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The Reformulated First Zagreb Index in relation to Generalized F-join graphs

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Abstract In chemical science, a topological graph index (TGI)/ molecular descriptor based on a graph yields the characterization of chemical compounds through its unique real index value. After representing a mathematical model from a molecular graph, it plays as a numeric parameter to analyze and develop the QSPR/ QSAR studies. The reformulated first Zagreb index (RFZI) is such a kind of parameter characterized as the sum of the squares of all edge degrees of a graph . Here we introduce two types of new four operations based on the generalized subdivision graph. We establish the key outcomes of RFZI in relation to generalized F-join graphs and also their relationships with Indu-Bala Product (IBP).

1 Introduction, Basic Definitions and Notations

The study of structural relationships between the vertices and edges of molecular graphs in chemical graph theory [5] has more applications in different disciplines in science. Moreover, this study plays an important role in characterizing the chemical compound or organic molecules by developing mainly the Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR) [26] studies. The TGI [8] is associated with a graph mathematically by a unique positive real number that characterizes its topology. It remains invariant under isomorphism.

All observed graphs are considered here simple, connected, and finite. Let us suppose that $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is such a graph with $V(\mathcal{G})$ as the vertex set as well as edge set $E(\mathcal{G})$ in $\mathcal{G}$. The notation $d_{\mathcal{G}}(v)$ stands for the degree (valency) of node (vertex) v in $\mathcal{G}$, that is the number of adjacent vertices with v in $\mathcal{G}$. The sum of the end-vertices degrees of an edge and then minus 2 is called the degree of an edge $(e = xz)$ and its expression follows as $d_{\mathcal{G}}(xz) = d_{\mathcal{G}}(x) + d_{\mathcal{G}}(z) - 2$. Here we follow some important topological indices. The earliest and vertex based TGIs are Zagreb indices [6, 7], which imply that $M_1(\mathcal{G}) = \sum_{\lambda \in V(\mathcal{G})} d^2_{\mathcal{G}}(\lambda)$ and $M_2(\mathcal{G}) = \sum_{\alpha\beta \in E(\mathcal{G})} d_{\mathcal{G}}(\alpha) d_{\mathcal{G}}(\beta)$ respectively. The edge version of classical Zagreb indices namely the reformulated Zagreb indices was founded by Milicevic et al. [20]. The reformulated Zagreb indices [15] are explained as $EM_1(\mathcal{G}) = \sum_{\varepsilon \in E(\mathcal{G})} d^2_{\mathcal{G}}(\varepsilon)$ and $EM_2(\mathcal{G}) = \sum_{\zeta \sim \eta \in E(\mathcal{G})} d_{\mathcal{G}}(\zeta) d_{\mathcal{G}}(\eta)$ respectively, where $\zeta \sim \eta$ manifests that ζ and η are adjacent. The F-index was characterized by Furtula et al. [9] and was characterized as $F(\mathcal{G}) = \sum_{\mu \in V(\mathcal{G})} d^3_{\mathcal{G}}(\mu) = \sum_{\mu\nu \in E(\mathcal{G})} [d^2_{\mathcal{G}}(\mu) + d^2_{\mathcal{G}}(\nu)]$. The reformulated F-index was initiated by Liu et al. [15]. In 2013, Shirdel et al. [25] instituted the hyper Zagreb index established as $HM(\mathcal{G}) = \sum_{\mu\nu \in E(\mathcal{G})} [d_{\mathcal{G}}(\mu) + d_{\mathcal{G}}(\nu)]^2$. This study also can be applied to investigating the indices of graphs in [12, 16, 18, 21]. We refer our interested readers to follow [1, 4, 13, 17, 23] for more study of the topological indices.

1.1 Preliminary

The joining $\mathcal{G}_1 \uplus \mathcal{G}_2$ is a new graph achieved from $\mathcal{G}_1$ and $\mathcal{G}_2$ by joining one by one node of $\mathcal{G}_1$ with every nodes of $\mathcal{G}_2$.
Let F be the set of derived graphs of graph subdivision $(S(\mathcal{G}))$, graph semi-total point $(R(\mathcal{G}))$, graph semi-total line $(Q(\mathcal{G}))$ and the graph total $(T(\mathcal{G}))$ that are derived from $\mathcal{G}$. Also $I(\mathcal{G})$ be the new nodes of $F(\mathcal{G})$ associated with every edge in $\mathcal{G}$ such that $V(F(\mathcal{G})) = I(\mathcal{G}) \cup V(\mathcal{G})$. The notation $L(\mathcal{G})$ [10] called line graph of $\mathcal{G}$ is a new derived graph associated with a new common vertex in every corresponding edges which are adjacent.

The vertex F-join of graphs [24] or F-vertex join graphs (FVJG) of $\mathcal{G}_1$ with $\mathcal{G}_2$ is a new graph derived from $F(\mathcal{G}_1)$ along with $\mathcal{G}_2$ that joins every vertex of $\mathcal{G}_1$ with each vertex of $\mathcal{G}_2$. It contains total number of vertices $V(F(\mathcal{G}_1))$ and $V(\mathcal{G}_2)$. The edge set includes in $E(\mathcal{G}_1) \cup E(\mathcal{G}_2) \cup \{xy : x \in V(\mathcal{G}_1), y \in V(\mathcal{G}_2)\}$. There are four types of FVJG discussed here namely S-Vertex Join Graphs (SVJG), R-Vertex Join Graphs (RVJG), Q-Vertex Join Graphs (QVJG) and T-Vertex Join Graphs (TVJG) (see Fig. 3).

The edge F-join of graphs or F-edge join graphs (FEJG) [24] of $\mathcal{G}_1$ with $\mathcal{G}_2$, symbolically $\mathcal{G}_1 \uplus_{\widehat{F}} \mathcal{G}_2$ is taken from $F(\mathcal{G}_1)$ and $\mathcal{G}_2$. Then it joins every vertex of $I(\mathcal{G}_1)$ with all vertices of $\mathcal{G}_2$. It consists of $V(F(\mathcal{G}_1)) \cup (\mathcal{G}_2)$ and edges $E(\mathcal{G}_1) \cup E(\mathcal{G}_1) \cup \{\mu v : \mu \in I(\mathcal{G}_1), v \in V(\mathcal{G}_2)\}$. There are four types of FEJG discussed here such as S-edge join graphs (SEJG), R-edge join graphs (REJG), Q-edge join graphs (QEJG) and T-edge join graphs (TEJG) (see Fig. 4). These derived graphs are led by the existence of inter-molecular forces between atoms and bonds besides among atoms interactions.
There exist four several generalized operations based on the subdivision of graphs that are defined in [14] are followed as:

•The graph $S_p(\mathcal{G})$ called the graph p-subdivision ($p \geq 1$ be an integer) is obtained by adding new p vertices in every edge of $\mathcal{G}$.

• In $R_p(\mathcal{G})$ known as the graph p-semi-total point, it is joined $S_p(\mathcal{G})$ with adjacent in $\mathcal{G}$.

•The graph p semi-total line $Q_p(\mathcal{G})$ is derived by the joint of new added vertices of the edges with one vertex colloquial in basic $\mathcal{G}$.

• The graph p total point $T_p(\mathcal{G})$ is obtained by both R_p and Q_p operations on $S_p(\mathcal{G})$.

Consider $F_p \in \{S_p, R_p, Q_p, T_p\}$ as an operator, where $p \in \mathbb{Z}^+$. The graph $F_p(\mathcal{G}_1)$ is derived by the operation F_p on $\mathcal{G}_1$. It contains vertex set $V(F_p(\mathcal{G}_1))$ and edge set $E(F_p(\mathcal{G}_1))$ respectively.

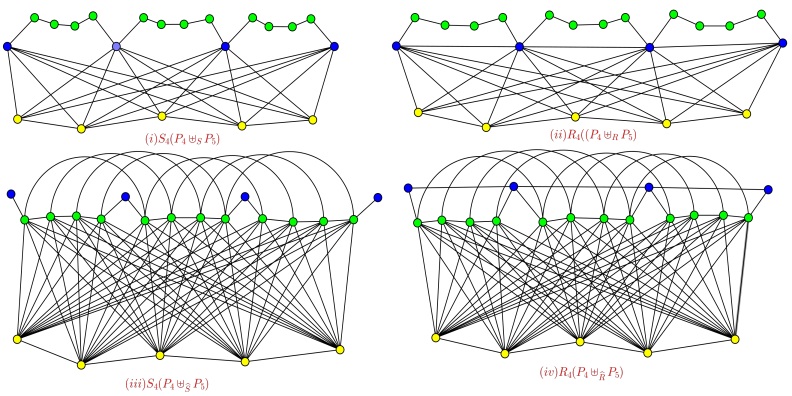


Figure 1. The g-FVJG of $S_4(P_4)$ with P_5 and $R_4(P_4)$ with P_5.

By inspiration on the work by Sarkar et al. [24], we have introduced two novel graph operations based on generalized subdivision graph.

Definition 1.1. [11] The generalized F-vertex join graphs (g-FVJG) of $\mathcal{G}_1$ with $\mathcal{G}_2$ is a new graph derived from generalized $F_p(\mathcal{G}_1)$ along with $\mathcal{G}_2$ that joins every vertex of $V(\mathcal{G}_1) = (F_p(\mathcal{G}_1) \setminus I_p(\mathcal{G}_1))$ with all vertices of $V(\mathcal{G}_2)$.

Definition 1.2. [11] The generalized F-Edge Join Graphs (g-FEJG) of $\mathcal{G}_1$ with $\mathcal{G}_2$ is a new graph derived from generalized $F_p(\mathcal{G}_1)$ and $\mathcal{G}_2$. Then it joins every vertex of $I_p(\mathcal{G}_1)$ with each of $V(\mathcal{G}_2)$.

