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Strongly irreducible ideals and S spectrum of commutative rings

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Abstract Let K be a proper ideal of a commutative ring $\mathcal{R}$. Then K is a strongly irreducible ideal if for any two ideals C and D of $\mathcal{R}$, $C \cap D \subseteq K$ implies $C \subseteq K$ or $D \subseteq K$. The S radical of an ideal I ($S.rad(I)$) of $\mathcal{R}$ is the intersection of all strongly irreducible ideals of $\mathcal{R}$ containing I. In this paper some new results concerning strongly irreducible ideals and $S.rad(I)$ are given and the relations between strongly irreducible ideals and some other types of ideals are studied. Some results obtained concerning families of strongly irreducible ideals. Additionally the concept of *CS* Zariski topology on $CS.spec\,(\mathcal{R})$ is introduced.

1 Introduction

In this paper, $\mathcal{R}$ is a commutative ring with identity. An ideal $K \neq \mathcal{R}$ of $\mathcal{R}$ is a strongly irreducible ideal if for any two ideals C and D of $\mathcal{R}$, $C \cap D \subseteq K$ implies $C \subseteq K$ or $D \subseteq K$ [1], and K is a completely strongly irreducible ideal if for any family of ideals $\{ K_t\}_{t\in\Omega}$ of $\mathcal{R}$, $\bigcap_{t\in\Omega} K_t \subseteq K$ implies $K_t \subseteq K$ for some $t \in \Omega$ [2]. Clearly, every completely strongly irreducible ideal is a strongly irreducible ideal. In Section 2, some properties of strongly irreducible ideals and their relation with some other types of ideals are given. In Section 3, we obtain some results concerning the S radical of ideals. In Section 4, the concepts of strongly irreducible spectrum $S.spec(\mathcal{R})$ and the S Zariski topology on $S.spec\,(\mathcal{R})$ are studied and some properties of such topologies are investigated. Finally, we study the *CS* Zariski topology on $CS.spec\,(\mathcal{R})$.

2 Strongly irreducible ideals

In this section, strongly irreducible ideals of a ring $\mathcal{R}$ are studied. We focus on the relations between strongly irreducible ideals and some other types of ideals and we obtain some results concerning strongly irreducible ideals. Finally, the construction of strongly irreducible ideals of a unique factorization domain is investigated.

Proposition 2.1. *[3] For an ideal K of $\mathcal{R}$, the following conditions are equivalent:*

1. *K is a strongly irreducible ideal.*
2. *If $\langle c\rangle \cap \langle d\rangle \subseteq K$, then $\langle c\rangle \subseteq K$ or $\langle d\rangle \subseteq K$, for any two principal ideals $\langle c\rangle$ and $\langle d\rangle$ of $\mathcal{R}$.*
3. *The set $\mathcal{R}\backslash K$ is an i-system; that is, $\forall\, c, d \in \mathcal{R}, c \in \mathcal{R}\backslash K$ and $d \in \mathcal{R}\backslash K$ implies $\langle c\rangle \cap \langle d\rangle \cap (\mathcal{R}\backslash K) \neq \varnothing$.*

The requisite definitions, all taken from [7], [8], and [9], are recalled below.

1. A proper ideal I of a ring $\mathcal{R}$ is
   a. prime, if $a, b \in \mathcal{R}$ with $ab \in I$ implies $a \in I$ or $b \in I$.
   b. weakly prime, if $a, b \in \mathcal{R}$ with $ab \in I\backslash\{0\}$ implies $a \in I$ or $b \in I$.

   c. almost prime, if $a, b \in \mathcal{R}$ with $ab \in I \backslash I^2$ implies $a \in I$ or $b \in I$.
   d. n-almost prime, if $a, b \in \mathcal{R}$ with $ab \in I \backslash I^n$, $n > 2$ implies $a \in I$ or $b \in I$.

2. A ring $\mathcal{R}$ is said to be a Laskerian ring, if every proper ideal of $\mathcal{R}$ has a primary decomposition.
3. A ring $\mathcal{R}$ is said to be a Zerlegung prime ideal ring (ZPI-ring) if every proper ideal of $\mathcal{R}$ can be written as a product of prime ideals of $\mathcal{R}$.
4. $\mathcal{R}$ is absolutely flat if every principal ideal is idempotent;
5. If $\mathcal{R}$ is a commutative ring and M is an $\mathcal{R}$-module, the idealization of $\mathcal{R}$ by M is the ring $\mathcal{R} \oplus M$ with addition $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. In this ring, M is identified with the ideal $\{0\} \oplus M$, whose square is zero.
6. A ring $\mathcal{R}$ is said to be arithmetical if any of the following equivalent conditions holds:
   a. For all ideals, I, J and K of $\mathcal{R}$, we have $I \cap (J + K) = (I \cap J) + (I \cap K)$
   b. For all ideals, I, J and K of $\mathcal{R}$, we have $I + (J \cap K) = (I + J) \cap (I + K)$.

Remark 2.2. [4, 5, 6]. Consider the ring $\mathcal{R}$. Then

1. Every strongly irreducible ideal is irreducible.
2. An ideal K is prime if and only if K is a strongly irreducible ideal and $K = rad(K)$.
3. Every strongly irreducible ideal is primal.
4. Every primary ideal of an arithmetical ring is irreducible and every irreducible ideal is a strongly irreducible ideal.
5. Every strongly irreducible ideal of a unique factorization domain or of a Laskerian ring, is primary.
6. Every principal primary ideal of a unique factorization domain is a strongly irreducible ideal.
7. In a ZPI-ring and in an absolutely flat ring, an ideal is primary if and only if it is a strongly irreducible ideal.

Remark 2.3. The concept of a strongly irreducible ideal is independent of the concepts of primary ideal, m-almost prime ideal, weakly prime ideal and almost prime ideal. For example, the zero ideal $I = \langle 0 \rangle$ of the ring $\mathbb{Z}_6$ is m-almost prime, weakly prime and almost prime ideal but it is not a strongly irreducible ideal and the ideal $\langle p^m \rangle$ of the ring $\mathbb{Z}_{p^n}$, $1 < m < n$ and $n > 2$ is a strongly irreducible ideal but it is not m-almost prime ideal, weakly prime ideal and almost prime ideal. Moreover, let $\mathcal{R}$ be a local Noetherian ring which has at least two minimal prime ideals and L be its maximal ideal and let $E = E(\mathcal{R}/L)$ be the injective envelope of the quotient field $\mathcal{R}/L$ of $\mathcal{R}$ considered as an $\mathcal{R}$ module. Then the ideal $\langle 0 \rangle$ of $A = \mathcal{R} + E$ the idealization is a strongly irreducible ideal but not primary [10]. Moreover, the ideal $I = \langle x, y \rangle^2$ of the ring $\mathcal{R} = \mathbb{Z}_2[x, y]$ is primary but not a strongly irreducible ideal.

