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PSEUDO-SPECTRAL METHOD WITH VIETA-LUCAS POLYNOMIALS FOR SOLVING STOCHASTIC VOLTERRA INTEGRAL EQUATIONS: ANALYSIS AND APPLICATIONS

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Abstract This paper develops a novel pseudo-spectral collocation method based on shifted Vieta–Lucas polynomials for the numerical solution of nonlinear stochastic Volterra integral equations. The proposed approach introduces Vieta–Lucas polynomials as a global orthogonal basis to approximate the unknown stochastic process, providing an alternative to classical spectral bases such as Legendre or Chebyshev polynomials. A direct collocation formulation is employed, which avoids the construction of stochastic operational matrices and significantly simplifies the resulting numerical scheme. Deterministic integrals are evaluated using Legendre–Gauss quadrature, while stochastic integrals are approximated through an Itô discretization framework. A rigorous theoretical analysis is established, including boundedness of the spectral coefficients, convergence results, and error estimates in the mean-square sense under standard Lipschitz and boundedness assumptions. The proposed method combines spectral accuracy with computational efficiency and stability. Numerical experiments are presented to validate the theoretical findings, illustrating the convergence behavior of the method and demonstrating its effectiveness compared with existing approaches for solving stochastic Volterra integral equations.

1 Introduction

Stochastic Differential Equations (SDEs) are differential equations that include terms involving stochastic processes, typically modeled using Brownian motion (Wiener process) [1, 4, 17, 28]. These equations are widely used in fields such as physics, finance, biology, and engineering to describe systems influenced by randomness. SDEs have wide applications, including:

- Finance: Black-Scholes model for option pricing, interest rate models [3].
- Physics: Langevin equations for Brownian motion and molecular dynamics .
- Biology: Population dynamics under environmental noise.
- Engineering: Control systems with noise and signal processing.

In recent years, stochastic functional equations have emerged in various applications. For instance, stochastic integral equations play a crucial role in the stochastic modeling of reactor dynamics [16], the study of biological population growth [10], and the analysis of automatic control systems, often leading to delay-differential equations. In many cases, obtaining exact solutions

to stochastic functional equations is infeasible. Hence, developing numerical techniques to approximate their solutions is essential [7,8,15,18,20,22,23,25,27,29,32] and in [10,11,14,33,34]. Recently, there has been growing interest among researchers in stochastic Volterra integral equations (SVIEs). In this work, we propose a pseudo-spectral collocation method based on shifted Vieta–Lucas polynomials for solving nonlinear stochastic Itô–Volterra integral equations. The unknown stochastic process is approximated by a truncated expansion in terms of shifted Vieta–Lucas polynomials, which serve as a global spectral basis. The choice of this polynomial family is motivated by its favorable analytical properties, including a simple recursive structure, good approximation capability, and fast convergence behavior. These characteristics allow the construction of stable and computationally efficient numerical algorithms for nonlinear stochastic problems. By exploiting these properties, the proposed method provides an accurate and efficient framework for approximating the solutions of stochastic integral equations. To illustrate the formulation of the method, we consider the following stochastic equation.

$$Y(t) = h(t) + \int_0^t k_1(t,s) f_1(Y(s)) ds + \int_0^t k_2(t,s) f_2(Y(s)) dW(s), \quad t \in [0,T], \tag{1.1}$$

where the kernels k_1 and k_2 are assumed continuous on $[0,T] \times [0,T]$, the nonlinearities f_1 and f_2 are stochastic processes adapted to the filtred probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfy Lipschitz conditions and linear growth condition to guarantee the existence and uniqueness of the solution. $h(t)$ is a deterministic function, $W(t)$ is a standard Brownian motion and Y_t is the unknown process to be determined. The main contributions of this work are summarized as follows: First, we propose, shifted Vieta–Lucas polynomials as a global orthogonal spectral basis for the numerical solution of stochastic Volterra integral equations (SVIEs). In contrast to classical bases such as Legendre or Chebyshev polynomials, Vieta–Lucas polynomials possess distinctive recurrence and structural properties that enhance stability and provide an efficient representation of nonlinear stochastic dynamics within a unified approximation framework. Second, unlike many existing approaches that rely on stochastic operational matrices, block-pulse functions, wavelets, or piecewise approximations, the present method adopts a direct pseudo-spectral collocation strategy. This eliminates the need for constructing operational matrices and reduces algebraic complexity while preserving high-order (spectral-type) accuracy. Third, the proposed scheme integrates global polynomial approximation, Gauss quadrature for deterministic integrals, and Itô discretization for stochastic terms within a consistent theoretical framework. A rigorous convergence and error analysis are established under standard boundedness, Lipschitz, and linear growth assumptions. A new collocation strategy is proposed by using Jacobi collocation points with the roots of Vieta-Lucas polynomials. This hybrid selection of nodes contributes to improved numerical stability and accuracy. A key theoretical contribution of this paper is the establishment of uniform boundedness for the coefficients arising in the Vieta-Lucas series representation of the approximate solution. This property guarantees the stability of the approximation and allows accurate solutions to be obtained using only a limited number of basis functions and also the error estimates for the Vieta–Lucas polynomial approximation is established. The structure of this article is as follows: Section 2 provides a review of essential results on stochastic calculus. In Section 3, we introduce a function approximation and collocation method to solve the original problem. The collocation method to solve Equation (1.1) using Vieta-Lucas polynomials is introduced in Section 4. Section 5 is dedicated to the convergence and error analysis of the proposed method. The accuracy of the method is demonstrated through numerical examples in Section 5. Finally, the article concludes with a summary in the last section.

2 Preliminaries

In this section, we outline the key definitions and properties essential for the subsequent analysis.

Definition 2.1 ([12], Brownian Motion Process). A real-valued stochastic process $W_t,\ t \in [0,T]$ is called a Brownian motion if it satisfies the following properties:

(i) **(Independence of increments)**: The increment $W_t - W_s$, for $t > s$, is independent of the past, i.e., of W_u for $0 \le u \le s$, or equivalently independent of $\mathcal{H}_s$, the σ-field generated by $\{W_u,\ u \le s\}$.

(ii) **(Normal increments)**: The increment $W_t - W_s$ follows a normal distribution with mean 0 and variance $t - s$.

(iii) **(Continuity of paths)**: The process W_t has continuous sample paths almost surely.

Definition 2.2. Let $\{\mathcal{M}_t\}_{t\geq 0}$ be an increasing family of σ-algebras on Ω. A process $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^d$ is called *$\mathcal{M}_t$-adapted* if, for each $t \geq 0$, the function $\omega \mapsto f(t,\omega)$ is $\mathcal{M}_t$-measurable.

Definition 2.3. Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that:

(i) The function $(t,\omega) \mapsto g(t,\omega)$ is $\mathcal{B} \times \mathcal{M}$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $[0,\infty)$ and $\mathcal{M}$ is the σ-algebra on Ω.

(ii) The process g is adapted to $\mathcal{M}_t$, where $\mathcal{M}_t$ is the σ-algebra generated by $\{W_s,\ s \leq t\}$.

(iii) $\mathbb{E}\left[\int_S^T g^2(t,\omega)dt\right] < \infty$.

Definition 2.4 (Itô Integral). Let $g \in \mathcal{V}(S,T)$. The Itô integral of g with respect to Brownian motion is defined as

$$\int_S^T g(t,\omega)dW_t(\omega) = \lim_{n\to\infty} \int_S^T \varphi_n(t,\omega)dW_t(\omega), \quad \text{(limit in } L^2(P)) \tag{2.1}$$

where φ_n is a sequence of elementary functions such that

$$\mathbb{E}\left[\int_S^T (g(t,\omega) - \varphi_n(t,\omega))^2 dt\right] \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Property 2.5 (Itô Isometry). Let $g \in \mathcal{V}(S,T)$. Then,

$$\mathbb{E}\left[\left(\int_S^T g(t,\omega)dW_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_S^T g^2(t,\omega)dt\right]. \tag{2.3}$$

Property 2.6 (Itô Approximation [9]). Suppose that g is a measurable stochastic process. Then, the Itô integral

$$\int_a^b g(\xi)\, dW(\xi)$$

can be approximated by

$$\int_a^b g(\xi)\, dB(\xi) \simeq \sum_{i=1}^{k} g(\xi_{i-1})\left(W(\xi_i) - W(\xi_{i-1})\right),$$

where k is a positive integer and

$$\xi_i = a + \frac{(b-a)i}{k}, \quad i = 0, 1, \ldots, k,$$

and the limit is in $L^2(P)$.

2.1 Legendre–Gauss Quadrature Formula

The integral of a function $V(s)$ over $[a,b]$ can be approximated as

$$\int_a^b V(s)\, ds \simeq \frac{b-a}{2} \sum_{p=0}^{M} w_p\, V(\Upsilon_p), \tag{2.4}$$

where

$$\Upsilon_p = \frac{b-a}{2}\,\xi_p + \frac{b+a}{2}, \quad p = 0, 1, \ldots, M,$$

and $\{\xi_p\}_{p=0}^{M}$ and $\{w_p\}_{p=0}^{M}$ are the Legendre–Gauss nodes and weights of order $M+1$. Let L_{M+1} denote the Legendre polynomial of degree $M+1$ on $[-1,1]$. The nodes ξ_p are the zeros of L_{M+1} and the corresponding weights are given by

$$w_p = \frac{2}{\left(1-\xi_p^2\right)\left[L'_{M+1}(\xi_p)\right]^2}, \quad p = 0, 1, \ldots, M.$$

Lemma 2.7. *The $(M+1)$-point Legendre–Gauss quadrature on $[a,b]$ is exact for all polynomials of degree less than or equal to $2M+1$.*

Theorem 2.8 ([30]). *Suppose that $V(x)$ is sufficiently smooth on $[a,b]$. Then the error of the Legendre–Gauss quadrature satisfies*

$$\left| \int_a^b V(x)\,dx - \frac{b-a}{2} \sum_{p=0}^{M} w_p V(\Upsilon_p) \right| \leq \frac{(b-a)^{2M+1}(M!)^4}{(2M+1)\left[(2M)!\right]^3}\,\rho_{2M},$$

where ρ_{2M} is an upper bound of $\left|V^{(2M)}(x)\right|$ on $[a,b]$.

Theorem 2.9. *[2, 5] let x, Ψ and χ be real continuous functions defined in $[a,b]$, $\chi(t) \geq 0$ for $t \in [a,b]$. We suppose that on $[a,b]$ we have the inequality*

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)x(s)ds$$

Then

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)\Psi(s)exp\left(\int_s^t \chi(u)du\right)ds.$$

3 Function Approximation and Collocation method

3.1 Function approximation

The weighted inner product and its corresponding norm in $L_w^2([a,b])$ are defined as

$$\langle f, g\rangle_w = \int_a^b f(x)g(x)w(x)\,dx, \quad \forall f, g \in L_w^2([a,b]), \tag{3.1}$$

$$\|f\|_{L_w^2} = \sqrt{\langle f, f\rangle_w}. \tag{3.2}$$

Theorem 3.1. *[6] For any function $f \in L_w^2([a,b])$ and any integer $n \in \mathbb{N}$, there exists a unique polynomial $p_N^* \in P_n$ (the space of polynomials of degree at most n) such that*

$$\|f - p_N^*\|_{L_w^2} = \inf_{p_n \in P_n} \|f - p_n\|_{L_w^2}. \tag{3.3}$$

This polynomial is given by

$$p_N^*(x) = \sum_{k=0}^{N} \alpha_k \psi_k(x), \quad \textit{where} \quad \alpha_k = \frac{\langle f, \psi_k\rangle_w}{\|\psi_k\|_w^2}, \tag{3.4}$$

and $\{\psi_k\}_{k=0}^{N}$ is an orthogonal basis.

The polynomial $p_N^*(x)$ is referred to as the best approximation of f, it is precisely the L_w^2–orthogonal projection of f onto the space spanned by $\{\psi_k\}_{k=0}^{N}$.