As an example these definitions are explained in the Figures 1 and 2.

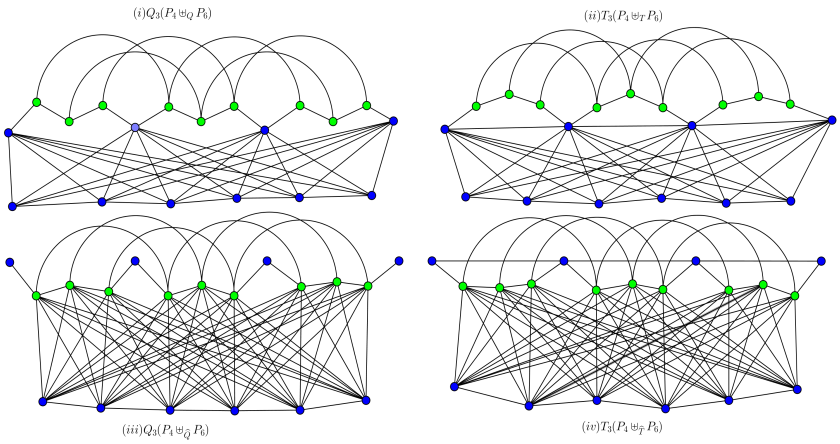


Figure 2. The g-FEJG of $Q_3(P_4)$ with P_6 and $T_3(P_4)$ with P_6.

2 Main Results and Discussions

The aim of this study is to establish certain sharp explanations of RFZI based on g-FVJG such as generalized S-vertex join graphs(g-SVJG), generalized R-vertex join graphs (g-RVJG), generalized Q-vertex join graphs (g-QVJG) and generalized T-vertex join graphs (g-TVJG) and otherwise g-FEJG like generalized S-edge join graphs(g-SEJG), generalized R-edge join graphs (g-REJG), generalized Q-edge join graphs (g-QEJG) and generalized T-edge join graphs (g-TEJG) respectively.

Definition 2.1. The g-SVJG of $\mathcal{G}_1$ and $\mathcal{G}_2$, symbolized as $\mathcal{G}_1 \uplus_{S_k} \mathcal{G}_2$, is achieved from $S_k(\mathcal{G}_1)$ together with $\mathcal{G}_2$ by connecting all vertices of $V(\mathcal{G}_1)$ to each and every one vertex of $\mathcal{G}_2$ (see Fig. 1(i)).

Firstly, we are going to establish the expression of RFZI in relation to g-SVJG by applying the Definition 1.1.

Theorem 2.2. *Suppose that the graph $\mathcal{G}_i$ is connected with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2) = F(\mathcal{G}_1) + F(\mathcal{G}_2) + 3n_2M_1(\mathcal{G}_1) + (5n_1 - 4) + 2M_2(\mathcal{G}_2) + 2m_1m_2^2 + 4(p-1)m_1 + 4(n_1-1)^2m_2 + n_1n_2(n_1+n_2-2)^2 + 4(m_1n_2 + m_2n_1)(n_2+n_1-2) + 8m_1n_2$.

Proof. For $\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2$, the edge set is are decomposed into three parts such as $E(S_p(\mathcal{G}_1))$, $E(\mathcal{G}_2)$ and also edges $\{\mu\nu \in E(\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2) : \mu \in V(\mathcal{G}_1) \text{ and } \nu \in V(\mathcal{G}_2)\}$. There exist three types of vertex degree distribution in $\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2$ such as $d_{\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2}(\mu) = d_{\mathcal{G}_1}(\mu) + n_2$ if $\mu \in V(\mathcal{G}_1)$; for $\nu \in V(\mathcal{G}_2)$, $d_{\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2}(\nu) = d_{\mathcal{G}_2}(\nu) + n_1$ and $d_{\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2}(\mu) = 2$ for $\mu \in I(S_p(\mathcal{G}_1))$. Then $EM_1(\mathcal{G}_1 \uplus_{S_p} \mathcal{G}_2)$

$$= \sum_{\mu\nu \in E(\mathcal{G}_1)} [d_{\mathcal{G}_1}(\mu) + n_2]^2 + \sum_{\mu\nu \in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(\mu) + n_1 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2$$

$$+ \sum_{\mu \in V(\mathcal{G}_1)} \sum_{\nu \in V(\mathcal{G}_2)} [d_{\mathcal{G}_1}(\mu) + n_2 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2$$

$$+ \sum_{\mu\nu \in E(S_p(\mathcal{G}_1)); \mu, \nu \in V(S_p(\mathcal{G}_1)) \setminus V(\mathcal{G}_1)} [2]^2$$

$$= \sum_{\nu\in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(\nu)(d^2_{\mathcal{G}_1}(\nu) + 2n_2 d_{\mathcal{G}_1}(\nu) + n_2^2) + \sum_{\mu\nu\in E(\mathcal{G}_2)} [(d^2_{\mathcal{G}_2}(\mu) + d^2_{\mathcal{G}_2}(\nu))$$
$$+ 2(2n_2 - 2)(d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(\nu)) + 4(n_1 - 1)^2 + 2d_{\mathcal{G}_2}(\mu)d_{\mathcal{G}_2}(\nu)]$$

$$+ \sum_{\mu\in V(\mathcal{G}_1)} \sum_{\nu\in V(\mathcal{G}_2)} [d^2_{\mathcal{G}_1}(\mu) + d^2_{\mathcal{G}_2}(\nu) + (n_1 - 2 + n_2)^2$$
$$+ 2(n_1 + n_2 - 2)d_{\mathcal{G}_1}(u) + 2(n_1 + n_2 - 2)d_{\mathcal{G}_2}(v) + 2d_{\mathcal{G}_1}(u)d_{\mathcal{G}_2}(v)] + (p-1) \sum_{\mu\nu\in E(S_p(\mathcal{G}_1))} 4$$
$$= F(\mathcal{G}_1) + 2n_2M_1(\mathcal{G}_1) + 2m_1n_2^2 + F(\mathcal{G}_2) + 4(n_2 - 1)M_1(\mathcal{G}_2) + 4(n_1 - 1)^2m_2$$
$$+ 2M_2(\mathcal{G}_2) + n_2M_1(\mathcal{G}_1) + n_1M_1(\mathcal{G}_2) + n_1n_2(n_1 - 2 + n_2)^2$$
$$+ 4m_1n_2(n_1 - 2 + n_2) + 4m_2n_1(n_2 - 2 + n_1) + 8m_1m_2 + 4(p-1)m_1.$$

Hence the result. □

For the general p-subdivision graph $S_p(\mathcal{G})$, each edge of G is replaced by a path of length $p+1$ by inserting p new vertices. In the particular case when $p = 1$, every edge is replaced by a path of length 2, i.e., each edge is subdivided once. The resulting graph is precisely the subdivision graph S(G) [22].

The vertex S-join of graphs (also called the S-vertex join graphs, SVJG) [24]) of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \uplus_S \mathcal{G}_2$, is obtained from $S(\mathcal{G}_1)$ together with $\mathcal{G}_2$ by joining every vertex of $V(\mathcal{G}_1)$ to every vertex of $\mathcal{G}_2$ (see Fig. 3(i)).

Corollary 2.3. *Let $\mathcal{G}_i$ be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(\mathcal{G}_1 \uplus_S \mathcal{G}_2) = F(\mathcal{G}_1) + F(\mathcal{G}_2) + 3n_2M_1(\mathcal{G}_1) + (5n_1 - 4) + 2M_2(\mathcal{G}_2) + 2m_1m_2^2 + 4(n_1 - 1)^2m_2 + n_1n_2(n_1 + n_2 - 2)^2 + 4(m_1n_2 + m_2n_1)(n_2 + n_1 - 2) + 8m_1n_2.$

The symbol $\otimes$ implies the IBP [11] of graphs. The F-sum of $\mathcal{G}_1$ and $\mathcal{G}_2$ [2, 3] is characterized as $(\mathcal{G}_1 \otimes_F \mathcal{G}_2)$ and is defined by $F(\mathcal{G}_1) \otimes \mathcal{G}_2$ having node-set $V(\mathcal{G}_1 \otimes_F \mathcal{G}_2) = V(\mathcal{G}_1) \cup E(\mathcal{G}_1) \cup V(\mathcal{G}_2)$ and edge-set $E(\mathcal{G}_1 \otimes_F \mathcal{G}_2) = E(F(\mathcal{G}_1) \otimes \mathcal{G}_2) \setminus E^\star$, where $E^\star = \{\mu\nu | \mu \in V(F(\mathcal{G}_1)) \setminus V(\mathcal{G}_1), \nu \in V(\mathcal{G}_2)\}$. After an investigation, we have seen that there exist an important relationship between g-FVJG $(\mathcal{G}_1 \uplus_F \mathcal{G}_2)$ or g-FEJG $(\mathcal{G}_1 \uplus_{\widehat{F}} \mathcal{G}_2)$ with $(\mathcal{G}_1 \otimes_F \mathcal{G}_2)$. The IBP $(\mathcal{G}_1 \otimes \mathcal{G}_2)$ is gained from two disjoint copies of join $(\mathcal{G}_1 + \mathcal{G}_2)$ by joining every corresponding vertices of two copies of $\mathcal{G}_2$.