By an ideal C is smaller than an ideal D of $\mathcal{R}$ or D is larger than C we mean C is contained in D properly.

Proposition 2.4. *Consider two ideals C and D of $\mathcal{R}$. Then*

1. *$C \cap D$ is strongly irreducible in $\mathcal{R}$ if and only if C and D are comparable and the smaller one is strongly irreducible.*
2. *$C \cup D$ is strongly irreducible in $\mathcal{R}$ if and only if C and D are comparable and the larger of the one ideal is strongly irreducible.*
3. *If C and D are distinct maximal ideals of $\mathcal{R}$, then for every $m, n \in \mathbb{Z}^+$ neither $C^m \cap D^n$ nor CD is strongly irreducible in $\mathcal{R}$.*

Proof. 1. Let $C \cap D$ be a strongly irreducible ideal of $\mathcal{R}$ and suppose that there are $c,\ d \in \mathcal{R}$ with $c \in C \setminus D$ and $d \in D \setminus C$. Then clearly $\langle c \rangle \cap \langle d \rangle \subseteq C \cap D$ but $\langle c \rangle, \langle d \rangle \nsubseteq C \cap D$, contradiction with $C \cap D$ is a strongly irreducible ideal. Then $C \subseteq D$ or $D \subseteq C$, consequently the smaller one is a strongly irreducible ideal. The converse is clear.

2. It is clear that if $C \cup D$ is an ideal of $\mathcal{R}$, then $C \subseteq D$ or $D \subseteq C$. Moreover, if $C \cup D$ is a strongly irreducible ideal, then the larger one is a strongly irreducible ideal.

3. Let C and D be two distinct maximal ideals of $\mathcal{R}$ and let $C^m \cap D^n$ be a strongly irreducible ideal of $\mathcal{R}$. Then by part (1), $C^m \subseteq D^n$ or $D^n \subseteq C^m$. If $C^m \subseteq D^n$, then for each $x \in C$, $x^m \in C^m \subseteq D^n \subseteq D$. Since D is a prime ideal, then $x \in D$. This means that $C \subseteq D$, contradiction with C and D are two maximal ideals. Therefore, $C^m \cap D^n$ is not a strongly irreducible ideal of $\mathcal{R}$. Clearly the ideals C, D are coprime. Then $C \cap D = CD$. Therefore, CD is not a strongly irreducible ideal of $\mathcal{R}$.

□

Proposition 2.5. *[4] Every proper ideal of $\mathcal{R}$ is a strongly irreducible ideal if and only if any two ideals of $\mathcal{R}$ are comparable.*

Theorem 2.6. *If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a family of strongly irreducible ideals of $\mathcal{R}$ ordered by inclusion, then $\bigcap_{\lambda \in \Lambda} S_\lambda$ and $\bigcup_{\lambda \in \Lambda} S_\lambda$ are strongly irreducible ideals of $\mathcal{R}$.*

Proof. Let $S = \bigcap_{\lambda \in \Lambda} S_\lambda$ and $\langle a \rangle \cap \langle b \rangle \subseteq S$ for some $a,\ b \in \mathcal{R}$. Then $\langle a \rangle \cap \langle b \rangle \subseteq S_\lambda$ for each λ. If $a,\ b \in S_\lambda$ for each λ, then the statement is true. So suppose that there is $\sigma \in \Lambda$ such that $a \in S_\sigma$ and $b \notin S_\sigma$. We claim that $a \in S_\lambda$ for each $\lambda \in \Lambda$. Contrary, suppose $a \notin S_\delta$ for some $\delta \in \Lambda$. Since S_δ is a strongly irreducible ideal, hence $b \in S_\delta$, this contradicts the fact that $\{S_\lambda\}_{\lambda \in \Lambda}$ is totally ordered by inclusion. Hence $a \in S_\lambda$ for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} S_\lambda$ is a strongly irreducible ideal. Now, let $T = \bigcup_{\lambda \in \Lambda} S_\lambda$. Clearly T is an ideal of $\mathcal{R}$, hence T is contained in a maximal ideal M of $\mathcal{R}$. So M is an upper bound of $\{S_\lambda\}_{\lambda \in \Lambda}$. By Zorn's lemma, $\{S_\lambda\}_{\lambda \in \Lambda}$ contains at least one maximal ideal say S_t for some $t \in \Lambda$. Since $\{S_\lambda\}_{\lambda \in \Lambda}$ is totally ordered by inclusion, S_t is the unique maximal ideal in $\{S_\lambda\}_{\lambda \in \Lambda}$, consequently $\bigcup_{\lambda \in \Lambda} S_\lambda = S_t$ which is a strongly irreducible ideal of $\mathcal{R}$. □

Theorem 2.7. *Let $f : \mathcal{R}_1 \to \mathcal{R}_2$ be a ring homomorphism and K be a strongly irreducible ideal of $\mathcal{R}_2$ with $K^c \neq \mathcal{R}_1$. If one of the following conditions is satisfied:*

(i) $(C \cap D)^e = C^e \cap D^e$ *for any two ideals C, D of $\mathcal{R}_1$, or*

(ii) $C \cap D = CD$ *for any two ideals C, D of $\mathcal{R}_2$,*

then K^c is a strongly irreducible ideal of $\mathcal{R}_1$.

Proof. Suppose $(C \cap D)^e = C^e \cap D^e$ for any two ideals C and D of $\mathcal{R}_1$. Clearly K^c is an ideal of $\mathcal{R}_1$. Let $\langle c \rangle \cap \langle d \rangle \subseteq K^c$ for $c, d \in \mathcal{R}_1$. Then $(\langle c \rangle \cap \langle d \rangle)^e \subseteq K^{ce}$. By [11, Proposition 1.17], $(\langle c \rangle \cap \langle d \rangle)^e \subseteq K$. From assumption $\langle c \rangle^e \cap \langle d \rangle^e \subseteq K$. Since K is a strongly irreducible ideal, $\langle c \rangle^e \subseteq K$ or $\langle d \rangle^e \subseteq K$, so $\langle c \rangle \subseteq K^c$ or $\langle d \rangle \subseteq K^c$, that is $K^c$is a strongly irreducible ideal. Now, suppose $C \cap D = CD$ for any two ideals C, D of $\mathcal{R}_2$ and $\langle c \rangle \cap \langle d \rangle \subseteq K^c$. Then $\langle c \rangle \langle d \rangle = \langle cd \rangle \subseteq K^c$ which implies $(\langle cd \rangle)^e \subseteq K$, so $\langle f(cd) \rangle = \langle f(c) f(d) \rangle \subseteq K$. Then $\langle f(c) \rangle \cap \langle f(d) \rangle \subseteq K$, that is $\langle c \rangle^e \cap \langle d \rangle^e \subseteq K$ implies $\langle c \rangle^e \subseteq K$ or $\langle d \rangle^e \subseteq K$. Hence $\langle c \rangle \subseteq K^c$ or $\langle d \rangle \subseteq K^c$, that is K^c is a strongly irreducible ideal. □

The following result is an improvement of [4, Corollary 2.3(2)].