3.2 Collocation Method

The collocation method is a versatile numerical technique for solving differential equations, including stochastic differential equations (SDEs), partial differential equations (PDEs), and integral equations. The fundamental idea is to approximate the exact solution by projecting it onto a finite-dimensional subspace spanned by carefully chosen basis functions. The method then enforces the governing equation to be satisfied exactly at a selected set of points, known as collocation points. For example, consider the model equation:

$$\varphi - A\varphi = f, \tag{3.5}$$

where A is a linear or nonlinear operator. We select a finite-dimensional subspace X_N with basis functions $\psi_1, \ldots, \psi_N$. The approximate solution is written as

$$\varphi_N(t) = \sum_{j=1}^{N} c_j \, \psi_j(t), \quad t \in I, \tag{3.6}$$

where the coefficients c_j are determined by imposing the condition that the residual of (3.5) vanishes at the chosen collocation points. In deterministic problems, collocation points are often chosen according to Gauss quadrature nodes (e.g., Gauss–Legendre, Gauss–Chebyshev), which maximize accuracy for polynomial-based approximations. In stochastic settings, the method can be combined with polynomial chaos expansion (PCE), spectral techniques, or other orthogonal polynomial approximations to represent the stochastic processes. One of the strengths of the collocation method is its adaptability. By selecting different sets of basis functions, it can handle a wide variety of problems elliptic, parabolic, and hyperbolic PDEs under different types of boundary or initial conditions. Moreover, adaptive strategies can be employed, refining the set of collocation points in regions where the solution exhibits steep gradients, oscillations, or singular behavior. From a computational perspective, the collocation method offers spectral-level accuracy when smooth solutions are involved, making it highly attractive for high-dimensional problems in areas such as fluid dynamics, financial modeling, and quantum mechanics. However, the method also comes with challenges. Choosing inappropriate collocation points or basis functions can result in loss of accuracy, instability, or slow convergence. Ill-conditioning in the resulting algebraic system may arise, especially for large N, requiring stabilization techniques such as preconditioning or the use of alternative solvers like iterative Krylov subspace methods. The collocation method stands as a powerful, flexible, and high-accuracy tool in numerical analysis. When paired with advanced quadrature rules and stochastic expansion techniques, it becomes particularly effective for solving complex deterministic and stochastic problems across diverse scientific and engineering domains. The collocation approximation developed in this section is formulated in the mean-square sense. The unknown stochastic process $Y(t, \omega)$ is approximated by a truncated spectral expansion in the time variable, while all stochastic integrals are interpreted in the Itô sense and analyzed in $L^2(P)$. In particular, the spectral truncation is applied only to the temporal dependence of $Y(t)$; the stochastic integrals are not expanded spectrally. Owing to the linearity of the Itô integral and the square-integrability of the truncated approximation, the substitution of the spectral approximation into the stochastic integral is justified in the mean-square sense, consistently with the Itô isometry recalled in Section 2. The Volterra structure of the integral equation plays a fundamental role in the collocation formulation. The presence of the upper integration limit t ensures causality, since at each collocation point the integrals involve only values of the approximation on the interval $[0, t_j]$. Moreover, standard collocation stability issues are mitigated by the combination of the Volterra operator, the boundedness of the kernels, and the use of orthogonal polynomial bases. Stochastic terms do not introduce additional instability, since all convergence and error estimates are carried out in the mean-square framework. The general collocation strategy introduced in this section is specialized in the next section by selecting shifted Vieta–Lucas polynomials as the approximation basis, whose orthogonality and recurrence properties make them particularly suitable for Volterra-type stochastic integral equations.

4 Numerical Method for Solving SDE (1.1)

Let consider the stochastic integral equation (1.1), applying the collocation method to solve the stochastic integral equation (1.1) using Vieta-Lucas (VL) polynomials. These polynomials exhibit a wide range of fascinating properties and have been extensively explored in number theory and algebraic geometry. VL polynomials also play a crucial role in various mathematical areas, including continued fractions, Diophantine equations, and algebraic geometry. Their deep structural properties and broad applications make them a subject of ongoing research in modern mathematics.

4.1 Vieta-Lucas Polynomials

Vieta–Lucas polynomials constitute a relatively less explored yet interesting class of polynomials that arise from the interplay between classical Lucas sequences and Vieta's formulas. These polynomials inherit structural properties from both recurrence relations associated with Lucas sequences and algebraic relationships between polynomial roots and coefficients described by Vieta's formulas. Consequently, Vieta–Lucas polynomials provide a useful framework for analyzing polynomial structures through recurrence-based representations. Their mathematical properties make them suitable for studying connections between linear recurrence sequences and polynomial approximations, with potential applications in several areas of mathematics and computational analysis. In the following, we briefly summarize the main definitions and fundamental properties of Vieta–Lucas polynomials. The Vieta-Lucas polynomials $L_n(t)$ are defined by

$$V_n(t) = 2\cos(n\theta), \quad \theta = \cos^{-1}(t/2), \quad |t| \leq 2. \tag{4.1}$$

They satisfy the recurrence relation

$$V_0(t) = 2, \quad V_1(t) = t, \quad V_n(t) = tV_{n-1}(t) - V_{n-2}(t), \quad n \geq 2. \tag{4.2}$$

Also, $V_n(t)$ can be determined by means of the following explicit formula for the corresponding power series

$$V_n(t) = \sum_{i=0}^{\lceil n/2 \rceil} (-1)^i \frac{n\Gamma(n-i)}{\Gamma(i+1)\Gamma(n+1-2i)} t^{n-2i}, \quad n = \{2, 3, \dots\}. \tag{4.3}$$

where $\lceil n/2 \rceil$ is the ceiling function. Furthermore, $V_n(t)$ are orthogonal polynomials with respect to the following inner product:

$$\langle V_m(t), V_n(t) \rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-t^2}} V_m(t)V_n(t)dt = \begin{cases} 0, & m \neq n \neq 0, \\ 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0. \end{cases} \tag{4.4}$$

Shifted Vieta–Lucas Polynomials

The shifted Vieta–Lucas polynomials, denoted by $V_n^*(t)$, are a novel class of orthogonal polynomials defined on the interval $[0, 1]$.

Definition 4.1. The shifted Vieta–Lucas polynomials of degree n on $[0, 1]$ can be obtained from $V_n(t)$ as follows

$$V_n^*(t) = V_n(4t-2) = V_{2n}(2\sqrt{t}). \tag{4.5}$$

Additionally, they satisfy the recurrence relation

$$V_{n+1}^*(t) = (4t-2)L_n^*(t) - V_{n-1}^*(t), \quad n = 1, 2, \dots, \tag{4.6}$$

with the initial polynomials

$$V_0^*(t) = 2, \quad V_1^*(t) = 4t - 2. \tag{4.7}$$

Furthermore, the analytical explicit formula for $V_n^*(t)$ is given by

$$V_n^*(t) = 2^n \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{4^{n-i}\Gamma(2n-i)}{\Gamma(i+1)\Gamma(2n-2i+1)} t^{n-i}, \quad n = 2, 3, \ldots. \tag{4.8}$$

The polynomials $V_n^*(t)$ satisfy the following orthogonality condition

$$\langle V_m^*(t), V_n^*(t) \rangle = \int_0^1 V_m^*(t) V_n^*(t) w(t) dt = \begin{cases} 0, & m \neq n \neq 0, \\ 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0. \end{cases} \tag{4.9}$$

where the weight function is given by $w(t) = \sqrt{t - t^2}$. For using VL polynomials over $[0, T]$ ($T > 0$), the VL polynomials, $\mathcal{VL}_{n,T}(t)$, can be introduced as

$$\mathcal{VL}_{n,T}(t) = V_n\left(\frac{4t}{T} - 2\right). \tag{4.10}$$

Then, we can write:

$$\begin{gathered} \mathcal{VL}_{n,T}(t) = \left(\frac{4t}{T} - 2\right) \mathcal{VL}_{n-1,T}(t) - \mathcal{VL}_{n-2,T}(t), \\ \mathcal{VL}_{0,T}(t) = 2, \quad \mathcal{VL}_{1,T}(t) = \frac{4t}{T} - 2, \quad n \geq 2. \end{gathered} \tag{4.11}$$

4.2 Approximation Using Shifted Vieta-Lucas Polynomials

If the function $Y(t)$ is Lebesgue-square-integrable on the interval $[0, 1]$, it can be represented as a linear combination of the independent power functions in terms of $V_n^*(t)$

$$Y(t) = \sum_{i=1}^{\infty} c_i V_i^*(t), \tag{4.12}$$

where c_i are the coefficients to be determined. In practical applications, only the first N terms of the series are considered, leading to the approximation

$$Y_N(t) = \sum_{i=0}^{N} c_i V_i^*(t). \tag{4.13}$$

The coefficients c_i can be computed as follows

$$c_i = \frac{1}{\delta_i \pi} \int_{-2}^{2} Y\left(\frac{t+2}{4}\right) V_i(t) \frac{dt}{\sqrt{4 - t^2}}, \tag{4.14}$$

or equivalently,

$$c_i = \frac{1}{\delta_i \pi} \int_0^1 Y(t) V_i^*(t) \frac{dt}{\sqrt{t - t^2}}, \tag{4.15}$$

where

$$\delta_i = \begin{cases} 4, & i = 0, \\ 2, & i = 1, 2, \ldots, N. \end{cases} \tag{4.16}$$

4.3 Derivation of the Numerical Method

Let us consider the stochastic differential equation (SDE)

$$Y(t) = h(t) + \int_0^t k_1(t, s) f_1(Y(s)) ds + \int_0^t k_2(t, s) f_2(Y(s)) dW(s), \quad t \in [0, T]. \tag{4.17}$$

The goal is to solve this equation numerically using the collocation method with shifted Lucas polynomials over the interval $[0, T]$. The idea is to expand the solution as

$$Y(t) = \sum_{i=0}^{\infty} c_i V_i^*(t), \tag{4.18}$$

where $c_i, i = 0, 1, \ldots$ are unknown coefficients. To determine these coefficients, we approximate the solution using a truncated series

$$Y_N(t) = \sum_{i=0}^{N} c_i V_i^*(t). \tag{4.19}$$

Substituting (4.19) into Eq (4.17), we obtain

$$\sum_{i=0}^{N} c_i V_i^*(t) = Y_0 + \int_0^t k_1(t,s) f_1 \left(\sum_{i=0}^{N} c_i V_i^*(s) \right) ds + \int_0^t k_2(t,s) f_2 \left(\sum_{i=0}^{N} c_i V_i^*(s) \right) dW(s). \tag{4.20}$$

Next, we collocate Eq (4.20) at the nodes t_j, $j = 0 : N$, which are the Jacobi points or Vieta-Lucas roots of order $N + 1$, we get

$$\begin{aligned} Y(t_j) = \sum_{i=0}^{N} c_i V_i^*(t_j) = Y_0 &+ \int_0^{t_j} k_1(t_j,s) f_1 \left(\sum_{i=0}^{N} c_i V_i^*(s) \right) ds \\ &+ \int_0^{t_j} k_2(t_j,s) f_2 \left(\sum_{i=0}^{N} c_i V_i^*(s) \right) dW(s). \end{aligned} \tag{4.21}$$

To approximate the integrals in equation (4.21), we use:

- Legendre-Gauss quadrature for the deterministic integral

$$\int_0^{t_j} k_1(t_j,s) f_1(Y(s)) ds \simeq \frac{t_j}{2} \sum_{k=1}^{n_1} k_1 \left(t_j, \frac{t_j}{2}(s_k + 1) \right) f_1 \left(\sum_i c_i V_i^* \left(\frac{t_j}{2}(s_k + 1) \right) \right). \tag{4.22}$$

- Itô approximation for the stochastic integral

$$\int_0^{t_j} k_2(t_j,s) f_2(Y(s)) dW(s) \simeq \sum_{k=1}^{n_2} k_2(t_j, d_k) f_2 \left(\sum_i c_i V_i^*(d_k) \right) (W(d_{k+1}) - W(d_k)). \tag{4.23}$$

where $\{d_k\}_{k=1}^{n_2}$ is a subdivision of the interval $[0, t_j]$. Substituting equations Eqs (4.22)-(4.23) into Eq (4.21) which provides a nonlinear system of algebraic equations, which can be solved by any appropriate approach, To solve the examples, we will solve the obtained system by Newton's method with tolerance 10^{-6}.