Corollary 2.4. *Let n_i and m_i $(i = 1, 2)$ be the vertices and edges of $\mathcal{G}_i$ respectively. Then we obtain the following relation* $EM_1(\mathcal{G}_1 \otimes_S \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_S \mathcal{G}_2) + 12M_1(\mathcal{G}_2) + 4n_1n_2(n_1 + n_2 - 2) + 16m_2(m_1 + n_1) + 4m_1^2n_2 + 2n_1n_2 + 8m_1.$

Proof. By the definition of F-sum of $\mathcal{G}_1$ and $\mathcal{G}_1$ graphs based on IBP, we have

$$EM_1(\mathcal{G}_1 \otimes_S \mathcal{G}_2) = \sum_{E_i=\mu\nu\in E(\mathcal{G}_1\otimes_S\mathcal{G}_2)}^{4} [d_{\mathcal{G}_1\otimes_S\mathcal{G}_2}(\mu) + d_{\mathcal{G}_1\otimes_S\mathcal{G}_2}(\nu) - 2]^2$$

There exist four types of edges in $\mathcal{G}_1 \otimes_S \mathcal{G}_2$.
$E_1 : \{\mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2) : \mu\nu \in E(\mathcal{G}_2)\}$; $E_2 : \{\mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2) : \mu \in V(\mathcal{G}_1); \nu \in V(\mathcal{G}_2)\}$
$E_3 : \{\mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2) : \mu\nu \in E(S(\mathcal{G}_1))\}$ and $E_4 : \{\mu\mu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2) : \mu \in V(\mathcal{G}_2)\}$.
Here the degree distribution in $\mathcal{G}_1 \otimes_S \mathcal{G}_2$ such as $d_{\mathcal{G}_1\otimes_S\mathcal{G}_2}(\mu) = d_{\mathcal{G}_1}(\mu) + n_2$ if $\mu \in V(\mathcal{G}_1)$; for $\nu \in V(\mathcal{G}_2)$, $d_{\mathcal{G}_1\otimes_S\mathcal{G}_2}(\nu) = d_{\mathcal{G}_2}(\nu) + n_1 + 1$ and $d_{\mathcal{G}_1\otimes_S\mathcal{G}_2}(\mu) = 2$ for $\mu \in I(\mathcal{G}_1)$.
Case I: $E_1 = \mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2)$ such that $\sum_{\mu\nu\in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(\mu) + n_1 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2 = 2[F(\mathcal{G}_2) + 4(n_2 - 1)M_1(\mathcal{G}_2) + 4(n_1 - 1)^2m_2 + 2M_2(\mathcal{G}_2) + \Big(4m_2 + 8m_2(n_1 - 1) + 4M_1(\mathcal{G}_2)\Big)]$.
Case II: $E_2 = \mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2)$ such that $\sum_{\mu\in E(\mathcal{G}_1);\nu\in E(\mathcal{G}_2)} [d_{\mathcal{G}_1}(\mu) + n_2 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2 = 2[n_2M_1(\mathcal{G}_1) + n_1M_1(\mathcal{G}_2) + n_1n_2(n_1 - 2 + n_2)^2 + 4m_1n_2(n_1 - 2 + n_2) + 4m_2n_1(n_2 - 2 + n_1) + 8m_1m_2 + \Big(n_1n_2 + 4(m_1 + m_2) + 2n_1n_2(n_1 + n_2 - 2)\Big)]$.
Case III: $E_3 = \mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2)$ such that $\sum_{\mu\nu\in E(S(\mathcal{G}_1))} [d_{\mathcal{G}_1}(\mu) + n_2]^2 = 2[F(\mathcal{G}_1) + 2n_2M_1(\mathcal{G}_1) +$

$2m_1n_2^2]$.
Case IV: $E_4 = \mu\nu \in E(\mathcal{G}_1 \otimes_S \mathcal{G}_2)$ such that $\sum_{\mu\in V(\mathcal{G}_2)} 4[d_{\mathcal{G}_2}(\mu) + n_1]^2 = 4[M_1(\mathcal{G}_2) + n_1^2n_2 + 4n_1m_2]$.
Using the Corollary 2.3 we simplify the result and get the relation between $EM_1(\mathcal{G}_1 \otimes_S \mathcal{G}_2)$ and $EM_1(\mathcal{G}_1 \uplus_S \mathcal{G}_2)$. □

Definition 2.5. The g-RVJG $(\mathcal{G}_1 \uplus_{R_p} \mathcal{G}_2)$ is obtained from $R_p(\mathcal{G}_1)$ and $\mathcal{G}_2$. It forms by connecting each $u \in V(\mathcal{G}_1)$ together with every $v \in V(\mathcal{G}_2)$ (see Fig. 1(ii)).

Now we determine the expression of RFZI for $\mathcal{G}_1 \uplus_{R_p} \mathcal{G}_2$.

Theorem 2.6. *Let $\mathcal{G}_y$ be a graph containing vertices n_i and edges m_i for $i = 1, 2$. Then we have* $EM_1(\mathcal{G}_1 \uplus_{R_p} \mathcal{G}_2) = 8F(\mathcal{G}_1) + F(\mathcal{G}_2) + 8(2n_2 - 1)M_1(\mathcal{G}_1) + (5n_1 - 4)M_1(\mathcal{G}_2) + 2M_2(\mathcal{G}_2) + 8M_2(\mathcal{G}_1) + 4m_1(n_2 - 1)^2 + 4m_2(n_1 - 1)^2 + n_1n_2(n_1 + n_2 - 2)^2 + 4(n_1 + n_2 - 2)(2m_1n_2 + n_1m_2) + 2m_1n_2^2 + 16m_1m_2 + 4(p - 1)m_1$.

Proof. We divide $E(\mathcal{G}_1 \uplus_{R_p} \mathcal{G}_2)$ into four distinct partitions that are E_y for $y \in \{1, 2, 3, 4\}$. The partition E_1 contains m_1 edges $\mu\nu$ such that $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(\mu) = 2d_{G_1}(\mu) + n_2$ and $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(\nu) = 2d_{G_2}(\nu) + n_2$. Also the E_2 consists of $2m_1$ edges $w\nu$ such that $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(w) = 2$ for $w \in I(R_p(\mathcal{G}_1))$ and $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(\nu) = d_{\mathcal{G}_2}(\nu) + n_1$. The third E_3 is of m_2 edges $\mu\nu$ such that $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(\mu) = d_{\mathcal{G}_2}(\mu) + n_1$ and $d_{\mathcal{G}_1\uplus_R\mathcal{G}_2}(\nu) = d_{\mathcal{G}_2}(\nu) + n_1$ and the last partition E_4 includes of n_1n_2 edges $\mu\nu$ such that $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(\mu) = 2d_{\mathcal{G}_2}(\mu) + n_2$ and $d_{\mathcal{G}_1\uplus_{R_p}\mathcal{G}_2}(\nu) = d_{\mathcal{G}_2}(\nu) + n_1$. Therefor the g-RVJG for $(\mathcal{G}_1 \uplus_{R_p} \mathcal{G}_2)$ is expressed as

$$EM_1(\mathcal{G}_1 \uplus_{R_p} \mathcal{G}_2) = \sum_{\mu\nu\in E(\mathcal{G}_1)} [2(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu)) + 2n_2 - 2]^2 + \sum_{\mu\in I(R_p(\mathcal{G}_1)),\nu\in V(\mathcal{G}_1)} [2d_{\mathcal{G}_1}(\nu) + n_2 + d_{I(\mathcal{G}_1)}(\mu) - 2]^2$$

$$+ \sum_{\mu\nu\in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(\mu) + n_1 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2$$

$$+ \sum_{\mu\in V(\mathcal{G}_1))} \sum_{\nu\in V(\mathcal{G}_2)} [2d_{\mathcal{G}_1}(\mu) + n_2 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2$$

$$+ \sum_{\mu\nu\in E(R_p(\mathcal{G}_1));\mu,\nu\in V(R_p(\mathcal{G}_1))\setminus V(\mathcal{G}_1)} [2]^2$$

$$= 4\Big[\sum_{\mu\nu\in E(\mathcal{G}_1)} [(d^2_{\mathcal{G}_1}(\mu) + d^2_{\mathcal{G}_1}(\nu)) + 2d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu) + (n_2 - 1)^2 + 2(n_2 - 1)(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu))]\Big]$$

$$+ \sum_{\nu\in V(\mathcal{G}_1)} [4d^2_{\mathcal{G}_1}(\nu) + n_2^2 + 4n_2d_{\mathcal{G}_1}(\nu)]d_{\mathcal{G}_1}(\nu)$$

$$+ \sum_{\mu\nu\in E(\mathcal{G}_2)} [(d^2_{\mathcal{G}_2}(\mu) + d^2_{\mathcal{G}_2}(\nu)) + 2d_{\mathcal{G}_2}(\mu)d_{\mathcal{G}_2}(\nu) + 4(n_1 - 1)d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(\nu) + 4(n_1 - 1)^2]$$

$$+ \sum_{\mu\in V(\mathcal{G}_1)} \sum_{\nu\in V(\mathcal{G}_2)} [4d^2_{\mathcal{G}_1}(\mu) + d^2_{\mathcal{G}_2}(\nu) + (n_1 + n_2 - 2)^2$$

$$+ 4(n_1 + n_2 - 2)d_{\mathcal{G}_1}(\mu) + 2(n_1 + n_2 - 2)d_{\mathcal{G}_2}(\nu) + 4d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_2}(\nu)]$$

$$+ (p - 1) \sum_{\mu\nu\in E(R_p(\mathcal{G}_1))} 4$$

$$= 4[F(\mathcal{G}_1) + 2M_2(\mathcal{G}_1) + m_1(n_2 - 1)^2 + 2(n_2 - 1)M_1(\mathcal{G}_1)] + 4F(\mathcal{G}_1) + 2m_1n_2^2 + 4n_2M_1(\mathcal{G}_1)$$
$$+ F(\mathcal{G}_2) + 2M_2(\mathcal{G}_2) + 4(n_1 - 1)M_1(\mathcal{G}_2) + 4(n_1 - 1)^2m_2 + 4n_2M_1(\mathcal{G}_1) + n_1M_1(\mathcal{G}_2)$$

$+ n_1n_2(n_1+n_2-2)^2 + 4(n_1+n_2-2)(4m_1n_2+n_1m_2) + 16m_1m_2 + 4(p-1)m_1.$

Thus the result. □

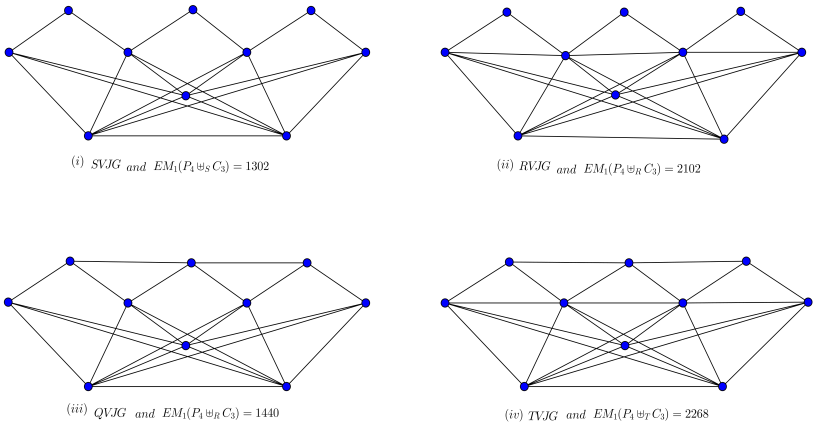


Figure 3. The several FVJG of P_4 and C_3.