Theorem 2.8. *Every strongly irreducible ideal of a unique factorization domain can be generated by a prime power element or by a set of distinct prime elements.*

Proof. Let K be a strongly irreducible ideal of a unique factorization domain $\mathcal{R}$. By [4, Corollary 2.3], there is a family A of prime power elements generating K. If $A = \{p^n\}$ for some prime p, that is A contains exactly one element, then $K = \langle p^n \rangle$. Suppose that A contains more than one element. Now, let $B = \{{p_i}^{\beta_i} : {p_i}^{\alpha_i} \in A$ for some α_i in $\mathbb{Z}^+$ and β_i is the minimum of all γ_i where ${p_i}^{\gamma_i} \in K\}$. Then clearly B generates the ideal K. Suppose that ${p_r}^{\beta_r}, {p_s}^{\beta_s} \in B$ such that $\beta_r > 1$. Then clearly $\langle {p_r}^{\beta_r - 1} (p_r + {p_s}^{\beta_s}) \rangle = \langle {p_r}^{\beta_r - 1} \rangle \langle p_r + {p_s}^{\beta_s} \rangle = \langle {p_r}^{\beta_r - 1} \rangle \cap \langle p_r + {p_s}^{\beta_s} \rangle \subseteq K$. Then $\langle {p_r}^{\beta_r - 1} \rangle \subseteq K$ or $\langle p_r + {p_s}^{\beta_s} \rangle \subseteq K$, that is ${p_r}^{\beta_r - 1} \in K$ or $p_r \in K$. This means that β_r is not the minimum of all powers of p_r *in* K, contradiction. So that $\langle {p_r}^{\beta_r - 1} \rangle, \langle p_r + {p_s}^{\beta_s} \rangle \nsubseteq K$ which is contradiction with the assumption K is a strongly irreducible ideal. Therefore, K can be generated by a set of distinct prime elements. □

Remark 2.9. A nonprincipal ideal generated by at least two prime elements of $\mathcal{R}$ need not be prime. For example, x and $x + y^2$ are two prime elements in $\mathcal{R}$=$\mathbb{Q}[x, y]$ but the ideal $I = \langle x, x + y^2 \rangle \neq \mathcal{R}$ is not prime. Since $y^2 \in I$ but $y \notin I$.

3 S radical of ideals and rings

Consider a commutative ring $\mathcal{R}$. The strongly irreducible spectrum $X = S.\text{spec}(\mathcal{R})$ of $\mathcal{R}$ is the set $S.\text{spec}(\mathcal{R}) = \{I : I$ is a strongly irreducible ideal of $\mathcal{R}\}$. In this section, we study some properties of the S radical of an ideal of $\mathcal{R}$, which is also an ideal of $\mathcal{R}$. For any subset E of $\mathcal{R}$, the S variety of E, denoted by $V_S(E)$, is the set $V_S(E) = \{I \in S.\text{spec}(\mathcal{R}) : E \subseteq I\}$[12]. *Note that* $V_S(E) = V_S(\langle E \rangle)$.

Definition 3.1. [12]. Let K be an ideal of $\mathcal{R}$. Then the S radical of K denoted $S.rad(K)$, is defined to be the intersection of all strongly irreducible ideals of $\mathcal{R}$ containing K. Moreover, the strongly irreducible radical of $\mathcal{R}$ denoted by $S.rad(\mathcal{R})$ is the intersection of all strongly irreducible ideals of $\mathcal{R}$.

Example 3.2. Consider the ring $\mathcal{R}$=$\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle$=$\{0, 1, 2, 3, x, 1+x, 2+x, 3+x\}$. The proper ideals of $\mathcal{R}$ are K_1=$\{0\}$, K_2=$\langle 2 \rangle$=$\{0, 2\}$, K_3=$\langle x \rangle$=$\{0, x\}$, K_4=$\langle 2{+}x \rangle$=$\{0, 2{+}x\}$ and K_5=$\langle 2, x \rangle$=$\{0, 2, x, 2{+}x\}$. Clearly K_5 is the only strongly irreducible ideal of $\mathcal{R}$ which is the maximal ideal of $\mathcal{R}$. So that for each $1 \leq i \leq 5$, $S.rad(\mathcal{R})$=$S.rad(K_i) = K_5$. This means that for each $1 \leq i < 5$, $S.rad(K_i) \neq K_i$. Moreover, for each $1 \leq j < k \leq 5$, $S.rad(K_j) = S.rad(K_k)$ but $K_j \neq K_k$.

Proposition 3.3. *Consider the two ideals* K_1 *and* K_2 *of* $\mathcal{R}$. *Then* $S.rad(K_1K_2) \subseteq S.rad(K_1 \cap K_2)$= $S.rad(K_1) \cap S.rad(K_2)$.

Proof. Consider the following three subsets of $S.spec(\mathcal{R})$; $A = \{H \in S.spec(\mathcal{R}) : K_1 \subseteq H\}$, $B = \{H \in S.spec(\mathcal{R}) : K_2 \subseteq H\}$, and $C = \{H \in S.spec(\mathcal{R}) : K_1 \cap K_2 \subseteq H\}$. Now, if $H \in C$, then $K_1 \subseteq H$ or $K_2 \subseteq H$ since H is a strongly irreducible ideal. Hence $C \subseteq A \cup B$ and clearly $A \cup B \subseteq C$. Then S. $rad(K_1 \cap K_2)$= Intersection of all strongly irreducible ideals in C. That is S. $rad(K_1 \cap K_2) = \bigcap_{H \in C} H = \bigcap_{H \in A \cup B} H = (\bigcap_{H \in A} H) \bigcap (\bigcap_{H \in B} H) =$ S. $rad(K_1) \cap$ S. $rad(K_2)$. Now, since $K_1K_2 \subseteq K_1 \cap K_2$, then $S.rad(K_1K_2) \subseteq S.rad(K_1 \cap K_2)$, which completes the proof. □

Corollary 3.4. *Let* $I_1, I_2, \ldots, I_n$ *be distinct ideals of* $\mathcal{R}$. *Then*

$$S.rad\Big(\prod_{i=1}^{n} I_i\Big) \subseteq S.rad\Big(\bigcap_{i=1}^{n} I_i\Big) = \bigcap_{i=1}^{n} S.rad(I_i).$$

Now, we give an example showing that the inclusion in Proposition 3.3, can not be an equality.