5 Convergence Analysis of the New Proposed Technique

In this section, we investigate the convergence properties of the proposed pseudo-spectral collocation method based on shifted Vieta–Lucas polynomials. First, Theorem 5.1 establishes the convergence of the truncated approximation $Y_N(t)$ to the exact solution $Y(t)$ under suitable smoothness assumptions, and provides bounds on the decay of the spectral coefficients. Based on this result, Theorem 5.2 derives an explicit error estimate in the L^2_ω-norm and determines the convergence rate of the approximation. Finally, le last Theorem 5.3 extends the analysis to the complete numerical scheme applied to the stochastic Volterra integral equation, proving that the numerical solution converges to the exact solution in the mean-square sense under suitable Lipschitz and boundedness conditions on the kernels and nonlinear terms.

Theorem 5.1. *Assume that $Y(t) \in L^2_\omega[0,1]$ and that its second derivative is bounded, i.e.,*

$$Y''(t) \leq L, a.s \quad t \in [0,1], \tag{5.1}$$

where L is a constant. The approximation $Y_N(t)$ in (4.19) converges to $Y(t)$ as $N \to \infty$. Furthermore, the unknown coefficients satisfy

$$|c_i| \leq \frac{L}{4i(i-1)}, \quad i > 2. \tag{5.2}$$

Proof. Let $Y(t)$ be Lebesgue-square-integrable on the interval $[0,1]$ and $Y_N(t)$ its approximation using Vieta-Lucas polynomials. Using the variable change $4t - 2 = 2\cos(\theta)$, the integral in equation (4.14) transforms into

$$c_i = \frac{2}{\delta_i \pi} \int_0^\pi Y\left(\frac{2 + 2\cos(\theta)}{4}\right) \cos(i\theta) d\theta. \tag{5.3}$$

Double integration by parts leads to

$$c_i = \frac{1}{4i\delta_i \pi} \int_0^\pi Y''\left(\frac{1 + \cos(\theta)}{2}\right) \sin(\theta) D_i(\theta) d\theta, \tag{5.4}$$

where

$$D_i(\theta) = \frac{\sin((i-1)\theta)}{i-1} - \frac{\sin((i+1)\theta)}{i+1}. \tag{5.5}$$

Applying absolute values, we obtain

$$|c_i| \leq \frac{L}{8i\pi} \int_0^\pi |D_i(\theta)| d\theta. \tag{5.6}$$

Using algebraic computations, we deduce

$$|c_i| \leq \frac{L}{4i(i-1)}, \quad i > 2. \tag{5.7}$$

□

Theorem 5.2 (Convergence Rate). *If $Y(t)$ satisfies the conditions of Theorem 5.1, and $Y_N(t) = \sum_{i=1}^N c_i V_i^*(t)$, then the error in $L^2_\omega[0,1]$ is given by*

$$\|Y(t) - Y_N(t)\|^2_\omega \leq \frac{\pi L^2}{24N^3}. \tag{5.8}$$

Proof. The error is given by

$$e_N = \|Y(t) - Y_N(t)\|^2_\omega = \int_0^1 |Y(t) - Y_N(t)|^2 \omega(t) dt. \tag{5.9}$$

Using (4.12), (4.13), and (4.9), we obtain

$$e_N \leq 2\pi \sum_{i=N+1}^{\infty} |c_i|^2. \tag{2.34}$$

From (5.7), it follows that

$$e_N \leq \frac{2\pi L^2}{16} \sum_{i=N+1}^{\infty} \frac{1}{i^2(i-1)^2}. \tag{2.35}$$

Since

$$i^2(i-1)^2 > (i-1)^4,$$

we deduce

$$e_N \leq \frac{\pi L^2}{8} \sum_{i=N+1}^{\infty} \frac{1}{(i-1)^4}. \tag{5.10}$$

Using the integral test,

$$\sum_{i=N+1}^{\infty} \frac{1}{(i-1)^4} \leq \int_N^{\infty} \frac{1}{x^4}\, dx = \frac{1}{3N^3}, \tag{5.11}$$

we obtain

$$e_N \leq \frac{\pi L^2}{24N^3}. \tag{2.36}$$

Thus, we conclude the convergence result. □

The following theorem shows the sufficient conditions to make the estimate error tend to zero.

Theorem 5.3. *Assume that $Y(t)$ is the exact solution of the nonlinear Itô–Volterra SIE (1.1) and $\hat{Y}(t)$ is the approximate solution given by the new proposed technique. Suppose that f_1 and f_2 are Lipschitzian functions with constants K_1 and K_2, respectively, and the functions k_i, $i = 1, 2$ satisfy the following bounded conditions*

$$|k_i(t,s)| \leq M_i, \quad i = 1,2, \quad \forall (t,s) \in [0,T]^2, \tag{5.12}$$

where M_i, $i = 1, 2$ are positive constants. Then, we get

$$||Y(t) - \hat{Y}(t)||_{L^2(0,T)} = \left(E|Y(t) - \hat{Y}(t)|^2\right)^{1/2} \to 0 \quad as \quad N \to \infty. \tag{5.13}$$

Proof. The exact solution is given by

$$Y(t) = h(t) + \int_0^t k_1(t,s) f_1(Y(s))ds + \int_0^t k_2(t,s) f_2(Y(s))dW(s), \tag{5.14}$$

and the numerical solution is as follows

$$\hat{Y}(t) = h_N(t) + \int_0^t k_1(t,s) f_1(\hat{Y}(s))ds + \int_0^t f_2(t,s) f_2(\hat{Y}(s))dW(s) - \epsilon_1 - \epsilon_2, \tag{5.15}$$

where ϵ_1 and ϵ_2 are the associated errors of Gauss quadrature (I) and Itô approximation (II) defined as follows

$$\epsilon_1(t) = \int_0^t k_1(t,s) f_1(Y(s))ds - \frac{t}{2} \sum_{k=1}^{n_1} k_1\left(t, \frac{t}{2}(s_k+1)\right) f_1\left(\sum_i c_i V_i^*\left(\frac{t}{2}(s_k+1)\right)\right). \tag{5.16}$$

and ϵ_2 is given as follows

$$\epsilon_2(t) = \int_0^t k_2(t,s) f_2(Y(s))dW(s) - \sum_{k=1}^{n_2} k_2(t,d_k) f_2\left(\sum_i c_i V_i^*(d_k)\right)(W(d_{k+1}) - W(d_k)). \tag{5.17}$$

Then, we have

$$\begin{aligned} |Y(t) - \hat{Y}(t)|^2 \leq \Bigg[&|h(t) - h_N(t)| + \int_0^t k_1(t,s)(f_1(Y(s)) - f_1(\hat{Y}(s)))ds \\ &+ \int_0^t k_2(t,s)(f_2(Y(s)) - f_2(\hat{Y}(s)))dW(s) + \epsilon_1 + \epsilon_2 \Bigg]^2. \end{aligned} \tag{5.18}$$

Since $(a+b+c)^2 \leq 4a^2 + 4b^2 + 4c^2$, we obtain

$$\begin{aligned} |Y(t) - \hat{Y}(t)|^2 \leq &4|h(t) - h_N(t)|^2 + 4\left|\int_0^t k_1(t,s)(f_1(Y(s)) - f_1(\hat{Y}(s)))ds\right|^2 \\ &+ 4\left|\int_0^t k_2(t,s)(f_2(Y(s)) - f_2(\hat{Y}(s)))dW(s) + \epsilon_1 + \epsilon_2)\right|^2. \end{aligned} \tag{5.19}$$

Using Hölder's inequality, Itô isometry, Lipschitz conditions, and the boundedness of f_1 and f_2 and the fact that $(a+b)^2 \leq a^2+b^2$, we obtain

$$\begin{aligned} E|Y(t)-\hat{Y}(t)|^2 \leq & 4E|h(t)-h_N(t)|^2 + 4K_1\int_0^t |k_1(t,s)|^2 E|Y(s)-\hat{Y}(s)|^2 ds \\ & + 8K_2\int_0^t |k_2(t,s)|^2 E|Y(s)-\hat{Y}(s)|^2 ds + 16E(|\epsilon_1|^2+|\epsilon_2|^2) \\ & \leq (4K_1M_1^2+8K_2M_2^2)\int_0^t E|Y(s)-\hat{Y}(s)|^2 ds \\ & + 4|h(t)-h_N(t)|^2 + 16E|\epsilon_1|^2 + 16E|\epsilon_2|^2. \end{aligned} \tag{5.20}$$

By taking in Gronwall's lemma 2.9 [2, 5], $\chi(t) = (4K_1M_1^2+8K_2M_2^2)$, $\Psi_N(t) = 4|h(t)-h_N(t)|^2 + 16E|\epsilon_1(t)|^2 + 16E|\epsilon_2(t)|^2$ and $x(t) = E|Y(t)-\hat{Y}(t)|^2$, we get

$$\begin{aligned} E|Y(t)-\hat{Y}(t)|^2 \leq & \, 4|h(t)-h_N(t)|^2 + 16E|\epsilon_1(t)|^2 + 16E|\epsilon_2(t)|^2 + (4K_1M_1^2+8K_2M_2^2) \\ & \times \int_0^t \left(4|h(s)-h_N(s)|^2 + 16E|\epsilon_1(s)|^2 + 16E|\epsilon_2(s)|^2\right) e^{(4K_1M_1^2+8K_2M_2^2)(t-s)} ds. \end{aligned} \tag{5.21}$$

The Inequality (5.21) provides a pointwise bound for the mean-square error at each $t \in [0,T]$. By inequality (5.21) and the fact that $|h(t)-h_N(t)|^2 \to 0 \quad \text{as} \quad N \to \infty$ and $E|\epsilon_1|^2$ and $E|\epsilon_2|^2 \to 0 \quad \text{as} \quad n_1, n_2 \to +\infty$, we get

$$E|Y(t)-\hat{Y}(t)|^2 \to 0 \quad \text{as} \quad N \to +\infty. \tag{5.22}$$

6 Numerical Experiments

In the numerical examples, the proposed method is implemented using two different sets of collocation points, the first set consists of Jacobi-Gauss nodes, specific choices of the parameters α and βcorrespond to the Legendre and Chebyshev nodes. The second type of collocation points is given by the Vieta-Lucas nodes. The algorithm associated with the numerical proposed approach was performed using Matlab 2014 software with hp laptop. The Newton method is used with tolerance 10^{-6}. The numerical experiments presented in this section are designed to validate the theoretical findings established in Section 5 and to illustrate the practical implementation of the Vieta–Lucas pseudo-spectral collocation scheme introduced in Section 4. In particular, they demonstrate the spectral accuracy and mean-square convergence behavior of the proposed method as the truncation order N increases. Special attention is given to the influence of key numerical parameters, including the choice of collocation points (Jacobi–Gauss nodes or Vieta–Lucas roots), the order of Gauss quadrature for deterministic integrals, and the stochastic time discretization used for the Itô integral. These experiments highlight the robustness and stability of the method with respect to these parameters and confirm that, when chosen consistently with the truncation degree, the numerical errors behave in accordance with the theoretical convergence estimates. Furthermore, comparisons with existing methods reported in the literature emphasize the effectiveness and competitiveness of the proposed Vieta–Lucas collocation approach for solving nonlinear stochastic Volterra integral equations. The numerical experiments are designed to validate different aspects of the proposed method. They assess the accuracy of the scheme through comparison with exact solutions and analysis of mean-square error decay, evaluate stability and robustness under varying collocation parameters and longer time intervals, demonstrate efficiency for nonlinear stochastic problems, investigate the influence of stochastic perturbations on numerical stability, and illustrate the applicability of the method to more complex problems, including comparisons with existing numerical techniques reported in the literature.