For non-trivial graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ the vertex R-join of graph [19] or the RVJG $(\mathcal{G}_1 \uplus_R \mathcal{G}_2)$ is achieved from $R(\mathcal{G}_1)$ and $\mathcal{G}_2$. It forms by connecting each $u \in V(\mathcal{G}_1)$ together with every $v \in V(\mathcal{G}_2)$ (see Fig. 3). Therefore g-RVJG becomes to RVJG when $p = 1$.

Corollary 2.7. *Let $\mathcal{G}_y$ be a graph containing vertices n_i and edges m_i for $i = 1, 2$. Then we have* $EM_1(\mathcal{G}_1 \uplus_R \mathcal{G}_2) = 8F(\mathcal{G}_1) + F(\mathcal{G}_2) + 8(2n_2-1)M_1(\mathcal{G}_1) + (5n_1-4)M_1(\mathcal{G}_2) + 2M_2(\mathcal{G}_2) + 8M_2(\mathcal{G}_1) + 4m_1(n_2-1)^2 + 4m_2(n_1-1)^2 + n_1n_2(n_1+n_2-2)^2 + 4(n_1+n_2-2)(2m_1n_2 + n_1m_2) + 2m_1n_2^2 + 16m_1m_2.$

Corollary 2.8. *Let n_i and m_i $(i = 1, 2)$ be the vertices and edges of $\mathcal{G}_i$ respectively. Then we obtain the following relation* $EM_1(\mathcal{G}_1 \otimes_R \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_R \mathcal{G}_2) + 12M_1(\mathcal{G}_2) + 4n_1n_2(n_1 + n_2 - 2) + 16m_2(m_1+n_1) + 4m_1^2n_2 + 2n_1n_2 + 16m_1.$

Definition 2.9. The generalized Q-vertex join graphs or g-QVJG $(\mathcal{G}_1 \uplus_{Q_p} \mathcal{G}_2)$ is obtained from $Q_p(\mathcal{G}_1)$ along with $\mathcal{G}_2$. It is constructed by adding each $u \in V(\mathcal{G}_1)$ with every $v \in V(\mathcal{G}_2)$(see Fig. 2(i)).

Here we calculate the expression of RFZI for $\mathcal{G}_1 \uplus_{Q_p} \mathcal{G}_2$.

Theorem 2.10. *If graph $\mathcal{G}_i$ be connected of order n_i and size m_i for $i = 1, 2$, then the RFZI of $(\mathcal{G}_1 \uplus_{Q_p} \mathcal{G}_2)$ can be expressed as follows:*

$EM_1(\mathcal{G}_1 \uplus_{Q_p} \mathcal{G}_2) = p[F(L(\mathcal{G}_1)) + 4EM_1(\mathcal{G}_1) + 2EM_2(\mathcal{G}_1)] + 4(p-1)[F(\mathcal{G}_1) + 2M_2(\mathcal{G}_1) - M_1(\mathcal{G}_1) + m_1] + 5F(\mathcal{G}_1) + HM(\mathcal{G}_2) + (7n_2 + 2p - 10)M_1(\mathcal{G}_1) + (5n_1-4)M_1(\mathcal{G}_2) + 8M_2(\mathcal{G}_1) + n_1n_2(n_1+n_2-2)^2 + 4(n_1+n_2-2)(m_1n_2+n_1m_2) + 8m_1m_2 + 4m_2(n_1-1)^2 + 2m_1(n_2-2)^2 - 4pm_1.$

Proof. With the help of formalization of RFZI, we have

$$\begin{aligned} EM_1(\mathcal{G}_1 \uplus_{Q_p} \mathcal{G}_2) = & \sum_{\mu\nu, \nu w \in E(Q_p(\mathcal{G}_1))} p[(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1)}(\nu) + (d_{\mathcal{G}_1}(\nu) + d_{\mathcal{G}_1}(w)) - 2]^2 \\ & + \sum_{\mu \in V(Q_p(\mathcal{G}_1)), e=\nu w \in I(Q_p(\mathcal{G}_1))} [d_{\mathcal{G}_1}(\mu) + n_2 + (d_{\mathcal{G}_1}(\nu) + d_{\mathcal{G}_1}(w)) - 2]^2 \\ & + \sum_{\mu\nu \in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(\mu) + n_1 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2 \\ & + \sum_{\mu \in V(Q_p(\mathcal{G}_1))} \sum_{\nu \in V(\mathcal{G}_2)} [d_{\mathcal{G}_1}(\mu) + n_2 + d_{\mathcal{G}_2}(\nu) + n_1 - 2]^2 \end{aligned}$$

$$+ \sum_{\mu\nu\in E(I(Q_p(\mathcal{G}_1)));\mu,\nu\in V(Q_p(\mathcal{G}_1))-V(\mathcal{G}_1)} 4[(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu)-1]^2$$

$$= \sum_{e,f\in L(\mathcal{G}_1),e=\mu\nu,f=\nu w} [(d_{L(\mathcal{G}_1)})(e)+(d_{L(\mathcal{G}_1)})(f)+2]^2$$
$$+ \sum_{\mu\nu\in E(\mathcal{G}_1)} [(2d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu)+(n_2-2))^2+(d_{\mathcal{G}_1}(\mu)+2d_{\mathcal{G}_1}(\nu)+(n_2-2))^2]$$
$$+ \sum_{\mu\nu\in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(\mu)+d_{\mathcal{G}_2}(\nu))+2(n_1-1)]^2$$
$$+ \sum_{\mu\in V(\mathcal{G}_1)}\sum_{\nu\in V(\mathcal{G}_2)} [d^2_{\mathcal{G}_2}(\nu)+d^2_{\mathcal{G}_1}(\mu)+2d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_2}(\nu)$$
$$+ (n_1+n_2-2)^2+2(n_1+n_2-2)(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_2}(\nu))]$$
$$+ \sum_{\mu\nu\in E(I(Q_p(\mathcal{G}_1)))} 4[(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu)-1]^2$$

$$= \sum_{ef\in E(L(\mathcal{G}_1))} [(d^2_{L(\mathcal{G}_1)}(e)+d^2_{L(\mathcal{G}_1)}(f))+2d_{L(\mathcal{G}_1)}(e)d_{L(\mathcal{G}_1)}(f)+4+\Big(d_{L(\mathcal{G}_1)}(e)+d_{L(\mathcal{G}_1)}(f)\Big)]$$
$$+ \sum_{\mu\nu\in E(\mathcal{G}_1)} [5(d^2_{\mathcal{G}_1}(\mu)+d^2_{\mathcal{G}_1}(\nu))+2(n_2-2)^2+6(n_2-2)(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))+8d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu)]$$
$$+ \sum_{\mu\nu\in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(\mu)+d_{\mathcal{G}_2}(\nu))^2+4(n_1-1)^2+4(n_1-1)(d_{\mathcal{G}_2}(\mu)+d_{\mathcal{G}_2}(\nu))]$$
$$+ \sum_{\mu\in V(\mathcal{G}_1)}\sum_{\nu\in V(\mathcal{G}_2)} d^2_{\mathcal{G}_2}(\nu)+\sum_{\nu\in V(\mathcal{G}_2)}\sum_{\mu\in V(\mathcal{G}_1)} d^2_{\mathcal{G}_1}(\mu)+2\sum_{\mu\in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(\mu)\sum_{\nu\in V(\mathcal{G}_2)} d_{\mathcal{G}_2}(\nu)$$
$$+ n_1n_2(n_1+n_2-2)^2+2(n_1+n_2-2)[\sum_{\mu\in V(\mathcal{G}_2)}\sum_{\nu\in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(\mu)+\sum_{\mu\in V(\mathcal{G}_1)}\sum_{\nu\in V(\mathcal{G}_2)} d_{\mathcal{G}_2}(\nu)]$$
$$+ 4(p-1)\sum_{\mu\nu\in E(Q_p(\mathcal{G}_1))} [(d^2_{\mathcal{G}_1}(\mu)+d^2_{\mathcal{G}_1}(\nu))+2(d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu))-2(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))+1]$$

$$= F(L(\mathcal{G}_1))+2EM_2(\mathcal{G}_1)+4[\frac{(M_1\mathcal{G}_1)}{2}-m_1]+4EM_1(\mathcal{G}_1)$$
$$+ 5F(\mathcal{G}_1)+2m_1(n_2-2)^2+6(n_2-2)M_1(\mathcal{G}_1)+8M_2(\mathcal{G}_1)$$
$$+ HM(\mathcal{G}_2)+4(n_1-1)^2m_2+4(n_1-1)M_1(\mathcal{G}_2)$$
$$+ n_1M_1(\mathcal{G}_2)+n_2M_1(\mathcal{G}_1)+8m_1m_2+n_1n_2(n_1+n_2-2)^2+4(n_1+n_2-2)(m_1n_2+n_1m_2)$$
$$+ 4(p-1)[F(\mathcal{G}_1)+2M_2(\mathcal{G}_1)-2M_1(\mathcal{G}_1)+m_1].$$