Example 3.5. Consider the ring of integers $\mathbb{Z}$. Let I=J=$\langle 2 \rangle$, then IJ=$\langle 4 \rangle$ and $S.rad(IJ) = \langle 4 \rangle$ but $S.rad(I \cap J) = \langle 2 \rangle$. Therefore, $S.rad(IJ) \neq S.rad(I \cap J)$.

Proposition 3.6. *Let* K *be an ideal of* $\mathcal{R}$. *Then*

1. $S.rad(K) \subseteq rad(K) \subseteq Jac(K)$; *In particular* $S.rad(\mathcal{R}) \subseteq rad(\mathcal{R}) \subseteq Jac(\mathcal{R})$.
2. *If* $x \in S.rad(\mathcal{R})$, *then* $x^n = 0$ *for some* $n \in \mathbb{Z}^+$.
3. *If* $x \in S.rad(\mathcal{R})$, *then* $1 - xy$ *is a unit in* $\mathcal{R}$ *for all* $y \in \mathcal{R}$.

Proof. 1. It is clear.

2. Let $x \in S.rad(\mathcal{R})$. Then by part (1), $x \in rad(\mathcal{R})$. By [11, Proposition 1.8], $x^n = 0$ for some $n \in \mathbb{Z}^+$.
3. Let $x \in S.rad(\mathcal{R})$. Then by part (1), $x \in Jac(\mathcal{R})$. By [11, Proposition 1.9], $1 - xy$ is a unit in $\mathcal{R}$ for all $y \in \mathcal{R}$.

□

The converse of part 2 and part 3 of Proposition 3.6, need not be true, for example, in the ring $\mathbb{Z}_8$, $S.rad(\mathbb{Z}_8) = \{0\}$ and $2 \notin S.rad(\mathbb{Z}_8)$ however, $2^3 = 0$. On the other hand, $1 - 4y$ is a unit in $\mathcal{R}$ for all $y \in \mathcal{R}$ but $4 \notin S.rad(\mathbb{Z}_8)$.
The following is easy to prove.

Proposition 3.7. *Let* K_1 *and* K_2 *be two ideals of* $\mathcal{R}$. *Then* $S.rad(K_1) = S.rad(K_2)$ *if and only if* $V_S(K_1) = V_S(K_2)$.

Proposition 3.8. *If* K_1 *and* K_2 *are two ideals of* $\mathcal{R}$, *then* $S.rad(K_1 + K_2) = S.rad\big(S.rad(K_1) + S.rad(K_2)\big)$.

Proof. Let $H \in V_s(K_1 + K_2)$. Then $H \in S.spec(\mathcal{R})$ and $K_1 + K_2 \subseteq H$. Implies that $K_1 \subseteq S.rad(K_1) \subseteq H$ and $K_2 \subseteq S.rad(K_2) \subseteq H$. Then clearly $K_1 + K_2 \subseteq S.rad(K_1) + S.rad(K_2) \subseteq H + H = H$, implies $H \in V_s(S.rad(K_1) + S.rad(K_2))$. Let $H \in V_s(S.rad(K_1) + S.rad(K_2))$. Then $H \in S.spec(\mathcal{R})$ and $S.rad(K_1) + S.rad(K_2) \subseteq H$. Since $K_1 \subseteq S.rad(K_1)$ and $K_2 \subseteq S.rad(K_2)$, then $K_1 + K_2 \subseteq S.rad(K_1) + S.rad(K_2) \subseteq H$, implies $H \in V_s(K_1 + K_2)$. Therefore, $V_s(K_1 + K_2) = V_s(S.rad(K_1) + S.rad(K_2))$. Then by Proposition 3.7, $S.rad(K_1 + K_2) = S.rad(S.rad(K_1) + S.rad(K_2))$. □

Recall that two ideals I and J of $\mathcal{R}$ are coprime or comaximal if $I + J = \mathcal{R}$.

Proposition 3.9. *Consider an ideal* K *of* $\mathcal{R}$. *Then*

1. $S.rad(S.rad(K)) = S.rad(K)$.
2. $S.rad(K) = \mathcal{R}$ *if and only if* $K = \mathcal{R}$.
3. *If* K *is a strongly irreducible ideal, then* $S.rad(K) = K$.
4. *Let* K_1*and* K_2 *be two ideals of* $\mathcal{R}$ *such that* $S.rad(K_1)$ *and* $S.rad(K_2)$ *are coprime. Then* K_1*and* K_2 *are coprime ideals.*
5. *Let* $a, b \in \mathcal{R}$. *If* $X_s(a) \subseteq X_s(b)$, *then* $a \in S.rad(b)$.

Proof. 1. By [12, Proposition 3.1(4)], $V_s(K) = V_s(S.rad(K))$, then by Proposition 3.7, $S.rad(S.rad(K)) = S.rad(K)$.

2. Let $S.rad(K) = \mathcal{R}$. Since $S.rad(K) = \bigcap_{S \in V_s(K)} S$, then $\bigcap_{S \in V_s(K)} S = \mathcal{R}$. So that $\mathcal{R} \subseteq S$ for each $S \in V_s(K)$. Since $V_s(K) \subseteq S.spec(\mathcal{R})$ and $\mathcal{R} \notin S.spec(\mathcal{R})$, then $V_s(K) = \emptyset$. Therefore, $K = \mathcal{R}$. Conversely, let $K = \mathcal{R}$. Then by [12, Proposition 3.2(1)], $V_s(K) = \emptyset$. So that $S.rad(K) = \bigcap_{S \in V_s(K)} S = \mathcal{R}$, since it is intersection of an empty family of sets.
3. It is Obvious.
4. By Proposition 3.8 and part (2), $S.rad(K_1 + K_2) = S.rad(S.rad(K_1) + S.rad(K_2)) = S.rad(\mathcal{R}) = \mathcal{R}$.