Example 6.1. [12] Consider the stochastic differential equation

$$dY(t) = \lambda Y(t)dt + \mu Y(t)dW(t), \quad Y(0) = Y_0, t \in [0,1]. \tag{6.1}$$

The exact solution is:

$$Y(t) = Y_0 \exp\left((\lambda - \frac{1}{2}\mu^2)t + \mu W(t)\right). \tag{6.2}$$

Table 1 presents the errors obtained by the proposed method by taking $\lambda = -3, \mu = 0.1$ and $Y_0 = 0.1$ with $N = 6, N_1 = 20, N_2 = 50$ and $M = 50$ simulations. Table 2 displays the absolute errors using Vieta-Lucas collocation points of the presented method for Example 6.1. Figures 1-3 show the exact solution, approximate solution and absolute errors obtained by the proposed method using Jacobi collocation points. Exact solution, approximate solution and absolute error using Vieta-Lucas collocation points with $N = 15$ are presented in Figure 4. Figure 5 illustrates the log-log decay of the L^2 error at final point $t = 1$ versus N using $\alpha = -1/2, \beta = 1/2$ and Vieta-Lucas nodes. Table 3 shows the mean-square error at selected points $t \in [0, 1]$ with $N = 5$ and $M = 20$ simulations.

Table 1. Absolute Errors for different values of α and β for Example 6.1

t	$\alpha = 0, \beta = 0$	$\alpha = 1/2, \beta = 1/2$	$\alpha = -1/2, \beta = 1/2$
0.1	0.0002543919765	0.0002006769421	0.0002946608258
0.2	0.000436366854	0.000459780236	0.0003917736946
0.3	0.0005193530079	0.0004957473094	0.0006437202145
0.4	0.0004072513677	0.0001821905257	0.000000043060103
0.5	0.000000630070	0.0004084144071	0.0004704640305
0.6	0.0004521141598	0.0001601845235	0.0001968200646
0.7	0.0004825116305	0.0008744926478	0.0005573626446
0.8	0.00001361183505	0.0006716417844	0.0008884087002
0.9	0.0005662107437	0.0004282546957	0.0001485084446
1	0.0003350166209	0.001552438807	0.000002581740468

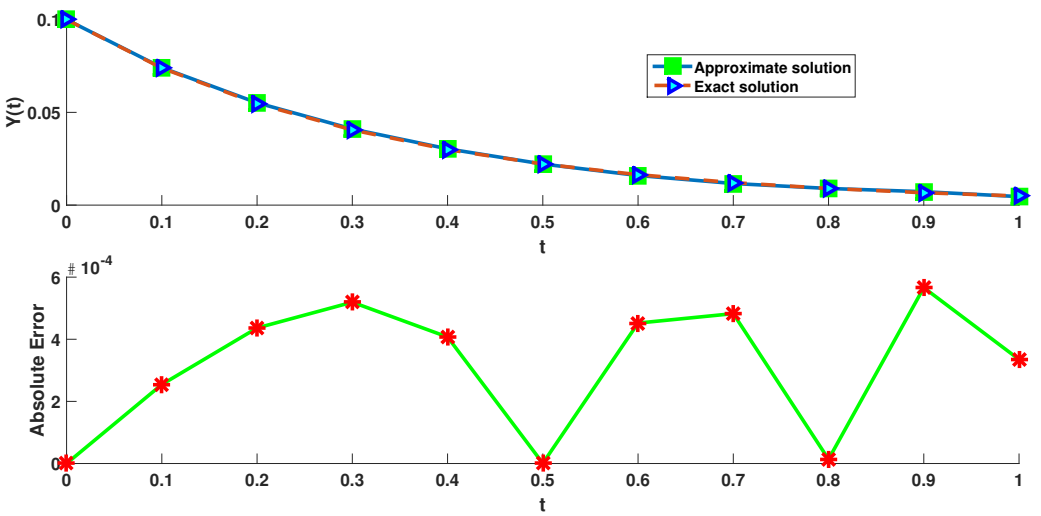


Figure 1. Exact solution, approximate solution and absolute error for $\alpha = 0$ and $\beta = 0$ of Example 6.1

Example 6.2. [12] Let consider the stochastic integral equation

$$Y(t) = Y_0 + a^2 \int_0^t cosh(Y(s)sinh^3(Y(s)ds - a \int_0^t sinh^2(Y(s)dW(s), t \in [0, T], \tag{6.3}$$

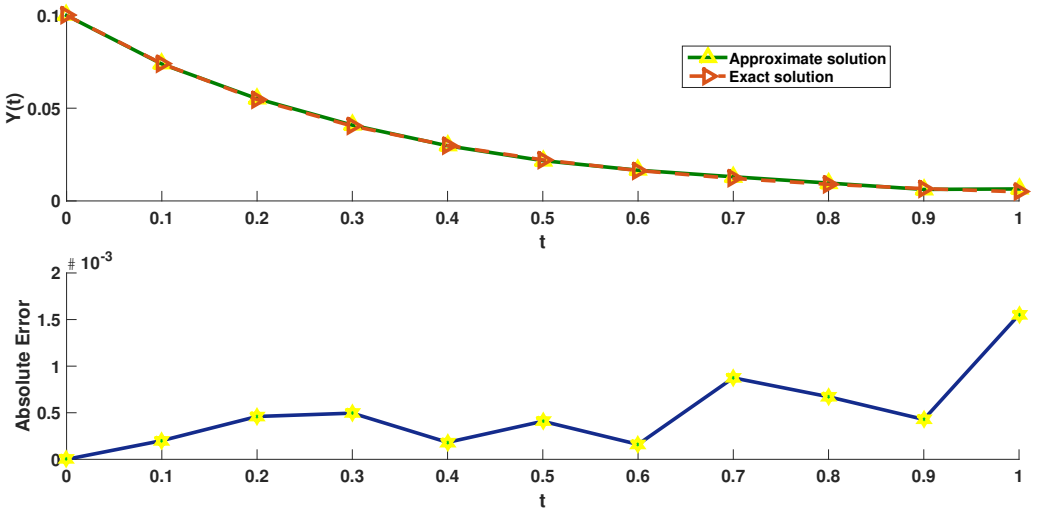


Figure 2. Exact solution, approximate solution and absolute error for $\alpha = 1/2$ and $\beta = 1/2$ of Example 6.1

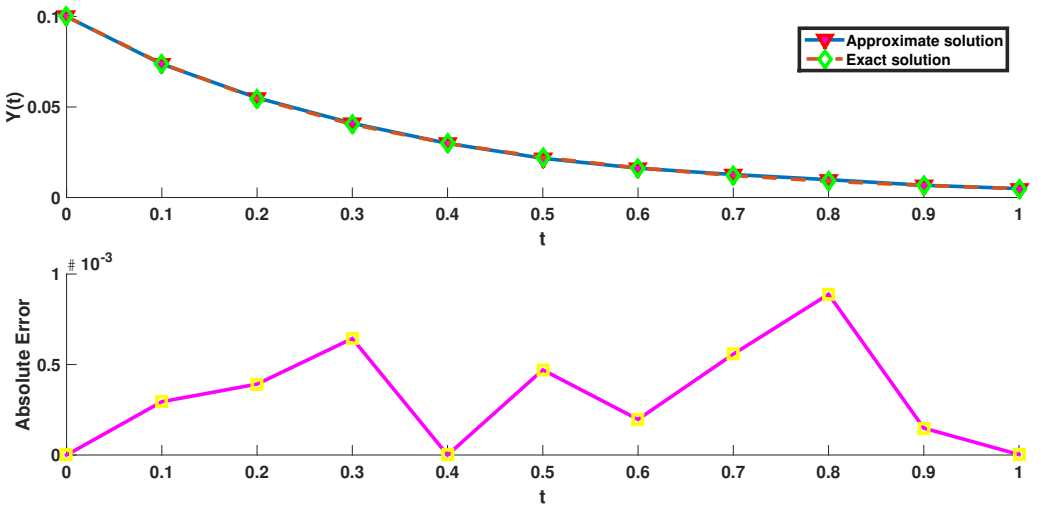


Figure 3. Exact solution, approximate solution and absolute error for $\alpha = 1/2$ and $\beta = -1/2$ of Example 6.1

Table 2. Absolute Errors using Vieta-Lucas collocation points for Example 6.1

t	$N = 6$	$N = 10$	$N = 15$
0.1	0.0006413546705	0.0004090663405	0.00005189328132
0.2	0.0005469159676	0.0000271960009	0.00002442506792
0.3	0.000153033481	0.0008270651965	0.0001170610103
0.4	0.0007987511249	0.001021972775	0.00009953717343
0.5	0.0007957177255	0.0001480686926	0.0004809140858
0.6	0.0001330060145	0.0005449335481	0.0008101764514
0.7	0.000662202041	0.0001320289242	0.0003448664077
0.8	0.0005968042063	0.00009564489433	0.0003249689278
0.9	0.0004527374067	0.001238947128	0.0004246044488
1	0.0002166620025	0.0002803024002	0.0002367706199

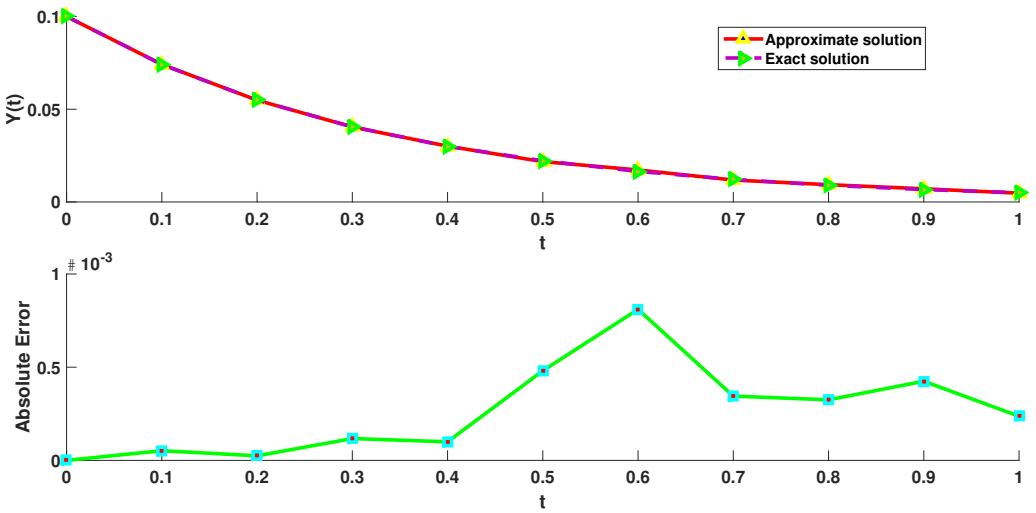


Figure 4. Exact solution, approximate solution and absolute error using Vieta-Lucas collocation points with $N = 15$ of Example 6.1

Table 3. Pointwise mean-square error for different values of t for Example 6.1

t	$\alpha = \beta = 0$	$\alpha = \beta = 1/2$	$\alpha = -1/2, \beta = 1/2$	Vieta-Lucas nodes
0.1	1.529 e-5	1.374 e-5	1.959 e-5	1.943 e-5
0.2	8.155 e-6	1.368 e-5	1.330 e-5	1.1103 e-5
0.3	1.029 e-5	1.236 e-5	5.518 e-6	1.410 e-5
0.4	1.094 e-5	1.133 e-5	8.586 e-6	1.020 e-5
0.5	6.681 e-6	7.578 e-6	1.094 e-5	2.576 e-6
0.6	8.182 e-6	5.341 e-6	9.210 e-6	7.147 e-6
0.7	4.914 e-6	8.762 e-6	5.373 e-6	1.053 e-5
0.8	4.499 e-6	1.567 e-5	8.981 e-6	6.955 e-6
0.9	8.921 e-6	7.939 e-6	1.996 e-5	8.008 e-6
1	6.165 e-5	1.525 e-6	3.421 e-5	4.101 e-5