Hence the required result follows. □

Corollary 2.11. *Let $\mathcal{G}_z$ be of order n_z and size m_z for $z=1,2$, the RFZI of $(\mathcal{G}_1 \uplus_Q \mathcal{G}_2)$ can be expressed as follows* $EM_1(\mathcal{G}_1 \uplus_Q \mathcal{G}_2) = F(L(\mathcal{G}_1))+4EM_1(\mathcal{G}_1)+2EM_2(\mathcal{G}_1)+5F(\mathcal{G}_1)+HM(\mathcal{G}_2)+(7n_2-10)M_1(\mathcal{G}_1)+(5n_1-4)M_1(\mathcal{G}_2)+8M_2(\mathcal{G}_1)+n_1n_2(n_1+n_2-2)^2+4(n_1+n_2-2)(m_1n_2+n_1m_2)+8m_1m_2+4m_2(n_1-1)^2+2m_1(n_2-2)^2-4m_1$.

Corollary 2.12. *Let n_i and m_i $(i=1,2)$ be the vertices and edges of $\mathcal{G}_i$ respectively. Then we obtain the following relation* $EM_1(\mathcal{G}_1 \otimes_Q \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_Q \mathcal{G}_2)+12M_1(\mathcal{G}_2)+4n_1n_2(n_1+n_2-2)+16m_2(m_1+n_1)+4m_1^2n_2+2n_1n_2+8m_1$.

Definition 2.13. The g-TVJG $(\mathcal{G}_1 \uplus_{T_p} \mathcal{G}_2)$ is derived from $T_p(\mathcal{G}_1)$ with $\mathcal{G}_2$ by linking each $u \in V(\mathcal{G}_1)$ with every $v \in V(\mathcal{G}_2)$(see Fig. 2(ii)).

The expression of RFZI for $\mathcal{G}_1 \uplus_{T_p} \mathcal{G}_2$ is derived here.

Theorem 2.14. *If $\mathcal{G}_i$ be a molecular graph of order n_i and size m_i, $i = 1, 2$. Then* $EM_1(\mathcal{G}_1 \uplus_{T_p} \mathcal{G}_2) = p[2EM_2(\mathcal{G}_1) + F(L(\mathcal{G}_1)) + 4EM_1(\mathcal{G}_1)] + 10F(\mathcal{G}_1) + HM(\mathcal{G}_2) + 4HM(\mathcal{G}_1) + 2(10n_2 + p - 12)M_1(\mathcal{G}_1) + (5n_1 - 4)M_1(\mathcal{G}_2) + 12M_2(\mathcal{G}_1) + 4(p-1)[2M_2(\mathcal{G}_1) + F(\mathcal{G}_1) - M_1(\mathcal{G}_1) + m_1] + n_1n_2(n_1 + n_2 - 2)^2 + 4(n_2 + n_1 - 2)(n_1m_2 + 2m_1n_2) + 4m_1(n_2 - 1)^2 + 16m_1m_2 + 4m_2(n_1 - 1)^2 + 2m_1(n_2 - 2)^2 - 4pm_1$.

Proof. The $RFZI$ of $(\mathcal{G}_1 \uplus_{T_p} \mathcal{G}_2)$ is calculated as

$$
\begin{aligned}
&EM_1(\mathcal{G}_1 \uplus_{T_p} \mathcal{G}_2)\\
&= \sum_{\mu,v\in I(T_p(\mathcal{G}_1))} [d_{I(\mathcal{G}_1)}(\mu) + d_{I(\mathcal{G}_1)}(v) - 2]^2\\
&+ \sum_{\mu\in V(\mathcal{G}_1),v\in I(T_p(\mathcal{G}_1))} [2d_{\mathcal{G}_1}(\mu) + n_2 + d_{I(\mathcal{G}_1)}(v) - 2]^2\\
&+ \sum_{\mu v\in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(\mu) + n_1 + d_{\mathcal{G}_2}(v) + n_1 - 2]^2\\
&+ \sum_{\mu\in V(\mathcal{G}_1)} \sum_{v\in V(\mathcal{G}_2)} [d_{\mathcal{G}_2}(v) + n_1 + 2d_{\mathcal{G}_1}(\mu) + n_2 - 2]^2\\
&+ \sum_{\mu v\in E(\mathcal{G}_1)} [2d_{\mathcal{G}_1}(\mu) + n_2 + 2d_{\mathcal{G}_1}(v) + n_2 - 2]^2\\
&+ \sum_{\mu\nu\in E(T_p(\mathcal{G}_1));\mu,\nu\in V(\mathcal{G}_1)\setminus V(\mathcal{G}_1)} 4[(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) - 1]^2\\
&= \sum_{ef\in E(L(\mathcal{G}_1))} [d^2_{L(\mathcal{G}_1)}(e) + d^2_{L(\mathcal{G}_1)}(f) + 2d_{L(\mathcal{G}_1)}(e)d_{L(\mathcal{G}_1)}(f) + 4 + 4(d_{L(\mathcal{G}_1)}(e) + d_{L(\mathcal{G}_1)}(f))]\\
&+ \sum_{\mu v\in E(\mathcal{G}_1)} [(3d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(v) + (n_2 - 2))^2 + (d_{\mathcal{G}_1}(\mu) + 3d_{\mathcal{G}_1}(v) + (n_2 - 2))^2]\\
&+ \sum_{\mu v\in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(v) + 2(n_1 - 1)]^2\\
&+ \sum_{\mu\in V(\mathcal{G}_1)} \sum_{v\in V(\mathcal{G}_2)} d^2_{\mathcal{G}_2}(v) + 4 \sum_{v\in V(\mathcal{G}_2)} \sum_{\mu\in V(\mathcal{G}_1)} d^2_{\mathcal{G}_1}(\mu) + (n_1 + n_2 - 2)^2 \sum_{\mu\in V(\mathcal{G}_1)} \sum_{v\in V(\mathcal{G}_2)} 1\\
&\quad + 2(n_2 + n_1 - 2)[\sum_{\mu\in V(\mathcal{G}_1)} \sum_{v\in V(\mathcal{G}_2)} d_{\mathcal{G}_2}(v) + 2 \sum_{v\in V(\mathcal{G}_2)} \sum_{\mu\in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(\mu)]\\
&\quad + 4 \sum_{\mu\in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(\mu) \sum_{v\in V(\mathcal{G}_2)} d_{\mathcal{G}_2}(v)\\
&\quad + \sum_{\mu v\in E(\mathcal{G}_1)} [2(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(v)) + 2(n_2 - 1)]^2\\
&\quad + 4 \sum_{\mu\nu\in E(T_p(\mathcal{G}_1))} [(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) - 1]^2\\
&= F(L(\mathcal{G}_1)) + 2M_2(L(\mathcal{G}_1)) + 4|E(L(\mathcal{G}_1))| + 4M_1(L(\mathcal{G}_1))\\
&+ \sum_{\mu v\in E(\mathcal{G}_1)} [10(d^2_{\mathcal{G}_1}(\mu) + d^2_{\mathcal{G}_1}(v)) + 2(n_2 - 2)^2 + 12d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(v)\\
&+ 8(n_2 - 2)(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(v))\}\\
&+ \sum_{\mu v\in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(v))^2 + 4(n_1 - 1)(d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(v)) + 4(n_1 - 1)^2]
\end{aligned}
$$

$$
\begin{aligned}
&+ n_1M_1(\mathcal{G}_2) + 4n_2M_1(\mathcal{G}_1) + n_1n_2(n_2+n_1-2)^2\\
&+ 2(n_1+n_2-2)(2n_1m_2+4m_1n_2) + 16m_1m_2\\
&+ 4\sum_{\mu v\in E(\mathcal{G}_1)}[(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(v))^2 + (n_2-1)^2 + 2(n_2-1)(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(v))]\\
&+ 4(p-1)\sum_{\mu\nu\in E(T_p(\mathcal{G}_1))}[(d^2_{\mathcal{G}_1}(\mu)+d^2_{\mathcal{G}_1}(\nu)) + 2(d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu))\\
&- 2(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu)) + 1]
\end{aligned}
$$

$$
\begin{aligned}
&= F(L(\mathcal{G}_1)) + 2EM_2(\mathcal{G}_1) + 2M_1(\mathcal{G}_1) - 4m_1 + 4EM_1(\mathcal{G}_1)\\
&+ 10F(\mathcal{G}_1) + 2m_1(n_2-2)^2 + 12M_2(\mathcal{G}_1) + 8(n_2-2)M_1(\mathcal{G}_1)\\
&+ HM(\mathcal{G}_2) + 4(n_1-1)M_1(\mathcal{G}_2) + 4m_2(n_1-1)^2\\
&+ n_1M_1(\mathcal{G}_2) + 4n_2M_1(\mathcal{G}_1) + n_1n_2(n_1+n_2-2)^2\\
&+ 4(n_1+n_2-2)(n_1m_2+2m_1n_2) + 16m_1m_2\\
&+ 4[HM(\mathcal{G}_1) + m_1(n_2-1)^2 + 2(n_2-1)M_1(\mathcal{G}_1)]\\
&+ 4(p-1)[2M_2(\mathcal{G}_1) + F(\mathcal{G}_1) - M_1(\mathcal{G}_1) + m_1].
\end{aligned}
$$

Hence this complete the result. □

When $p = 1$, g-TVJG converts to TVJG.