5. Suppose that $a \notin S.rad(b)$. Then there is $S \in S.spec(\mathcal{R})$ such that $a \notin S$, but $b \in S$. Since $X_s(a) \subseteq X_s(b)$, this means that for each $S \in S.spec(\mathcal{R})$, if $a \notin S$, then $b \notin S$, Contradiction.

□

Proposition 3.10. *Let $\mathcal{R}$ be a unique factorization domain. Then*

1. *If K is a principal ideal of $\mathcal{R}$, then $S.rad(K) = K$.*
2. *If K is a finitely generated ideal of $\mathcal{R}$ and for any two ideals C and D, $S.rad(C + D) = S.rad(C) + S.rad(D)$, then $S.rad(K) = K$.*

Proof. 1. Let K be a principal ideal of $\mathcal{R}$. Then $K = \langle p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} \rangle = \langle \prod_{i=1}^{k} p_i^{\alpha_i} \rangle$ for some distinct primes p_i with $k \geq 1$ and $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$. Clearly $\langle p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} \rangle = \bigcap_{i=1}^{k} \langle p_i^{\alpha_i} \rangle$. By Corollary 3.4, $S.rad(K) = S.rad\Big(\bigcap_{i=1}^{k} \langle p_i^{\alpha_i} \rangle\Big) = \bigcap_{i=1}^{k} S.rad\langle p_i^{\alpha_i} \rangle$. By [4, Theorem 2.2(iii)], for each i the ideal $\langle p_i^{\alpha_i} \rangle$ is strongly irreducible, so by Proposition 3.9(3), $S.rad\langle p_i^{\alpha_i} \rangle = \langle p_i^{\alpha_i} \rangle$. Consequently $\bigcap_{i=1}^{k} S.rad\langle p_i^{\alpha_i} \rangle = \bigcap_{i=1}^{k} \langle p_i^{\alpha_i} \rangle = K$. Therefore, $S.rad(K) = K$.

2. Let $K = \langle a_1, a_2, \dots, a_n \rangle$ be a finitely generated ideal of $\mathcal{R}$. Clearly $K = \langle a_1 \rangle + \langle a_2 \rangle + \dots + \langle a_n \rangle = \sum_{i=1}^{n} \langle a_i \rangle$. Then $S.rad(K) = S.rad\Big(\sum_{i=1}^{n} \langle a_i \rangle\Big) = \sum_{i=1}^{n} S.rad\langle a_i \rangle$. By part (1), for each i we have $S.rad\langle a_i \rangle = \langle a_i \rangle$. Therefore, $S.rad(K) = K$.

□

4 Strongly irreducible spectrum $S.spec(\mathcal{R})$

Let $\mathcal{R}$ be a commutative ring. From [12, Proposition 3.2], the family $F = \{V_s(E) : E \subseteq \mathcal{R}\}$ is closed under finite union and arbitrary intersection, so that F satisfy the axioms for closed sets of a topology on $X = S.spec(\mathcal{R})$. This topology is said to be the S Zariski topology on $S.spec(\mathcal{R})$ or $S.spec(\mathcal{R})$ topology. So that the collection of subsets $X_s(E) = S.spec(\mathcal{R}) \setminus V_s(E)$ of $S.spec(\mathcal{R})$ for all subsets E of $\mathcal{R}$ makes $S.spec(\mathcal{R})$ into a topological space. In this section, we obtain some results concerning these concepts. Moreover, we introduce and discuss the concept of the CS Zariski topology on $CS.spec(\mathcal{R})$.

The following remark is a direct consequence of Propositions 3.1 and 3.2 [12].

Remark 4.1. Let K_1, K_2 be two ideals of $\mathcal{R}$. Then

1. $V_s(K_i) \subseteq V_s(K_1K_2)$ and $X_s(K_1K_2) \subseteq X_s(K_i)$, $i \in \{1, 2\}$.
2. $V_s(K_1 \cup K_2) \subseteq V_s(K_i)$ and $X_s(K_i) \subseteq X_s(K_1 \cup K_2)$, $i \in \{1, 2\}$.
3. $V_s(K_1 + K_2) \subseteq V_s(K_i)$ and $X_s(K_i) \subseteq X_s(K_1 + K_2)$, $i \in \{1, 2\}$.
4. For each c, $d \in \mathcal{R}$, if $c|d$, then $V_s(c) \subseteq V_s(d)$ and $X_s(d) \subseteq X_s(c)$.

Remark 4.2. Let K_1, K_2be two ideals of $\mathcal{R}$. Then $V_s(K_1) \cap V_s(K_2) = V_s(K_1 \cup K_2) = V_s(K_1 + K_2)$ equivalently $X_s(K_1) \cup X_s(K_2) = X_s(K_1 \cup K_2) = X_s(K_1 + K_2)$.

Proof. By [12, Proposition 3.2(2)], $V_s(K_1) \cap V_s(K_2) = V_s(K_1 \cup K_2)$. Now, let $S \in V_s(K_1 \cup K_2)$. Then $K_1 \cup K_2 \subseteq S$. Then $K_1 \subseteq S$ and $K_2 \subseteq S$. Then $K_1 + K_2 \subseteq S + S = S$. Then $S \in V_s(K_1 + K_2)$. Therefore, $V_s(K_1 \cup K_2) \subseteq V_s(K_1 + K_2)$. Let $S \in V_s(K_1 + K_2)$. Then $K_1 + K_2 \subseteq S$. Then $K_1 \subseteq S$ and $K_2 \subseteq S$. Then $K_1 \cup K_2 \subseteq S$. Then $S \in V_s(K_1 \cup K_2)$. Therefore, $V_s(K_1 + K_2) \subseteq V_s(K_1 \cup K_2)$.

□

Proposition 4.3. *Let $N_1,\ N_2 \in S.spec(\mathcal{R})$. Then*

1. *$\{N_1\}$ is a closed subset of $S.spec(\mathcal{R})$ if and only if N_1 is a maximal ideal of $\mathcal{R}$.*
2. *$\overline{\{N_1\}} = V_s(N_1)$.*
3. *$N_2 \in \overline{\{N_1\}}$ if and only if $N_1 \subseteq N_2$.*
4. *The space $S.spec(\mathcal{R})$ is T_0.*
5. *For each subset $E \subseteq \mathcal{R}$, if $V_s(E) \neq \emptyset$, then there is a maximal ideal N_1 of $\mathcal{R}$ contained in $V_s(E)$.*
6. *If $\langle 0\rangle \in S.spec(\mathcal{R})$, then every nonempty open subset of $S.spec(\mathcal{R})$ contains the zero ideal.*

Proof. 1. Assume that $\{N_1\}$ is closed subset of $S.spec(\mathcal{R})$, then there an ideal I such that $V_s(I) = \{N_1\}$. So that $I \subseteq N_1$. If the ideal N_1 is not a maximal ideal, then there is a maximal ideal L such that $I \subseteq N_1 \subset L$, implies $L \in V_s(I)$, contradiction. Now, let N_1 be a maximal ideal. Therefore, $V_s(N_1) = \{N_1\}$, this means $\{N_1\}$ is a closed subset of $S.spec(\mathcal{R})$.