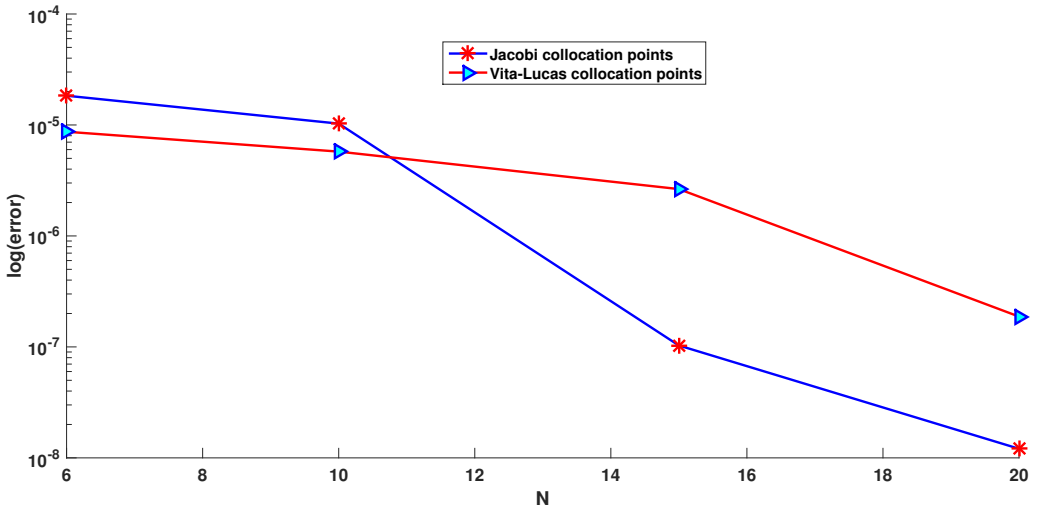


Figure 5. Log-log plot of the L^2-error at final point $t = 1$ versus N for $\alpha = -1/2$ and $\beta = 1/2$ and Vieta-Lucas nodes of Example 6.1

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where $Y(0) = Y_0$ is the initial condition and a is a paramater. The exact solution is given by $Y(t) = arccot(aW(t) + coth(Y_0))$. Table 4 gives the absolute errors of the proposed method using different values of Jacobi collocation points for $N = 5, Y_0 = 1/20, a = 1/20, N_1 = 20, N_2 = 50, T = 1$ and with $M = 100$ simulations. Table 5 displays the absolute errors given by proposed method by using Vieta-Lucas collocation points with various values of N. The errors obtained by the proposed method using Vieta-Lucas collocation with $N = 10$ and using Jacobi Gauss collocation points with $N = 5, Y_0 = 1/20, a = 1/20, N_1 = 20, N_2 = 50, T = 50$ and with $\alpha = \beta = 0$ are represented in Figures 6-7. Table 6 presents the numerical results obtained by the proposed method and compared with the results in [24]. The results given by the proposed method are better than those in [24]. Figure 8 presents the convergence behavior log-log error versus N at final points $t = 1$ using $\alpha = -1/2, \beta = 1/2$ and Vieta-Lucas nodes. Table 7 reports the mean-square error evaluated at some points $t \in [0, 1]$ with $N = 8$ and $M = 20$ simulations.

Table 4. Absolute Errors for different values of α and β for Example 6.2

t	$\alpha = 0, \beta = 0$	$\alpha = 1/2, \beta = 1/2$	$\alpha = -1/2, \beta = 1/2$
0.1	0.000006380896652	0.000003777198778	0.00001711661251
0.2	0.0000083167484	0.000004117528666	0.00001914042206
0.3	0.000008648347702	0.00001668806365	0.000008295358864
0.4	0.0000008759130666	0.00002142675862	0.000004382967829
0.5	0.0000063064633	0.000007695779115	0.000006894387383
0.6	0.000008727432195	0.00001187353006	0.000001630207448
0.7	0.000002184247548	0.00002560825357	0.000009585401491
0.8	0.00002138560208	0.00002405720511	0.00001282243502
0.9	0.00002468332722	0.000006919819951	0.00001112460334
1	0.00002561198292	0.00001220769109	0.00002114175433

Example 6.3. [12] Consider the the nonlinear stochastic equation

$$dY(t) = Y_0 - r^2 Y(t)(1 - Y^2(t))dt + r(1 - Y^2(t))dW(t), Y(0) = Y_0, t \in [0, 1]. \quad (6.4)$$

Table 5. Absolute Errors using Vieta-Lucas collocation points for Example 6.2

t	$N = 5$	$N = 8$	$N = 10$
0.1	0.00001180802481	0.00001589729599	0.00001147642818
0.2	0.000003334802028	0.000008824936588	0.00001361089248
0.3	0.000002450951654	0.000002358310846	0.00001334272455
0.4	0.00001514570709	0.000003269915668	0.00000588220554
0.5	0.00001664713906	0.000008300803068	0.00001866177439
0.6	0.00002688552518	0.00001028524326	0.000000288160912
0.7	0.00004215417474	0.00000310257672	0.00002283589988
0.8	0.00006841731913	0.000009516561643	0.0000146394294
0.9	0.00005810487037	0.00001727256903	0.00001202482514
1	0.00007731277892	0.00001859543056	0.00001242788021

Table 6. Absolute Errors using Jacobi Gauss collocation points and Vieta-Lucas collocation points for Example 6.2

t	$N = 10, \alpha = \beta = 0$	$N = 10$ Vieta Collocation points	[24]$M = 4, n = 11$	[24]$M = 4, n = 21$
0.1	0.000311834753	0.0001999810689	0.000111704	7.79416E-05
0.2	0.0004937567195	0.0003312523748	0.000161716	0.000144864
0.3	0.000149650746	0.000270331982	0.000183429	0.000140913
0.4	0.00009936910825	0.00002520606718	0.000197508	0.000189930
0.5	0.00004286249337	0.0002242913225	0.000233779	0.000248139
0.6	0.000181669407	0.0002436410595	0.000284752	0.000269361
0.7	0.00009035776256	0.00000257779695	0.000279789	0.000282680
0.8	0.0003530290413	0.0001565528622	0.000400146	0.000321173
0.9	0.0001260686729	0.000176863691	0.000431512	0.000354345
1	0.0001649547616	0.0003968492147	0.000411184	0.000391983

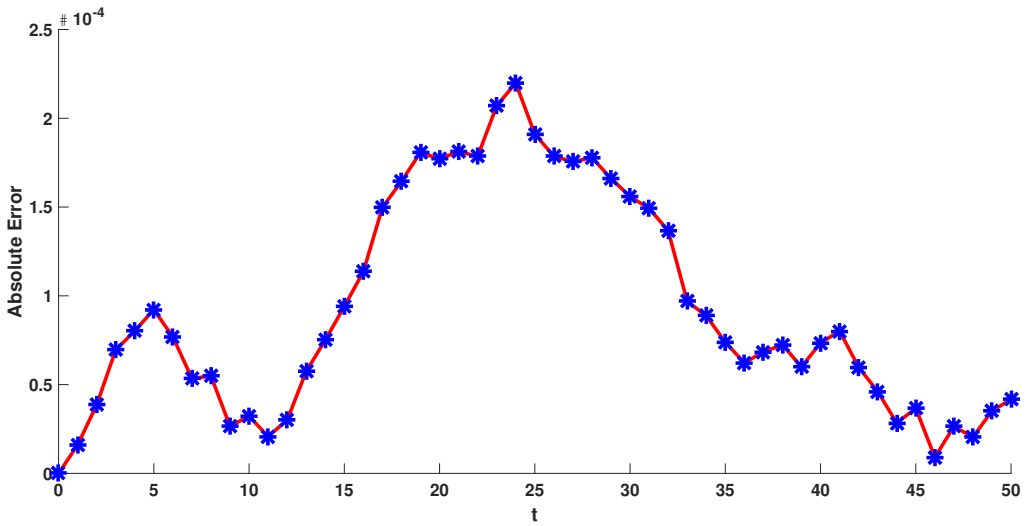


Figure 6. Absolute error using Vieta-Lucas collocation points with $N = 10$ in the interval $[0, 50]$ of Example 6.2

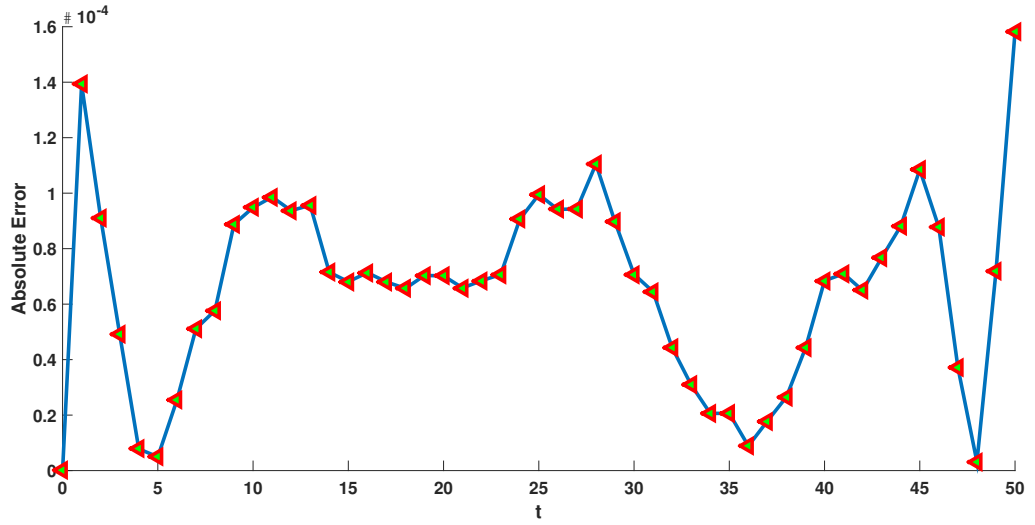


Figure 7. Absolute error using Jacobi Gauss collocation points with $\alpha = \beta = 0$ and $N = 5$ in the interval $[0, 50]$ of Example 6.2

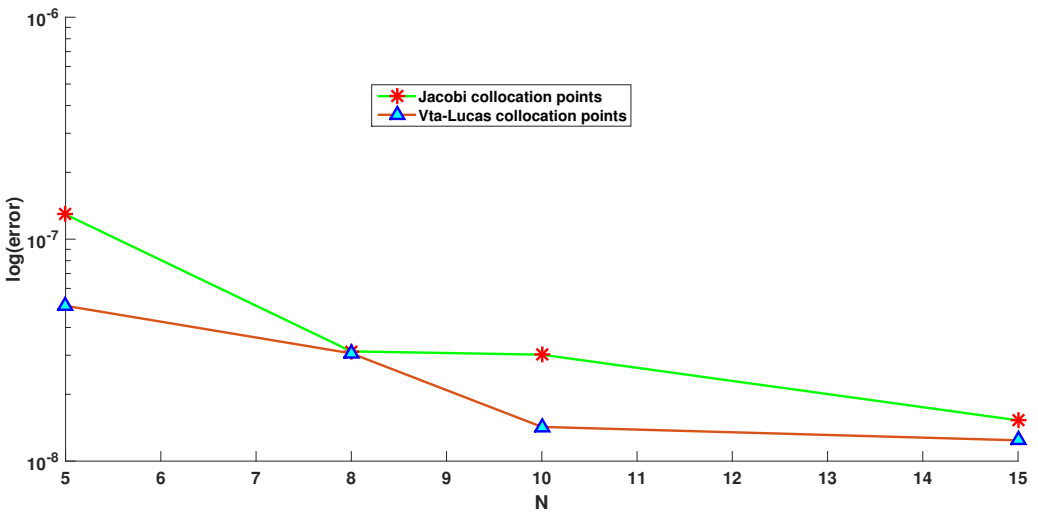


Figure 8. Log-log plot of the L^2-error at final point $t = 1$ versus N with $\alpha = -1/2, \beta = 1/2$ and Vieta-Lucas nodes of Example 6.2