Corollary 2.15. *Let $\mathcal{G}_i$ be a molecular graph of order n_i and size m_i, $i = 1, 2$. Then* $EM_1(\mathcal{G}_1 \uplus_T \mathcal{G}_2) = F(L(\mathcal{G}_1)) + 2EM_2(\mathcal{G}_1) + 4EM_1(\mathcal{G}_1) + 10F(\mathcal{G}_1) + HM(\mathcal{G}_2) + 4HM(\mathcal{G}_1) + 2(10n_2 - 11)M_1(\mathcal{G}_1) + (5n_1 - 4)M_1(\mathcal{G}_2) + 12M_2(\mathcal{G}_1) + n_1n_2(n_1+n_2-2)^2 + 4(n_2+n_1-2)(n_1m_2 + 2m_1n_2) + 4m_1(n_2-1)^2 + 16m_1m_2 + 4m_2(n_1-1)^2 + 2m_1(n_2-2)^2 - 4m_1$.

Corollary 2.16. *Let n_i and m_i $(i = 1, 2)$ be the vertices and edges of $\mathcal{G}_i$ respectively. Then we obtain the following relation* $EM_1(\mathcal{G}_1 \otimes_T \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_T \mathcal{G}_2) + 12M_1(\mathcal{G}_2) + 4n_1n_2(n_1 + n_2 - 2) + 16m_2(m_1+n_1) + 4m_1^2n_2 + 2n_1n_2 + 16m_1$.

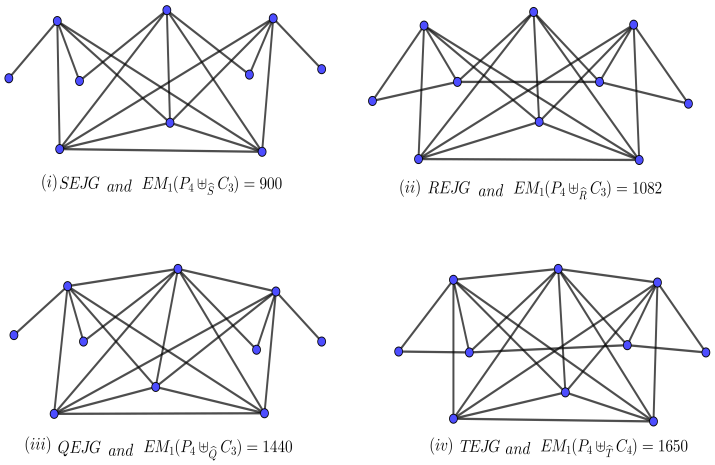


Figure 4. The several FEJG of P_4 and C_3.

Definition 2.17. The g-SEJG of $\mathcal{G}_1$ with $\mathcal{G}_2$, characterized as $\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2$, is derived from $\widehat{S}_p(\mathcal{G}_1)$ and $\mathcal{G}_2$. Then each and every vertex of $I(\widehat{S}_p(\mathcal{G}_1))$ is joined to all vertices of $\mathcal{G}_2$.

As an example we have depicted the Fig. 1(iii).
The degree distribution of the vertices in $\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2$ are followed as $d_{\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2}(\mu) = d_{\mathcal{G}_1}(\mu)$ if

$\mu \in V(\widehat{S}_p(\mathcal{G}_1))$; for $\nu \in V(\widehat{S}_p(\mathcal{G}_2))$, $d_{\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2}(\nu) = d_{\mathcal{G}_2}(\nu) + m_1$ and $d_{\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2}(\mu) = 2 + n_2$ for $\mu \in I(\widehat{S}_p(\mathcal{G}_2))$. Here we have calculated the RFZI of g-SEJG of $\mathcal{G}_1$ together with $\mathcal{G}_2$.

Theorem 2.18. *Suppose that $\mathcal{G}_i$, for $i = \{1, 2\}$, be a non-trivial connected graph of order n_i and size m_i. Thus* $EM_1(\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2) = F(\mathcal{G}_1) + 2m_1n_2^2 + 2n_2M_1(\mathcal{G}_1) + HM(\mathcal{G}_2) + 4(m_1 - 1)M_1(\mathcal{G}_2) + 4m_2(m_1 - 1)^2 + pm_1[M_1(\mathcal{G}_2) + 4m_2(n_2 + m_1) + n_2(m_1 + n_2)^2] + 4m_1(p - 1)(n_2 + 1)^2$.

We calculate here the manifest expression of RFZI for $\mathcal{G}_1 \uplus_{\widehat{S}} \mathcal{G}_2$.

Proof. The characterization of RFZI implies that

$$\begin{aligned} EM_1(\mathcal{G}_1 \uplus_{\widehat{S}_p} \mathcal{G}_2) = & \sum_{v \in V(\widehat{S}_p(\mathcal{G}_1))} [d_{\mathcal{G}_1}(v) + n_2]^2 d_{\mathcal{G}_1}(v) \\ & + \sum_{uv \in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(u) + m_1 + d_{\mathcal{G}_2}(v) + m_1 - 2]^2 \\ & + p[\sum_{u \in I(\widehat{S}_p(\mathcal{G}_1))} \sum_{w \in V(\mathcal{G}_2)} (d_{\mathcal{G}_2}(w) + m_1 + n_2)^2] \end{aligned}$$

$$\begin{aligned} & + \sum_{uv \in E(\widehat{S}_p(\mathcal{G}_1))} [2(n_2 + 2) - 2]^2 \\ & = \sum_{v \in V(\mathcal{G}_1)} [(d^2_{\mathcal{G}_1}(v) + n_2^2 + 2n_2 d_{\mathcal{G}_1}(v))] d_{\mathcal{G}_1}(v) \\ & + \sum_{uv \in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(u) + d_{\mathcal{G}_2}(v)) + 2(m_1 - 1)]^2 + pm_1 \sum_{v \in V(\mathcal{G}_2)} [d_{\mathcal{G}_2}(v) + m_1 + n_2]^2 \\ & + 4(p - 1) \sum_{uv \in E(\mathcal{G}_1)} [n_2 + 1]^2 \\ & = F(\mathcal{G}_1) + 2m_1n_2^2 + 2n_2M_1(\mathcal{G}_1) + HM(\mathcal{G}_2) + 4m_2(m_1 - 1)^2 \\ & + 4(m_1 - 1)M_1(\mathcal{G}_2) + pm_1 \sum_{v \in V(\mathcal{G}_2)} [d^2_{\mathcal{G}_2}(v) + 2(m_1 + n_2)d_{\mathcal{G}_2}(v) + (m_1 + n_2)^2] \\ & = F(\mathcal{G}_1) + 2m_1n_2^2 + 2n_2M_1(\mathcal{G}_1) + HM(\mathcal{G}_2) + 4(m_1 - 1)M_1(\mathcal{G}_2) \\ & + 4m_2(m_1 - 1)^2 + pm_1[M_1(\mathcal{G}_2) + 4m_2(n_2 + m_1) + n_2(m_1 + n_2)^2] \\ & + 4m_1(p - 1)(n_2 + 1)^2. \end{aligned}$$

Thus the desired result follows. □

The edge S-join of graphs [24] or S-edge join graphs (SEJG) of $\mathcal{G}_1$ with $\mathcal{G}_2$, characterized as $\mathcal{G}_1 \uplus_{\widehat{S}} \mathcal{G}_2$, is derived from $S(\mathcal{G}_1)$ with $\mathcal{G}_2$. The it conjoins each and every vertex of $I(\mathcal{G}_1)$ to all vertices of Y_2. As an example we have depicted the Fig. 4(i).
For p=1, g-SEJG becomes to SEJG.

Corollary 2.19. *Suppose that $\mathcal{G}_i$, for $i = \{1, 2\}$, be a non-trivial connected graph of order n_i and size m_i. Thus* $EM_1(\mathcal{G}_1 \uplus_{\widehat{S}} \mathcal{G}_2) = F(\mathcal{G}_1) + HM(\mathcal{G}_2) + 2n_2M_1(\mathcal{G}_1) + (5m_1 - 4)M_1(\mathcal{G}_2) + m_1n_2(m_1 + n_2)^2 + 4m_1m_2(n_2 + m_1) + 4m_2(m_1 - 1)^2 + 2m_1n_2^2$.

Corollary 2.20. *Let $\mathcal{G}_1$, $\mathcal{G}_2$ be two graphs having n_i vertices and m_i edges, where $i = \{1, 2\}$, Then* $EM_1(\mathcal{G}_1 \otimes_{\widehat{S}} \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_{\widehat{S}} \mathcal{G}_2) + 12M_1(\mathcal{G}_2) + 40m_1m_2 + 4m_1^2n_2 + 4m_1n_2(m_1 + n_2) + 2m_1n_2 - 8m_2$.

Definition 2.21. The g-REJG $\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2$ is derived from $\widehat{R}_p(\mathcal{G}_1)$ and $\mathcal{G}_2$ under the join operation in such way that every vertex of $I(\widehat{R}_p(\mathcal{G}_1))$ with all vertices of $\mathcal{G}_2$ (see Fig. 1(iv)).

The vertex-degrees in $\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2$ are followed as $d_{\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2}(\mu) = 2d_{\mathcal{G}_1}(\mu)$ when $\mu \in V(\widehat{R}_p(\mathcal{G}_1))$; when $\nu \in V(\widehat{R}_p(\mathcal{G}_2))$, $d_{\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2}(\nu) = m_1 + d_{\mathcal{G}_2}(\nu)$ and $d_{\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2}(\mu) = n_2 + 2$ for $\mu \in I(\widehat{R}_p(\mathcal{G}_2))$.