2. Since $V_s(N_1)$ is the smallest closed subset of $S.spec(\mathcal{R})$ containing N_1, then $\overline{\{N_1\}} = \bigcap_{N_1 \in V_s(E)} V_s(E) = V_s(N_1)$.
3. It is obvious.
4. Let $N_1 \neq N_2$. Then $N_1 \nsubseteq N_2$ or $N_2 \nsubseteq N_1$, say $N_1 \nsubseteq N_2$. Then $N_2 \notin \overline{\{N_1\}}$, otherwise by part (3), $N_1 \subseteq N_2$. By part (2), $N_2 \notin V_s(N_1)$. Therefore, $N_2 \in X_s(N_1)$ and $N_1 \notin X_s(N_1)$.
5. Let $V_s(E) \neq \emptyset$ for some $E \subset \mathcal{R}$. Then $\langle E\rangle$ contained in a maximal ideal N_1. Consequently $N_1 \in V_s(E)$.
6. Let $\langle 0\rangle \in S.spec(\mathcal{R})$ and $X_s(E) \neq \varnothing$ be an open subset of $S.spec(\mathcal{R})$. Then $E \neq \{0\}$. This means $\langle 0\rangle \notin V_s(E)$. Then $\langle 0\rangle \in X_s(E)$.

□

Recall that an ideal $N \in S.spec(\mathcal{R})$ is said to be the minimum ideal of $S.spec(\mathcal{R})$ if $N \subseteq I$, for each $I \in S.spec(\mathcal{R})$.

Remark 4.4. Let $H \in S.spec(\mathcal{R})$. Then the following are equivalent:

1. H is the minimum ideal of $S.spec(\mathcal{R})$.
2. $V_s(H) = S.spec(\mathcal{R})$ equivalently $X_s(H) = \emptyset$.
3. H is a generic point of $S.spec(\mathcal{R})$.

Proof. (1) ⇒ (2) It is obvious.

(2) ⇒ (3) . Let $V_s(H) = S.spec(\mathcal{R})$. Then by Proposition 4.3(2), $\overline{\{H\}} = V_s(H) = S.spec(\mathcal{R})$. Therefore, H is a generic point. (3) ⇒ (1) Let H be a generic point. Then $\overline{\{H\}} = S.spec(\mathcal{R})$. By Proposition 4.3(2), $\overline{\{H\}} = V_s(H) = S.spec(\mathcal{R})$. So that $H \subseteq S$ for each $S \in S.spec(\mathcal{R})$. Therefore, H is the minimum ideal in $S.spec(\mathcal{R})$. □

Remark 4.5. Let a be an element of $\mathcal{R}$. Then

1. $V_s(a) = \varnothing$ equivalently $X_s(a) = \mathrm{S}.spec(\mathcal{R})$ if and only if a is a unit element of $\mathcal{R}$.
2. If $V_s(E) = \mathrm{S}.spec(\mathcal{R})$ equivalently $X_s(E) = \varnothing$, then the elements of $E \subseteq \mathcal{R}$ are nilpotent.
3. If $r \neq 0$ is not a zero divisor regular element, then $V_s(r) = \varnothing$.
4. Let E be the set of all nontrivial idempotent elements of $\mathcal{R}$. If $E \neq \emptyset$, then $V_s(E) = \varnothing$.

Proof. 1. See [4, pp. 152, Theorem 4.1(iii)].
2. Let $V_s(E) = \text{S}.spec(\mathcal{R})$. By [12, Proposition 3.1(1)], for each $a \in E$, $V_s(E) \subseteq V_s(a)$. So that $\text{S}.spec(\mathcal{R}) = V_s(E) \subseteq V_s(a)$, implies $V_s(a) = \text{S}.spec(\mathcal{R})$. By [12, Proposition 3.4)], a is a nilpotent element. Therefore, the elements of $E \subseteq \mathcal{R}$ are nilpotent.
3. Let $r \neq 0$ be a nonzero divisor regular element. It is well known that r is a unit. Then by part (1), $V_s(r) = \varnothing$.
4. Let $E \neq \emptyset$. Then clearly there are two elements $e(\neq 0, 1)$, $1 - e \in E$. Then $\langle E \rangle = \mathcal{R}$. Therefore, $V_s(E) = \varnothing$.

□

Definition 4.6. [13]. If K is an ideal of $\mathcal{R}$ and $x + K$ is an idempotent element of $\mathcal{R}/K$, implies that $x + K = e + K$ for some idempotent e of $\mathcal{R}$, then we say idempotents lift modulo K.
Now, we have to mention that, if e is an idempotent element of $\mathcal{R}$ and K is an ideal of $\mathcal{R}$, then $\overline{e} = e + K$ is an idempotent of $\mathcal{R}/K$. The converse is not true, for example, consider the ideal $K = \langle 6 \rangle$ of $\mathbb{Z}$. Then $4 + \langle 6 \rangle$ is an idempotent element of the ring $\mathbb{Z}/\langle 6 \rangle$, however, 4 is not an idempotent element of $\mathbb{Z}$.

Proposition 4.7. *[13] If K is a nil ideal of $\mathcal{R}$, then idempotents lift modulo K.*

By Proposition 4.7 and the fact that $S.rad(\mathcal{R})$ is a nil ideal**,** we obtain the following result.