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Table 7. Mean-square error comparaison at selected points t for Example 6.2

t	$\alpha = -1/2, \beta = 1/2$	Vieta-Luca nodes
0.1	1.674 e-8	1.187 e-8
0.2	2.603 e-8	2.470 e-8
0.3	1.957 e-8	1.975 e-8
0.4	1.117 e-8	1.111 e-8
0.5	1.209 e-8	9.841 e-9
0.6	1.347 e-8	1.071 e-8
0.7	2.149 e-8	1.407 e-8
0.8	2.173 e-8	1.635 e-8
0.9	1.114 e-8	1.687 e-8
1	2.759 e-8	2.892 e-8

The analytical solution is as follows $Y(t) = tanh(rW(t) + arctanh(Y_0))$. Table 8 presents the errors obatined by the proposed method using Jacobi Gauss collocation points with $N = 10, N_1 = 20, N_2 = 50; Y_0 = 1/10, a = 1/30$ and taking $M = 100$ simulations and compared with the results in [24] $M = 4, n = 21$, where the results given by the proposed method are better than those in [24]. Table 9 displays a comparaison of Absolute Errors between the proposed method by taking Vieta-Lucas collocation points with $N = 6, M = 100$ and the method in [24] for $M = 4, n = 11$ for Example 6.3. Table 10 illustrates the mean-square error at different points $t \in [0, 1]$ for $Y_0 = 1/10, r = 1/50$, $N = 10$ and $M = 20$ simulations.

Table 8. Absolute Errors for different values of α and β using Jacobi Gauss collocation points and Vieta-Lucas collocation points with $N = 10$ and $M = 100$ for Example 6.3

t	$VietaN = 10$	$\alpha = \beta = 0$	$\alpha = -1/2, \beta = 1/2$	$\alpha = -3/4, \beta = -1/2$	[24] $M = 4, n = 21$
0.1	0.004152	0.007988	0.0026370	0.00080504	0.0032940
0.2	0.004083	0.001635	0.0010094	0.0024696	0.003717
0.3	0.000328	0.004696	0.00485418	0.0011880	0.0054210
0.4	0.0031833	0.002189	0.00171796	0.0028830	0.005574
0.5	0.0008829	0.0080862	0.006173	0.001753	0.007144
0.6	0.00870	0.0079921	0.00784187	0.0000356	0.007704
0.7	0.009738	0.002709	0.003549	0.0019125	0.007246
0.8	0.006564	0.0023580	0.0041623	0.0010511	0.008032
0.9	0.004924	0.0008461	0.0040876	0.002988	0.007679
1	0.00871	0.002472	0.0017901	0.0034051	0.007947

Example 6.4. Stochastic Logistic Model

In 1838, Verhulst introduced a model to examine the population size $Y(t)$ over time, known as the deterministic logistic equation [31], expressed as

$$\frac{dY(t)}{dt} = r(t)Y(t)\left(1 - \frac{Y(t)}{k}\right), \quad Y(0) = Y_0 > 0, \tag{6.5}$$

where $k > 0$ represents the external carrying capacity and $r(t)$ denotes the growth rate. The primary objective was to elucidate the exponential growth of species described by Malthus in 1798, incorporating both the inherent self-limitation of the process and the external environmental conditions. These ideas were subsequently examined by several authors, including Volterra. The above model assumes that the growth rate and carrying capacity remain constant and deterministic; however, fluctuations in the growth rate occur in real environments. In 1958, Elton

Table 9. Comparaison of Absolute Errors between proposed method by taking Vieta-Lucas collocation points with $N = 6, M = 100$ with the method in [24]$M = 4, n = 11$ for Example 6.3

t	$Vieta\, N = 6$	method in [24]$M = 4, n = 11$
0.1	0.0005281054	0.003583100
0.2	0.0000005399	0.006582441
0.3	0.0002941107	0.006824461
0.4	0.0012067251	0.007230631
0.5	0.0001387826	0.008646804
0.6	0.0006606858	0.009745649
0.7	0.0024824043	0.012108328
0.8	0.0036150428	0.012320197
0.9	0.0020302811	0.011858844
1	0.0044919722	0.011812619

Table 10. Pointwise mean-square error at selected points t for Example 6.3

t	$\alpha = 0, \beta = 0$	$\alpha = -1/2, \beta = 1/2$	Vieta-Lucas nodes
0.1	3.339 e-4	1.255 e-4	2.691 e-4
0.2	2.481 e-4	2.221 e-4	3.463 e-4
0.3	2.539 e-4	3.291 e-4	3.789 e-4
0.4	3.777 e-4	4.308 e-4	2.725 e-4
0.5	6.498 e-4	4.234 e-4	4.396 e-4
0.6	4.491 e-4	3.697 e-4	5.004 e-4
0.7	4.104 e-4	2.640 e-4	3.550 e-4
0.8	4.705 e-4	2.642 e-4	5.101 e-4
0.9	6.383 e-4	5.426 e-4	3.313 e-4
1	1.721 e-3	1.090 e-3	4.681 e-4

observed that one of the primary causes of variation is environmental instability. In many countries, climatic change, driven by phenomena such as global warming, disrupts growth rates and carrying capacity. This phenomenon is referred to as environmental noise. Assume the growth rate is influenced by ambient noise such that

$$r(t) \to r(t) + \alpha(t)\xi(t),$$

where $\xi(t)$ denotes white noise and $\alpha^2(t)$ signifies the intensity of the noise. Hence, we obtain the generalization of the integral form of (6.5), which is called the *stochastic logistic equation*, given as

$$Y(t) = Y_0 + \int_0^t r(s)Y(s)\left(1 - \frac{Y(s)}{k}\right)ds + \int_0^t c(s)Y(s)\left(1 - \frac{Y(s)}{k}\right)dW(s), \tag{6.6}$$

where $W(t) = \int_0^t \xi(s)ds$ is referred to as Brownian motion. The stability of (6.6) is explored by Liu and Wang [13]. In all numerical tests, we take $k = 1$. Figures 9-10 present the numerical solutions with $Y_0 = 0.1$ and $r = 0.2$ with two values of $c(t)$ and with $N = 12$. The approximate solution with $Y_0 = 0.2$ and $r = 0.2$ with two values of $c(t)$ and with $N = 12$ is presented in Figure 11. An approximation is presented for the case with $Y_0 = 0.3$, $r(t) = 0.2$, $k = 1$, and two different values of $c(t)$, namely $c(t) = 0.3 + 0.3\sin(t)$ and $c(t) = 0$, for $N = 12$ is ploted in Figure 12. Table 11 displays absolute errors obtained by the proposed method for different values of $c(t)$ and with $r = 0.2, Y_0 = 0.3$ and $N = 12$. In the previous analysis, it becomes evident that the function $c(t)$ plays a crucial role in determining the stability of the numerical solution. More precisely, the behavior of $c(t)$ directly affects the interaction between the stochastic perturbations and the underlying deterministic dynamics. This influence highlights the delicate balance between randomness and determinism: while the deterministic part governs the general trend and structure of the solution, the stochastic terms introduce fluctuations whose intensity and impact are modulated by $c(t)$. Consequently, understanding the role of $c(t)$ is essential not only for analyzing the stability properties of the scheme but also for providing deeper insights into how stochastic effects propagate and interact with deterministic components in the system.

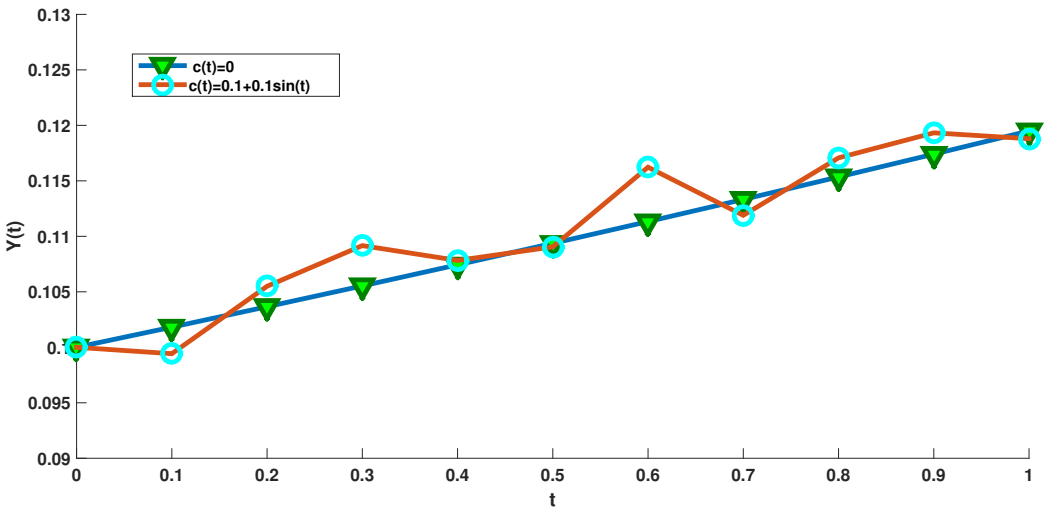


Figure 9. Approximate solutions for different values of $c(t)$ of Example 6.4

Example 6.5. [12] Consider the following stochastic integral equation:

$$Y(t) = Y_0 - \frac{a^2}{2}\int_0^t tanh(Y(s))sech^2(Y(s))\,ds + a\int_0^t sech(X(s))dW(s), t \in [0, 0.55]. \tag{6.7}$$

The exact solution of this problem is $Y(t) = arcsinh(aW(t) + sinh(Y_0))$. We have solved this problem by the proposed method for $N_1 = 20, N_2 = 50, a = 1/20, Y_0 = 1/10$ and $M = 20$