Theorem 2.22. *Let $\mathcal{G}_j$ be a molecular graph having n_j vertices and m_j edges, for $j = \{1, 2\}$, then* $EM_1(\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2) = 4HM(\mathcal{G}_1) - 8M_1(\mathcal{G}_1) + 4F(\mathcal{G}_1) + 4n_2M_1(\mathcal{G}_1) + 2m_1n_2^2 + pm_1[M_1(\mathcal{G}_2) + n_2(m_1 + n_2)^2 + 4m_2(m_1 + n_2)] + HM(\mathcal{G}_2) + 4m_2(m_1 - 1)^2 + 4(m_1 - 1)M_1(\mathcal{G}_2) + 4m_1(p - 1)(n_2 + 1)^2 + 4m_1$.

Proof. From the definition of RFZI of graph, we can be expressed as

$$
\begin{aligned}
EM_1(\mathcal{G}_1 \uplus_{\widehat{R}_p} \mathcal{G}_2) = & \sum_{\mu\nu \in E(\widehat{R}_p(\mathcal{G}_1))} [2d_{\mathcal{G}_1}(\mu) + 2d_{\mathcal{G}_1}(\nu) - 2]^2 \\
& + \sum_{\mu \in I(\widehat{R}_p(\mathcal{G}_1)), \nu \in V(\widehat{R}_p(\mathcal{G}_1))} [2d_{\mathcal{G}_1}(\nu) + n_2]^2 \\
& + \sum_{\mu \in I(\widehat{R}_p(\mathcal{G}_1)), \nu \in V(\widehat{R}_p(\mathcal{G}_2))} p[d_{\mathcal{G}_2}(\nu) + m_1 + n_2]^2 \\
& + \sum_{\mu\nu \in E(\widehat{R}_p(\mathcal{G}_2))} [d_{\mathcal{G}_2}(u) + m_1 + d_{\mathcal{G}_2}(\nu) + m_1 - 2]^2 \\
& + \sum_{\mu\nu \in E(I(\widehat{R}_p(\mathcal{G}_1)))} 4[n_2 + 1]^2 \\
& = 4 \sum_{\mu\nu \in E(\mathcal{G}_1)} [(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu))^2 - 2(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu)) + 1] \\
& + \sum_{\nu \in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(\nu)[4d_{\mathcal{G}_1}^2(\nu) + 4n_2 d_{\mathcal{G}_1}(\nu) + n_2^2] \\
& + pm_1 \sum_{\nu \in V(\mathcal{G}_2)} [d_{\mathcal{G}_1}^2(\nu) + (m_1 + n_2)^2 + 2(m_1 + n_2)d_{\mathcal{G}_2}(\nu)] \\
& + \sum_{\mu\nu \in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(\nu))^2 + 4(m_1 - 1)^2 + 4(m_1 - 1)(d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(\nu))] \\
& + 4(p - 1) \sum_{\mu\nu \in E(\mathcal{G}_1)} [n_2 + 1]^2 \\
& = 4[HM(\mathcal{G}_1) - 2M_1(\mathcal{G}_1) + m_1] \\
& + 4F(\mathcal{G}_1) + 4n_2M_1(\mathcal{G}_1) + 2m_1n_2^2 \\
& + pm_1[M_1(\mathcal{G}_2) + n_2(m_1 + n_2)^2 + 4m_2(m_1 + n_2)] \\
& + HM(\mathcal{G}_2) + 4m_2(m_1 - 1)^2 + 4(m_1 - 1)M_1(\mathcal{G}_2) \\
& + 4m_1(p - 1)(n_2 + 1)^2.
\end{aligned}
$$

By simplifying the above results we reach to the goal. □

The edge R-join for two graphs [24] or REJG of $\mathcal{G}_1$ and $\mathcal{G}_2$ signified by $\mathcal{G}_1 \uplus_{\widehat{R}} \mathcal{G}_2$ is formed by $R(\mathcal{G}_1)$ and $\mathcal{G}_2$. It is obtained by joining every vertex of $I(\mathcal{G}_1)$ with all vertices of $\mathcal{G}_2$ (see Fig. 4(ii)).

Corollary 2.23. *Let n_j and m_j be the vertices and edges in order of $\mathcal{G}_j$, for $j = \{1, 2\}$. Then* $EM_1(\mathcal{G}_1 \uplus_{\widehat{R}} \mathcal{G}_2) = 4HM(\mathcal{G}_1) + HM(\mathcal{G}_2) + 4F(\mathcal{G}_1) + 4(n_2 - 2)M_1(\mathcal{G}_1) + (5m_1 - 4)M_1(\mathcal{G}_2) + m_1n_2(n_2 + m_1)^2 + 4m_1m_2(n_2 + m_1) + 4m_2(m_1 - 1)^2 + 2m_1n_2^2 + 4m_1$.

Corollary 2.24. *If $\mathcal{G}_1$, $\mathcal{G}_2$ be two graphs having n_i vertices and m_i edges, where $i = \{1, 2\}$, Then* $EM_1(\mathcal{G}_1 \otimes_{\widehat{R}} \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_{\widehat{R}} \mathcal{G}_2) + 12M_1(\mathcal{G}_2) + 40m_1m_2 + 4m_1^2n_2 + 4m_1n_2(m_1 + n_2) + 2m_1n_2 - 8m_2$.

Definition 2.25. The g-QEJG of two disjoint graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, characterized as $\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2$, is derived from $\widehat{Q}_p(\mathcal{G}_1)$ as well as $\mathcal{G}_2$ by adding one and all vertices of $I(\widehat{Q}_p(\mathcal{G}_1))$ to all vertices of $\mathcal{G}_2$, (see Fig. 2(iii)).

The valency of the nodes of $\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2$ are $d_{\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2}(\mu) = d_{\mathcal{G}_1}(\mu)$ or $(d_{\mathcal{G}_2}(\mu) + m_1)$ or $(d_{\mathcal{G}_2}(\psi) + d_{\mathcal{G}_2}(\omega) + n_2)$ when $\mu \in V(\mathcal{G}_1)$ or $\mu \in V(\mathcal{G}_2)$ or $\psi\omega = e \in I(\widehat{Q}_p(\mathcal{G}_1))$ respectively.

Theorem 2.26. *Consider $\mathcal{G}_k$ for $k = \{1, 2\}$ be a (n_k, m_k) graph. Then* $EM_1(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2) = p[F(L(\mathcal{G}_1)) + 2EM_2(\mathcal{G}_1) + 4(n_2+1)EM_1(\mathcal{G}_1)] + n_2 HM(\mathcal{G}_1) + HM(\mathcal{G}_2) + 5F(\mathcal{G}_1) + \Big(2p(n_2+1)^2 + 2n_2^2 + 2m_1n_2 + 2n_2 + 4m_2 - 12\Big)M_1(\mathcal{G}_1) + (5m_1 - 4)M_1(\mathcal{G}_2) + 8M_2(\mathcal{G}_1) + m_1n_2(n_2 + m_1 - 2)^2 + 4n_2m_1(m_1 + n_2 - 2) + 4m_2(m_1 - 1)^2 + 2m_1(n_2 - 2)^2 - 4pm_1(n_2 + 1)^2.$

Proof. $E(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2)$ is partitioned into five subsets like
$\{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2) : \mu\nu \in E(I(\widehat{Q}_p(\mathcal{G}_1))$ for similar edges of $I(\widehat{Q}_p)\}$;
$\{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2) : \mu\nu \in E(I(\widehat{Q}_p(\mathcal{G}_1))$ for two dissimilar but also adjacent edges of $I(\widehat{Q}_p)\}$;
$\{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2) : \mu \in V(\widehat{Q}_p(\mathcal{G}_1)), \nu \in I(\widehat{Q}_p(\mathcal{G}_1))\}$;
$\{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2) : \mu \in I(\widehat{Q}_p(\mathcal{G}_1)), \nu \in V(\mathcal{G}_2)\}$;
$\{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2) : \mu\nu \in E(\mathcal{G}_2)\}$.