Proposition 4.8. *If $K = S.rad(\mathcal{R})$, then idempotents lift modulo K.*

Proposition 4.9. *Let $C \subseteq S.spec(\mathcal{R})$. Then the following are equivalent:*

1. *C is a clopen subset of $S.spec(\mathcal{R})$.*
2. *$C = V_s(e)$ for some idempotent e of $\mathcal{R}$.*

Proof. (1) $\Rightarrow$ (2). Assume that $C = V_s(H)$ is a clopen subset of $S.spec(\mathcal{R})$, where H is an ideal of $\mathcal{R}$. Then there is an ideal L such that $V_s(H) \cap V_s(L) = \emptyset$ and $V_s(H) \cup V_s(L) = S.spec(\mathcal{R})$. By Proposition 4.2, $V_s(H) \cap V_s(L) = V_s(H \cup L) = V_s(H + L)$, then $V_s(H + L) = \emptyset$. So that $H + L = \mathcal{R}$. Since $HL \subseteq H \cap L$, then if $S \in S.spec(\mathcal{R})$ contains one of H or L, then S contains HL and since $V_s(H) \cup V_s(L) = S.spec(\mathcal{R})$, then $HL \subseteq S.rad(\mathcal{R})$. Hence there exist $c \in H$ and $1 - c \in L$ and $c(c - 1) \in S.rad(\mathcal{R})$. This means $\overline{c} = c + S.rad(\mathcal{R})$ is an idempotent of $\mathcal{R}/S.rad(\mathcal{R})$. By Proposition 4.8, there is an idempotent $e \in \mathcal{R}$ with $\overline{c} = \overline{e}$. So that $V(c) = V(e)$ and $V(1 - c) = V(1 - e)$. Therefore, $C = V(e)$.
(2) $\Rightarrow$ (1) Let $e \in \mathcal{R}$ be an idempotent element. Then $1 - e$ is also an idempotent element. Then $V_s(e) \cap V_s(1 - e) = \emptyset$, otherwise there is $S \in S.spec(\mathcal{R})$ containing both e, $1 - e$, which implies $1 \in S$, contradiction. Now, we show that $V_s(e) \cup V_s(1 - e) = S.spec(\mathcal{R})$. Since $(e) + (1 - e) = \mathcal{R}$, then the two ideals (e), $(1 - e)$ are coprime. So that $(e) \cap (1 - e) = (e)(1 - e) = (0) \subseteq S$, for each $S \in S.spec(\mathcal{R})$, implies $(e) \subseteq S$ or $(1 - e) \subseteq S$. So that $S \in V_s(e)$ or $S \in V_s(1 - e)$. Therefore, $V_s(e) \cup V_s(1 - e) = S.spec(\mathcal{R})$, which means $C = V_s(e)$ is a clopen subset of $S.spec(\mathcal{R})$. □

Proposition 4.10. *For a ring $\mathcal{R}$, the following statements are equivalent:*

1. *There is an idempotent $e \in \mathcal{R}$ with $e \neq 0, 1$.*
2. *$\mathcal{R}$ is the direct product of two nonzero rings.*
3. *$S.spec(\mathcal{R})$ is disconnected.*
4. *$Spec(\mathcal{R})$ is disconnected.*

Proof. (1) $\Rightarrow$ (2) It is well known that if $e \neq 0, 1$ is an idempotent element of $\mathcal{R}$, then $\mathcal{R} = e\mathcal{R} \times (1 - e)\mathcal{R}$ [14, p. 746]
(2) $\Rightarrow$ (3) Suppose that $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$. Then $(1_{\mathcal{R}_1}, 0)$ is an idempotent element of $\mathcal{R}$ and $(1_{\mathcal{R}_1}, 0) \neq (1_{\mathcal{R}_1}, 1_{\mathcal{R}})$, $(0, 0)$. By Proposition 4.9, $V_s(1_{\mathcal{R}_1}, 0)$ is a clopen nonempty proper subset of $S.spec(\mathcal{R})$.
(3) $\Rightarrow$ (1) Follows from Proposition 4.9. (4) $\Leftrightarrow$ (1) The proof follows directly from Exercise 15 in [11, p. 14]. □

Proposition 4.11. *Let $\mathcal{R}$ be a ring. Then*

1. *The space $S.spec(\mathcal{R})$ is compact;*
2. *If $S.rad(\mathcal{R})$ is a strongly irreducible ideal, then $S.spec(\mathcal{R})$ is irreducible.*
3. *If $\mathcal{R}$ is Noetherian, then the space $S.spec(\mathcal{R})$ is Noetherian.*

Proposition 4.12. *For a ring $\mathcal{R}$, the following statements are equivalent:*

1. *There are no comparable strongly irreducible ideals of $\mathcal{R}$.*
2. *If $S\in$S.$spec(\mathcal{R})$, then S is a maximal ideal of $\mathcal{R}$*
3. *If $S\in$S.$spec(\mathcal{R})$, then $V_s(S) = \{S\}$.*
4. *If $S\in$S.$spec(\mathcal{R})$, then $\overline{\{S\}} = \{S\}$, equivalently $\{S\}$ is a closed subset of* S.$spec(\mathcal{R})$.
5. *The $S.spec(\mathcal{R})$ topology T_s is T_1.*

Proof. $(1) \Rightarrow (2)$. Let $S\in$S.$spec(\mathcal{R})$. Then there is no maximal ideal M such that $S \subset M$. This means that S is a maximal ideal.
$(2) \Rightarrow (3)$. It is obvious.
$(3) \Rightarrow (4)$. Let $S\in$S.$spec(\mathcal{R})$. By part(3), $V_s(S) = \{S\}$. This means that $\{S\}$ is a closed subset of S.$spec(\mathcal{R})$. Consequently, $\overline{\{S\}}=\{S\}$. $(4) \Rightarrow (1)$. Let H, L be two distinct strongly irreducible ideals of $\mathcal{R}$. Then clearly $H,\ L \in S.spec(\mathcal{R})$. So that $\{H\},\ \{L\}$ are closed subsets of $S.spec(\mathcal{R})$. By Proposition 4.3 (1), H and L are maximal ideals of $\mathcal{R}$. Then $H \not\subseteq L$ and $L \not\subseteq H$.
$(4) \Leftrightarrow (5)$ It is obvious. □

Corollary 4.13. *If the space S.$spec(\mathcal{R})$ is T_1, then*

1. *Either $\mathcal{R}$ is a field or the zero ideal is not a strongly irreducible ideal.*
2. *If the S.$spec(\mathcal{R})$ topology is an Alexandrov topology, then it is the discrete topology.*

Proof. 1. Suppose that $\mathcal{R}$ is not a field. Then $\mathcal{R}$ has a nonzero maximal ideal M. If the zero ideal is a strongly irreducible ideal, then being $\langle 0\rangle \subset M$ we get a contradiction with Proposition 4.12.

2. Let $S.spec(\mathcal{R})$ topology be an Alexandrov topology, $I\in S.spec(\mathcal{R})$ and $B=\{\ X_s\in T_s;\ I\in X_s\}$. Then $\bigcap_{X_s\in B} X_s \in T_s$. Since the space is T_1, then for each $J\in S.spec(\mathcal{R})$ there is $X_s \in B$ such that $J \notin X_s$. Then $J \notin \bigcap_{X_s\in B} X_s$, for each $J \in$ S.$spec(\mathcal{R}) \setminus\{I\}$. Therefore, $\bigcap_{X_s\in B} X_s = \{I\}$ which is an open set. Hence the topology is discrete.

