prime p and $k \in \mathbb{N}$ is obtained in the following proposition.

Proposition 5.5. *For any prime p and $k \in \mathbb{N}$, the total number of pseudo ordered groups on $\mathbb{Z}_{p^k}$ with positive part containing 2 elements is k.*

Proof. Let $n = p^k$. Then $X_2 = \{\{0, x\}; x = 1, 2, \cdots, p^k - 1\}$.

For any $a \in \mathbb{Z}_{p^k}^*$ and $P = \{0, x\} \in X_2$,

$$\begin{aligned}P \in (X_2)_a &\Leftrightarrow aP = P \\ &\Leftrightarrow ax \equiv x \pmod{p^k} \\ &\Leftrightarrow p^k \mid (a-1)x\end{aligned}$$

This can happen in several ways. In the table below, the first column specifies some conditions on p and a, while the next three columns list the total number of possible choices for a,x, and the pair (a, x), respectively, under those conditions.

(1)	(2)	(3)	(4)= (2) × (3)
$p^k \mid (a-1)$	1	$p^k - 1$	$p^k - 1$
$p^{k-1} \mid (a-1)$ and $p^k \nmid (a-1)$	$p-1$	$p^{k-1} - 1$	$p^k - p^{k-1} - p + 1$
$p^{k-2} \mid (a-1)$ and $p^{k-1} \nmid (a-1)$	$p^2 - p$	$p^{k-2} - 1$	$p^k - p^{k-1} - p^2 + p$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$p^2 \mid (a-1)$ and $p^3 \nmid (a-1)$	$p^{k-2} - p^{k-3}$	$p^2 - 1$	$p^k - p^{k-1} - p^{k-2} + p^{k-3}$
$p \mid (a-1)$ and $p^2 \nmid (a-1)$	$p^{k-1} - p^{k-2}$	$p-1$	$p^k - p^{k-1} - p^{k-1} + p^{k-2}$
$p \nmid (a-1)$	$p^k - p^{k-1}$	0	0

Thus adding up all elements of column (4) gives $\sum_{a \in (\mathbb{Z}_{p^k})^*} |(X_2)_a| = k(p^k - p^{k-1})$. Hence the

total number of pseudo ordered groups on $\mathbb{Z}_{p^k}$ with positive part containing 2 elements is

$$\frac{1}{|(\mathbb{Z}_{p^k})^*|}\sum_{a\in(\mathbb{Z}_{p^k})^*}|(X_2)_a| = \frac{1}{\phi(p^k)}k(p^k-p^{k-1}) = k.$$

□

Remark 5.6. By similar arguments, it can be shown that the total number of pseudo ordered groups on $\mathbb{Z}_{2^k}$ with positive part containing 2 elements is $k-1$

The following table gives the total number N of non-isomorphic pso-groups on $\mathbb{Z}_n$ for small values of n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
N	1	1	2	2	3	5	6	10	17	21	26	70	63	125	290

n	16	17	18	19	20	21	22	23	24	25
N	314	411	1121	1098	2558	5005	5909	8054	23210	26589

6 Conclusion

In this paper, we have obtained an elementary expression for counting the total number of non-isomorphic pseudo orders on the cyclic group $\mathbb{Z}_p$, where p is a prime. The values for all primes $p \leq 40$ are computed using the formula provided in Theorem 4.3, and these values are listed in a table. We have not been able to get an elementary expression to count the number of non-isomorphic pseudo orders on the general cyclic group $\mathbb{Z}_n$, for n composite. However, we prove some basic theorems in this case, with the help of which the total number of non-isomorphic pseudo orders on the $\mathbb{Z}_n$ is obtained for any $n \leq 25$. The computations were carried out using a basic SCILAB program. Since the enumeration of pseudo orders on arbitrary groups remains unresolved, we propose the following open problems for future research:

Open Problems:

(i) Derive an elementary expression to determine the total number of non-isomorphic pseudo orders on $\mathbb{Z}_n$ for an arbitrary $n \in \mathbb{N}$.

(ii) Count the number of non-isomorphic pseudo ordered groups of any given finite order n.

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