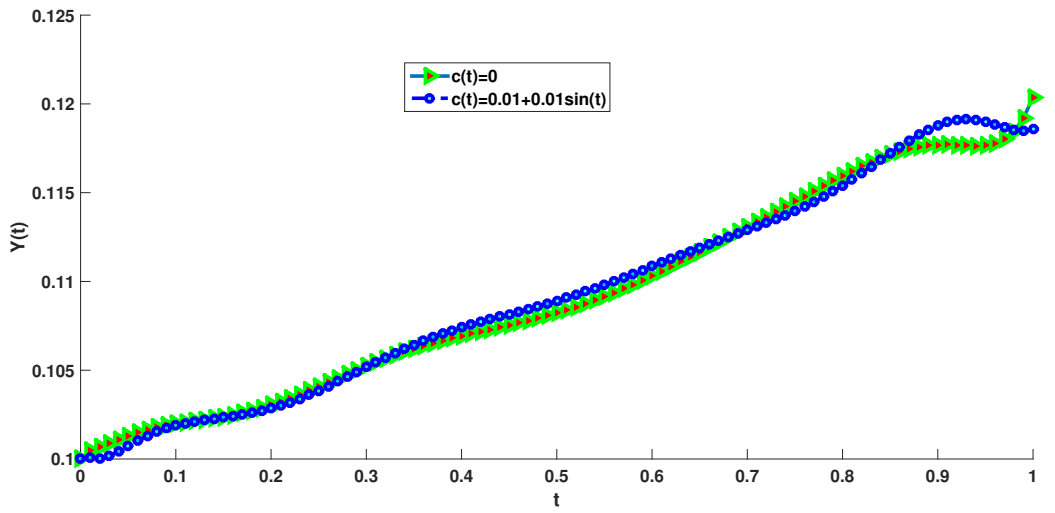


Figure 10. Approximate solutions for different values of $c(t)$ of Example 6.4

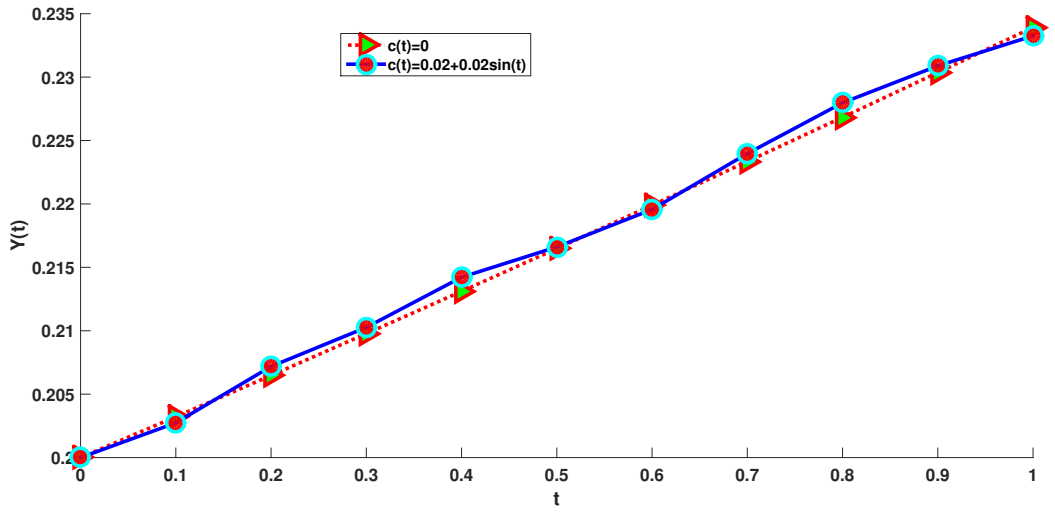


Figure 11. Exact solution, approximate solutions for different values of $c(t)$ of Example 6.4

Table 11. Comparaison of Absolute Errors by taking Vieta-Lucas collocation points with different values of $c(t)$ for Example 6.4

t	$c(t) = 0.3sin(t)$	$c(t) = 0.03sin(t)$	$c(t) = 0.003sin(t)$	$c(t) = 0.0003sin(t)$
0.1	0.00069943	0.00098822	0.000057454	2.4852e-6
0.2	0.0031077	0.00012288	0.000059562	4.7954e-6
0.3	0.01057	0.00045845	0.000041157	0.000010352
0.4	0.0042015	0.00110666	0.00021873	6.6375e-7
0.5	0.029825	0.00053508	5.831e-6	7.3905e-6
0.6	0.024645	0.00034378	0.00015432	1.1654e-6
0.7	0.0015384	0.0010619	0.000069153	4.0188e-6
0.8	0.0075468	0.00073502	0.00010622	3.294e-6
0.9	0.0099199	0.001068	0.0001509	0.000026468
1	0.01829	0.00023726	0.00027427	0.000021163

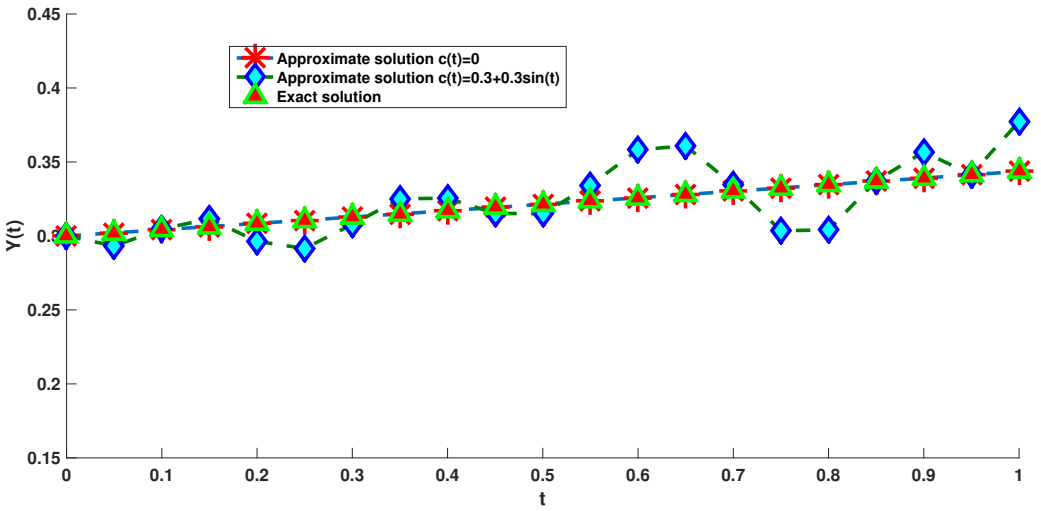


Figure 12. Approximate solutions for different values of $c(t)$ of Example 6.4

simulations. The graphs of the exact solution and approximate solution for $N = 10$ is shown in Figure 13. The plot of the mean square error is illustrated in Figure 14 for $N_1 = 20, N_2 = 50, N = 7$ and $M = 50$ simulations using Vieta-Lucas collocation points, Figure 15 illustrates the mean square error obtained using $N_1 = 20, N_2 = 50, N = 7$ and $M = 50$ based on Jacobi collocation points with $\alpha = -1/2 = -\beta$. Figure 16 illustrates the mean square error obtained using $N_1 = 20, N_2 = 50, N = 7$ and $M = 50$ based on Jacobi collocation points with $\alpha = \beta = 0$. The absolute errors at some different points $t \in [0, 1]$ and compared with those reported in [21] and [15] and [7] are reported in Table 12.

Table 12. The absolute errors of the approximate solutions for Example 6.5

t	Method of [15]	Method of [7]	Method in [21] m=15	Our method N=10	Our method N=12
0.05	4.0577e-04	6.9699e-05	2.1217e-05	0.0010425	0.000803
0.15	1.0331e-01	2.7414e-02	5.9221e-05	0.0014765	0.009229
0.25	2.4059e-01	1.5083e-01	1.7371e-03	0.0020701	0.003769
0.35	7.9036e-02	7.7907e-02	2.5274e-02	0.0024163	0.002403
0.45	3.5273e-02	1.4149e-01	4.8804e-02	0.0046008	0.006864
0.55	8.1839e-02	1.3064e-02	4.2837e-02	0.0036318	0.000215

Example 6.6. [26] Consider the following problem

$$Y(t) = Y_0 + \int_0^t r^2 cos(Y(s)) sin^3(s) ds - \int_0^t r sin^2(s) dW(s),\ t \in [0, 1]. \tag{6.8}$$

where the exact solution $Y(t) = arccot(rW(t) + cot(Y_0)$. We consider $Y_0 = \pi/32, r = 1/8$. The mean-square error illustrating the convergence of the method using Jacobi collocation points with $\alpha = \beta = 0$, $\alpha = -1/2, \beta = 1/2$ and $\alpha = \beta = 1/2$ with $N = 10, N_1 = 20, N_2 = 50$ and with $M = 50$ Monte-Carlo simulations are reported in Figures 17-18-19. Figure 20 illustrates the mean-square error computed at selected points t based on Vieta-Lucas collocation points.

7 Conclusion

This study has introduced and thoroughly investigated a novel pseudo-spectral approach for the numerical solution of functional stochastic Volterra integral equations. By constructing an approximation scheme based on Vita–Lucas polynomials in conjunction with the Gauss quadrature

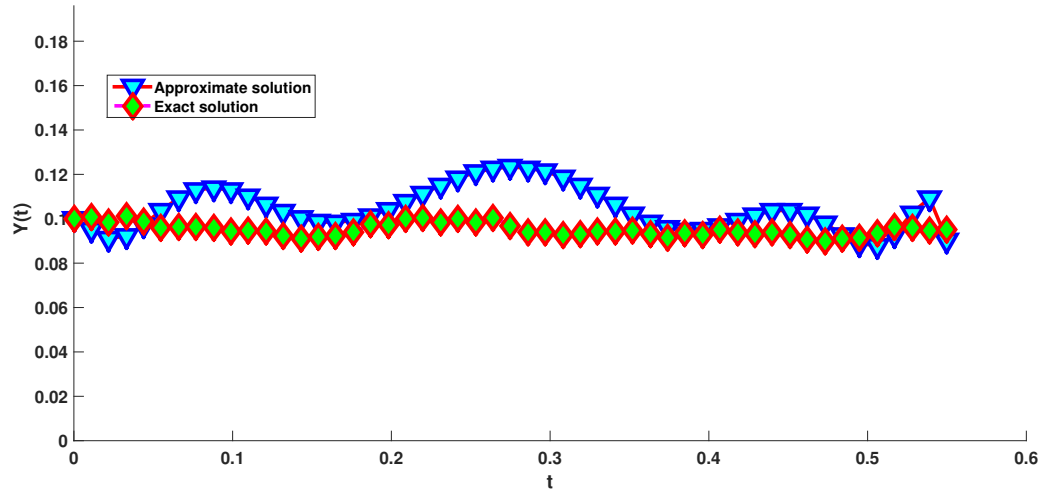


Figure 13. Exact and approximate solutions with $N = 10$ of Example 6.5

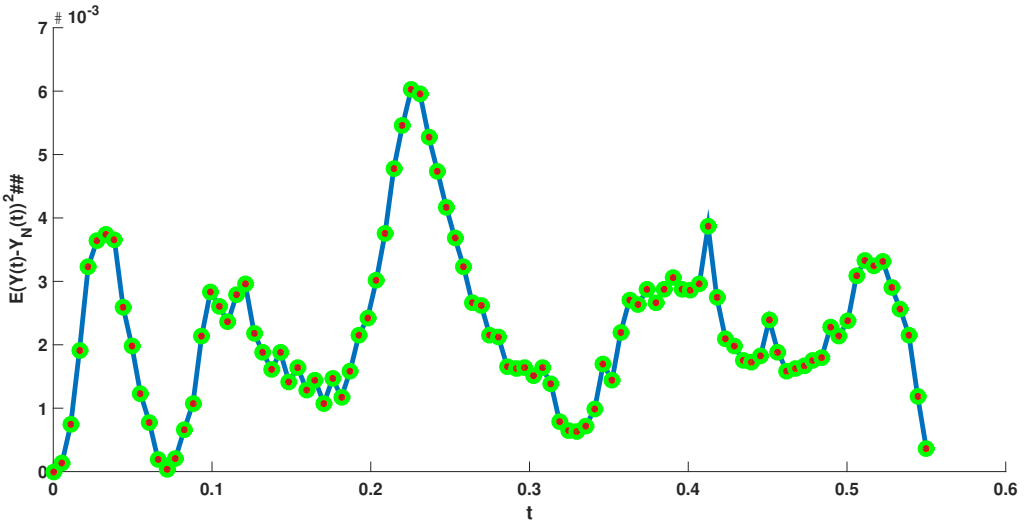


Figure 14. Mean-square error obtained using Vieta-Lucas collocation points of Example 6.5