□

Remark 4.14. The $S.spec(\mathcal{R})$ topology need not be an Alexandrov topology. For example, let $\mathcal{R} = \mathbb{Z}_{\langle 3\rangle}$ (the localization of $\mathbb{Z}$ at the prime ideal $\langle 3\rangle$). The non-zero ideals of $\mathcal{R}$ form the chain

$$\langle 3\rangle \supset \langle 3^2\rangle \supset \langle 3^3\rangle \supset \ldots.$$

Then $\bigcup_{k\in\mathbb{Z}^+} V_S(\langle 3^k\rangle) = S.spec(\mathcal{R}) \setminus \{\langle 0\rangle\}$. Hence $\bigcup_{k\in\mathbb{Z}^+} V_S(\langle 3^k\rangle)$ is not a closed subset of $S.spec(\mathcal{R})$.

Finally we study completely strongly irreducible ideals and we introduce the CS.$spec(\mathcal{R})$ topology.

Remark 4.15. The concept of completely strongly irreducible ideal is independent of each of the following concepts Maximal ideal, prime ideal, and weakly prime ideal, for example, the ideal $\langle p\rangle$ of $\mathbb{Z}$, p is a prime number is a maximal ideal consequently it is a prime ideal and a weakly prime ideal but it is not a completely strongly irreducible ideal. Moreover, the ideal $\langle 3^4\rangle$ of the ring $\mathbb{Z}_{3^5}$ is a completely strongly irreducible ideal but it is not a Maximal ideal, not a prime ideal, and not a weakly prime ideal.

Definition 4.16. The *completely strongly irreducible spectrum* of $\mathcal{R}$ is the set

$$X = \text{CS.spec}(\mathcal{R}) = \{I : I \text{ is a completely strongly irreducible ideal of } \mathcal{R}\}.$$

For any subset $E \subseteq \mathcal{R}$, the *CS variety* of E, denoted $V_{\text{cs}}(E)$, is

$$V_{\text{cs}}(E) = \{I \in \text{CS.spec}(\mathcal{R}) \mid E \subseteq I\}.$$

Note that $V_{\text{cs}}(E) = V_{\text{cs}}(\langle E\rangle)$ and $\text{CS.spec}(\mathcal{R}) \subseteq \text{S.spec}(\mathcal{R})$.
The family

$$\mathcal{F} = \{V_{\text{cs}}(E) : E \subseteq \mathcal{R}\}$$

is closed under finite unions and arbitrary intersections, hence satisfies the axioms for the closed sets of a topology on $X = \text{CS.spec}(\mathcal{R})$. This topology is called the *CS Zariski topology* on $\text{CS.spec}(\mathcal{R})$ (or the $\text{CS.spec}(\mathcal{R})$ topology).

Consequently, the collection of open subsets

$$X_{\text{cs}}(E) = \text{CS.spec}(\mathcal{R}) \setminus V_{\text{cs}}(E), \quad E \subseteq \mathcal{R},$$

makes $\text{CS.spec}(\mathcal{R})$ into a topological space.

In the following example we illustrate $CS.spec\,(\mathcal{R})$.

Example 4.17. Consider the ring $\mathcal{R} = \mathbb{Z}_{\langle p\rangle}$, the localization of $\mathbb{Z}$ at the prime ideal $\langle p\rangle$. Then

$$\text{S.spec}(\mathcal{R}) = \{\langle 0\rangle, \langle p\rangle, \langle p^2\rangle, \langle p^3\rangle, \dots\}$$

but

$$\text{CS.spec}(\mathcal{R}) = \text{S.spec}(\mathcal{R}) \setminus \{\langle 0\rangle\},$$

since $\bigcap_{n\in\mathbb{Z}^+} \langle p^n\rangle = \langle 0\rangle$ while $\langle p^n\rangle \not\subseteq \langle 0\rangle$ for every $n \in \mathbb{Z}^+$.

For each $n \in \mathbb{Z}^+$ we have

$$V_S\big(\langle p^n\rangle\big) = V_{\text{cs}}\big(\langle p^n\rangle\big) = \{\langle p\rangle, \langle p^2\rangle, \dots, \langle p^n\rangle\}$$

and

$$V_S\big(\langle 0\rangle\big) = \text{S.spec}(\mathcal{R}), \qquad V_{\text{cs}}\big(\langle 0\rangle\big) = \text{CS.spec}(\mathcal{R}).$$

Regarding $\text{CS.spec}(\mathcal{R})$ as a subspace of $\text{S.spec}(\mathcal{R})$, we obtain

$$\overline{\text{CS.spec}(\mathcal{R})} = \text{S.spec}(\mathcal{R}),$$

so $\text{CS.spec}(\mathcal{R})$ is dense in $\text{S.spec}(\mathcal{R})$. The space $\text{S.spec}(\mathcal{R})$ is sober, T_0 and quasi-compact, hence by [15, Proposition 4] it is a spectral space. Because $\text{CS.spec}(\mathcal{R})$ has no generic point, it is not spectral and therefore not sober.

Proposition 4.18. *The* $\text{CS.spec}(\mathcal{R})$ *topology is an Alexandrov topology.*

Proof. Let $\{V_{\text{cs}}(K_\alpha)\}_{\alpha\in\Lambda}$ be a family of closed subsets of $\text{CS.spec}(\mathcal{R})$, where each K_α is an ideal of $\mathcal{R}$. It is well known that $\bigcup_{\alpha\in\Lambda} V_{\text{cs}}(K_\alpha) \subseteq V_{\text{cs}}\Big(\bigcap_{\alpha\in\Lambda} K_\alpha\Big)$. Now, let $I \in V_{\text{cs}}\big(\bigcap_{\alpha\in\Lambda} K_\alpha\big)$. Then $\bigcap_{\alpha\in\Lambda} K_\alpha \subseteq I$. Since I is completely strongly irreducible, $K_\alpha \subseteq I$ for some $\alpha \in \Lambda$; hence $I \in V_{\text{cs}}(K_\alpha)$ for that α, and so $I \in \bigcup_{\alpha\in\Lambda} V_{\text{cs}}(K_\alpha)$. Therefore , $\bigcup_{\alpha\in\Lambda} V_{\text{cs}}(K_\alpha) = V_{\text{cs}}\Big(\bigcap_{\alpha\in\Lambda} K_\alpha\Big)$, which shows that an arbitrary union of closed sets is closed. Thus the $\text{CS.spec}(\mathcal{R})$ topology is an Alexandrov topology. □

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