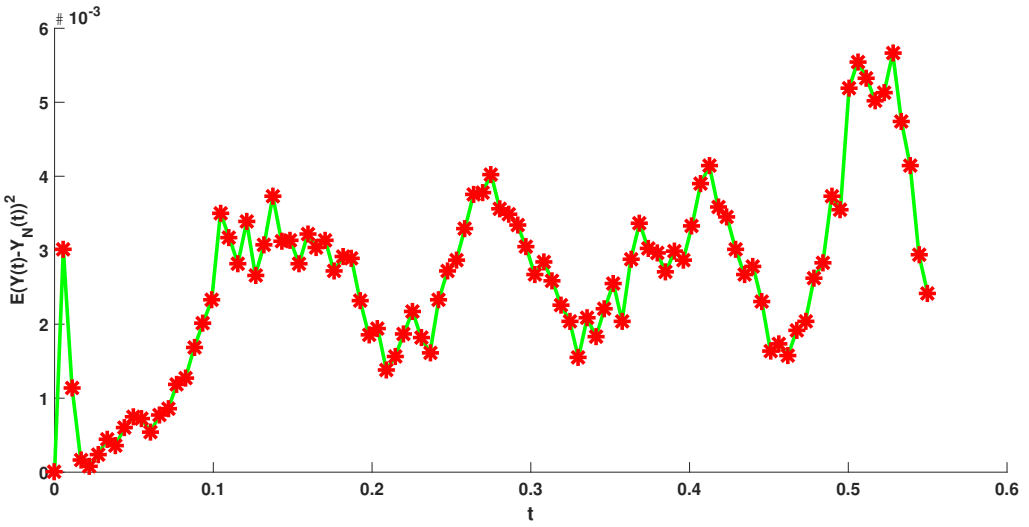


Figure 15. Plot of the mean-square error for the proposed method using Jacobi collocation points with $\alpha = -1/2 = -\beta$ of Example 6.5

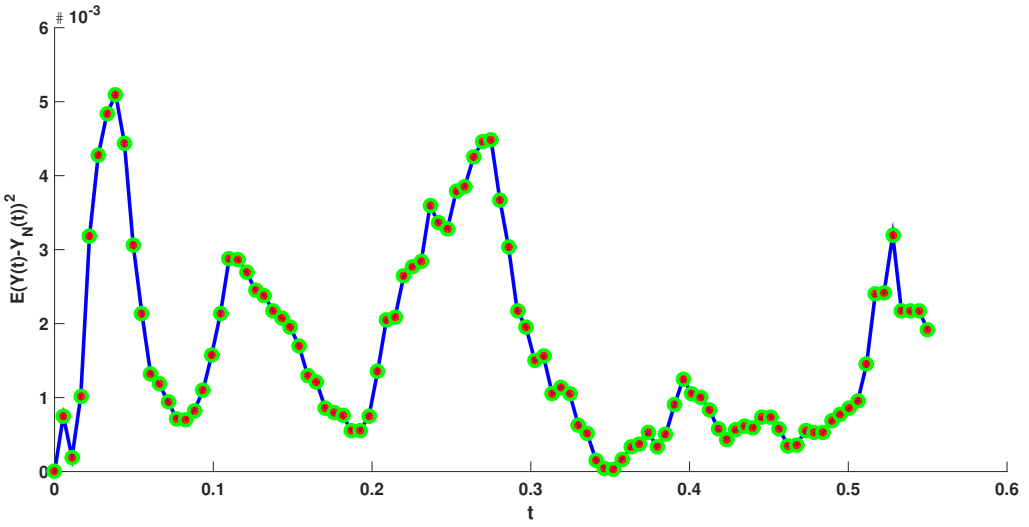


Figure 16. Mean-square error illustrating the convergence of the method using Jacobi collocation points with $\alpha = \beta = 0$ of Example 6.5

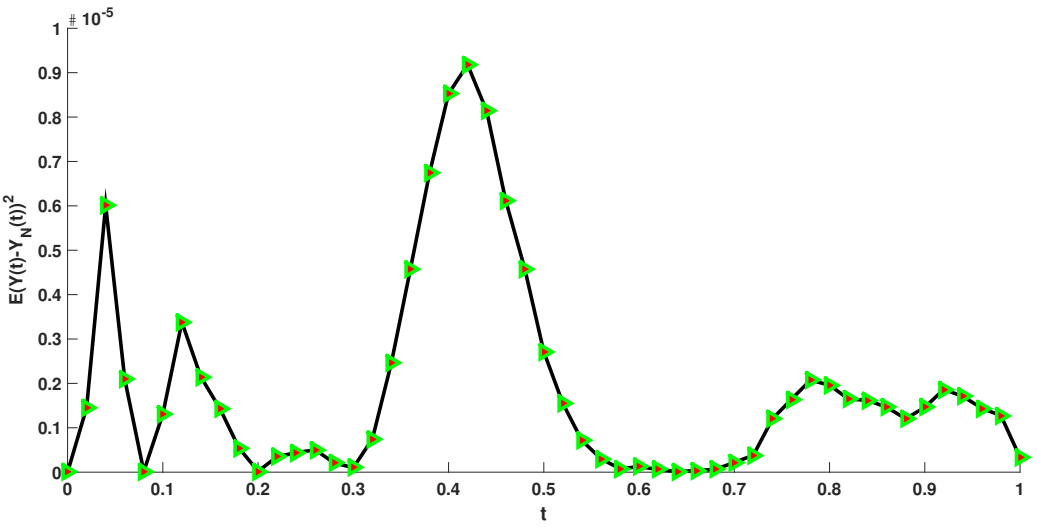


Figure 17. Mean-square error illustrating the convergence of the method using Jacobi collocation points with $\alpha = \beta = 0$ of Example 6.6

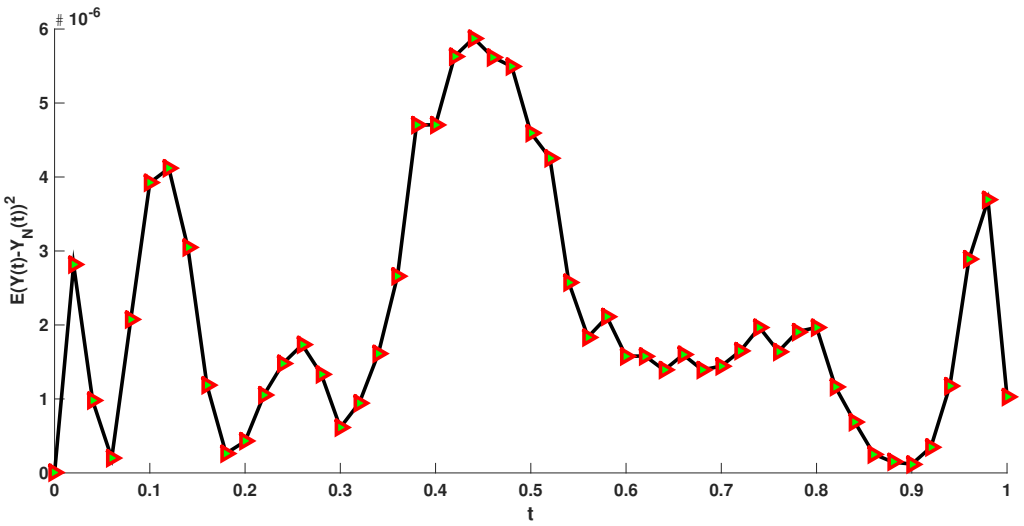


Figure 18. Mean-square error illustrating the convergence of the method using Jacobi collocation points with $\alpha = -1/2, \beta = 1/2$ of Example 6.6

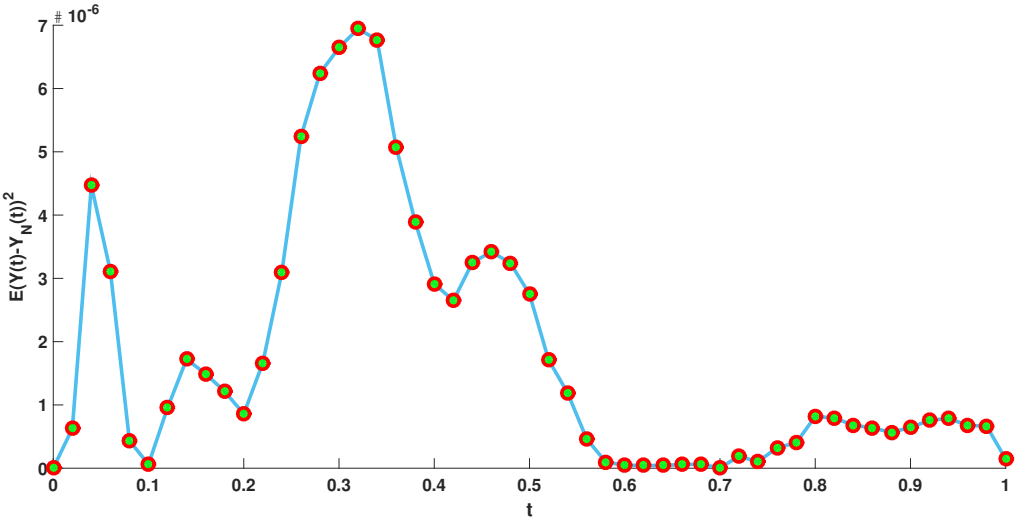


Figure 19. Mean-square error illustrating the convergence of the method using Jacobi collocation points with $\alpha = \beta = 1/2$ of Example 6.6

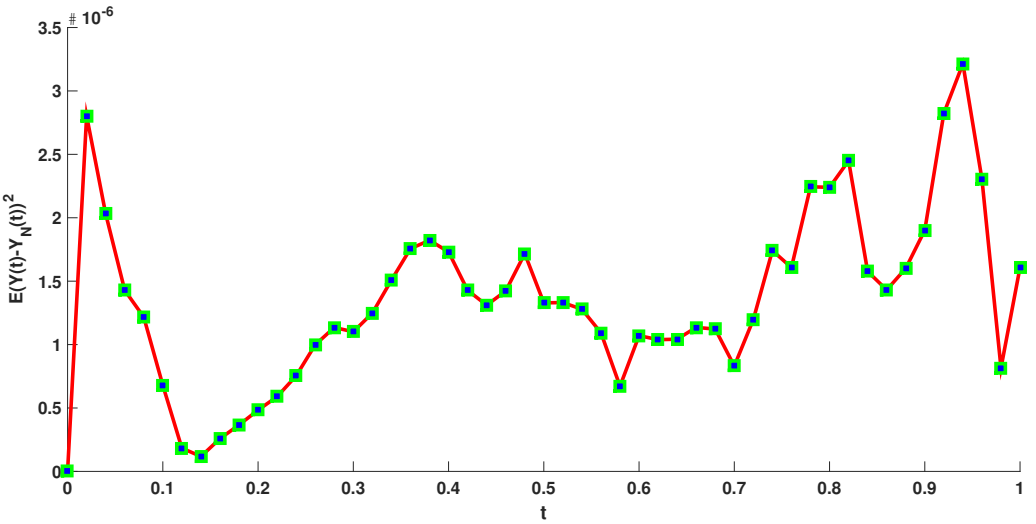


Figure 20. Mean-square error obtained using Vieta-Lucas collocation points of Example 6.6

rule, the proposed method demonstrates several important advantages. First, the use of orthogonal Vita–Lucas polynomials provides a flexible and stable basis for approximation, enabling efficient handling of the functional and stochastic components of the problem. Second, the incorporation of Gauss quadrature ensures accurate numerical integration, thereby enhancing the overall reliability of the scheme. A rigorous error analysis was carried out, establishing that the method achieves spectral convergence rates under suitable regularity conditions on the solution and data. This theoretical guarantee highlights the robustness of the proposed technique and confirms its potential as a powerful alternative to classical methods. Furthermore, the numerical experiments reported not only support the theoretical findings but also illustrate the efficiency, accuracy, and computational competitiveness of the algorithm when applied to representative test problems. The results of this work underscore the effectiveness of spectral methods, and in particular of Vita–Lucas polynomial expansions, for the treatment of stochastic integral equations of Volterra type. Beyond validating the proposed approach, this study opens new perspectives for further research. Future investigations may explore the extension of this framework to multidimensional problems, nonlinear settings, or equations driven by more general stochastic processes such as fractional Brownian motion. Additionally, adaptive strategies and hybrid techniques combining Vita–Lucas expansions with other approximation tools could be developed to improve flexibility and performance. The proposed pseudo-spectral method constitutes a reliable and accurate computational tool for functional stochastic Volterra integral equations, bridging theoretical advances with practical efficiency. It provides both a rigorous analytical foundation and promising numerical evidence, thereby contributing to the ongoing development of high-precision algorithms in stochastic numerical analysis.

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