$$
\begin{aligned}
&EM_1(\mathcal{G}_1 \uplus_{\widehat{Q}_p} \mathcal{G}_2)\\
&= \sum_{\mu\nu,\nu w \in E(I(\widehat{Q}_p(\mathcal{G}_1))} [d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + n_2 + d_{\mathcal{G}_1}(\nu) + d_{\mathcal{G}_1}(w) + n_2 - 2]^2\\
&+ \sum_{\mu\nu \in E(I(\widehat{Q}_p(\mathcal{G}_1)))} 4[(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + n_2 - 1]^2\\
&+ \sum_{\mu \in V(\widehat{Q}_p(\mathcal{G}_1)), e,f \in I(\widehat{Q}_p(\mathcal{G}_1))} [(d_{\mathcal{G}_1}(\mu) + d_{I(\mathcal{G}_1)}(e) - 2)^2 + (d_{\mathcal{G}_1}(\mu) + d_{I(\mathcal{G}_1)}(f) - 2)^2]\\
&+ \sum_{\nu \in V(\mathcal{G}_2)} \sum_{\mu \in I(\widehat{Q}_p(\mathcal{G}_1))} [(d_{I(\mathcal{G}_1)}(\mu) + n_2) + (d_{\mathcal{G}_2}(\nu) + m_1) - 2]^2\\
&+ {}_{\mu\nu \in E(\mathcal{G}_2)}[(d_{\mathcal{G}_2}(\mu) + m_1) + d_{\mathcal{G}_2}(\nu) + m_1 - 2]^2\\
&= \sum_{e=\mu\nu, f=\nu w \in E(L(\mathcal{G}_1))} p[d_{L(\mathcal{G}_1)}(e) + d_{L(\mathcal{G}_1)}(f) + 2(n_2 + 1)]^2\\
&+ 4(p-1) \sum_{\mu\nu \in E(\mathcal{G}_1)} [(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + n_2 - 1]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\mu\nu \in E(\mathcal{G}_1)} [(2d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + (n_2 - 2))^2 + (d_{\mathcal{G}_1}(\mu) + 2d_{\mathcal{G}_1}(\nu) + (n_2 - 2))^2]\\
&+ \sum_{\mu\nu \in E(\mathcal{G}_1)} \sum_{w \in V(\mathcal{G}_2)} [(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu)) + d_{\mathcal{G}_2}(w) + (m_1 + n_2 - 2)]^2\\
&+ \sum_{\mu\nu \in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(\mu) + d_{\mathcal{G}_2}(\nu) + 2(m_1 - 1))]^2\\
&= \sum_{ef \in E(L(\mathcal{G}_1))} p[d^2_{L(\mathcal{G}_1)}(e) + d^2_{L(\mathcal{G}_1)}(f) + 2d_{L(\mathcal{G}_1)}(e)d_{L(\mathcal{G}_1)}(f) + 4(n_2 + 1)^2\\
&+ 4(n_2 + 1)(d_{L(\mathcal{G}_1)}(e) + d_{L(\mathcal{G}_1)}(f))]\\
&+ 4(p-1) \sum_{\mu\nu \in E(\mathcal{G}_1)} [(d^2_{\mathcal{G}_1}(\mu) + d^2_{\mathcal{G}_1}(\nu)) + 2d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(n_2-1)(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))+(n_2-1)^2]\\
&+ 4(n_2+1)(d_{L(\mathcal{G}_1)}(e)+d_{L(\mathcal{G}_1)}(f))]+5\sum_{\mu\nu\in E(\mathcal{G}_1)}(d^2_{\mathcal{G}_1}(\mu)+d^2_{\mathcal{G}_1}(\nu))+\sum_{\mu\nu\in E(\mathcal{G}_1)}2(n_2-2)^2\\
&+ 6(n_2-2)\sum_{\mu\nu\in E(\mathcal{G}_1)}(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))+8\sum_{\mu\nu\in E(\mathcal{G}_1)}(d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu))\\
&+ \sum_{w\in V(\mathcal{G}_2)}\sum_{\mu\nu\in E(\mathcal{G}_1)}(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))^2+\sum_{\mu\nu\in E(\mathcal{G}_1)}\sum_{w\in V(\mathcal{G}_2)}d^2_{\mathcal{G}_2}(w)\\
&+ \sum_{w\in V(\mathcal{G}_2)}\sum_{\mu\nu\in E(\mathcal{G}_1)}(m_1+n_2-2)^2\\
&+ 2\sum_{w\in V(\mathcal{G}_2)}d_{\mathcal{G}_2}(w)\sum_{\mu\nu\in E(\mathcal{G}_1)}(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))\\
&+ 2(m_1+n_2-2)\sum_{w\in V(\mathcal{G}_2)}\sum_{\mu\nu\in E(\mathcal{G}_1)}(d_{\mathcal{G}_1}(\mu)+d_{\mathcal{G}_1}(\nu))\\
&+ 2(m_1+n_2-2)\sum_{w\in V(\mathcal{G}_2)}\sum_{\mu\nu\in E(\mathcal{G}_1)}d_{\mathcal{G}_2}(w)\\
&+ \sum_{\mu\nu\in E(\mathcal{G}_2)}[(d_{\mathcal{G}_2}(\mu)+d_{\mathcal{G}_2}(\nu))^2+4(m_1-1)(d_{\mathcal{G}_2}(\mu)+d_{\mathcal{G}_2}(\nu))+4(m_1-1)^2]\\
&= p[F(L(\mathcal{G}_1))+2EM_2(\mathcal{G}_1)+4(n_2+1)^2\Big(\frac{M_1(\mathcal{G}_1)}{2}-m_1\Big)+4(n_2+1)EM_1(\mathcal{G}_1)]\\
&+ 4(p-1)[F(\mathcal{G}_1)+2M_2(\mathcal{G}_1)+2(n_2-1)M_1(\mathcal{G}_1)+m_1(n_2-1)^2]\\
&+ 5F(\mathcal{G}_1)+2m_1(n_2-2)^2+6(n_2-2)M_1(\mathcal{G}_1)+8M_2(\mathcal{G}_1)\\
&+ n_2HM(\mathcal{G}_1)+m_1M_1(\mathcal{G}_2)+m_1n_2(m_1+n_2-2)^2+4m_2M_1(\mathcal{G}_1)\\
&+ 2n_2(m_1+n_2-2)M_1(\mathcal{G}_1)+4m_1n_2(m_1+n_2-2)\\
&+ HM(\mathcal{G}_2)+4(m_1-1)M_1(\mathcal{G}_2)+4m_2(m_1-1)^2.
\end{aligned}
$$

By adding the above, we have the desired result. □

The edge Q-join of graphs [24] or Q-edge join graphs (QEJG) of two disjoint graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, characterized as $\mathcal{G}_1 \uplus_{\widehat{Q}} \mathcal{G}_2$, is derived from $Q(\mathcal{G}_1)$ as well as $\mathcal{G}_2$ by adding one and all vertices of $V(\mathcal{G}_1)$ to all vertices of $\mathcal{G}_2$, (see the Fig. 4(iii)).

Corollary 2.27. *Assume that $\mathcal{G}_k$ be a graph with n_k vertices and m_k edges, for $k=\{1,2\}$. Then* $EM_1(\mathcal{G}_1 \uplus_{\widehat{Q}} \mathcal{G}_2) = F(L(\mathcal{G}_1))+2EM_2(\mathcal{G}_1)+4(n_2+1)EM_1(\mathcal{G}_1)+n_2HM(\mathcal{G}_1)+HM(\mathcal{G}_2)+5F(\mathcal{G}_1)+(4n_2^2+2m_1n_2+6n_2+4m_2-10)M_1(\mathcal{G}_1)+(5m_1-4)M_1(\mathcal{G}_2)+8M_2(\mathcal{G}_1)+m_1n_2(n_2+m_1-2)^2+4n_2m_1(m_1+n_2-2)+4m_2(m_1-1)^2+2m_1(n_2-2)^2-4m_1(n_2+1)^2$.

Corollary 2.28. *Let $\mathcal{G}_1$, $\mathcal{G}_2$ be two graphs containing n_i vertices and m_i edges, where $i=\{1,2\}$. Then* $EM_1(\mathcal{G}_1 \otimes_{\widehat{Q}} \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_{\widehat{Q}} \mathcal{G}_2)+4(n_2+3)M_1(\mathcal{G}_2)+2m_1n_2(4m_1+2n_2-3)+32m_1m_2$.

Definition 2.29. The g-TEJG $\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2$ is created by adding each and every vertex of $T(\mathcal{G}_1)$ to one and all vertices of $\mathcal{G}_2$ (see Fig. 2(iv)).

$E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2)$ is divided into six subsets such as

$\mathcal{P}_1 : \{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) : \mu\nu \in E(I(\widehat{T}_p(\mathcal{G}_1))$ for similar edges of $I(\widehat{T}_p)\}$;

$\mathcal{P}_2 : \{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) : \mu\nu \in E(I(\widehat{T}_p(\mathcal{G}_1))$ for two dissimilar but adjacent edges of $I(\widehat{T}_p)\}$;

$\mathcal{P}_3 : \{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) : \mu \in V(\mathcal{G}_1), \nu \in I(\widehat{T}_p(\mathcal{G}_1))\}$;

$\mathcal{P}_4 : \{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) : \mu \in I(\widehat{T}_p(\mathcal{G}_1)), \nu \in V(\mathcal{G}_2)\}$;

$\mathcal{P}_5 : \{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) : \mu\nu \in E(\mathcal{G}_1)\}$;

$\mathcal{P}_6 : \{\mu\nu \in E(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) : \mu\nu \in E(\mathcal{G}_2)\}$.

Now we establish the exact formula of $RFZI$ for $\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2$.

Theorem 2.30. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two disjoint graphs containing n_1, n_2 vertices and edges m_1, m_2, separately. Then* $EM_1(\mathcal{G}_1 \uplus_{\widehat{T}} \mathcal{G}_2) = p[F(L(\mathcal{G}_1)) + 2EM_2(\mathcal{G}_1) + 4(n_2+1)EM_1(\mathcal{G}_1)] + (n_2+4)HM(\mathcal{G}_1) + HM(\mathcal{G}_2) + 10F(\mathcal{G}_1) + \Big(2p(n_2+1)^2 + 2n_2^2 + 2m_1n_2 + 4n_2 + 4m_2 - 24\Big)M_1(\mathcal{G}_1) + (5m_1-4)M_1(\mathcal{G}_2) + 12M_2(\mathcal{G}_1) + 4(p-1)[F(\mathcal{G}_1) + 2M_2(\mathcal{G}_1) + 2(n_2-1)M_1(\mathcal{G}_1) + m_1(n_2-1)^2] + n_2m_1(m_1+n_2-2)^2 + 4n_2m_1(n_2+m_1-2) + 2m_1(n_2-2)^2 + 4m_2(m_1-1)^2 - 4pm_1(n_2+1)^2 + 4m_1.$

Proof. Applying the definition of $RFZI$, we get

$$
\begin{aligned}
EM_1(\mathcal{G}_1 \uplus_{\widehat{T}_p} \mathcal{G}_2) = & \sum_{xv \in E(I(\widehat{T}_p(\mathcal{G}_1))} [d_{I(\mathcal{G}_1)}(x) + d_{I(\mathcal{G}_1)}(v) - 2]^2 \\
& + \sum_{\mu\nu,\nu w \in E(I(\widehat{T}_p(\mathcal{G}_1))} [d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + n_2 + d_{\mathcal{G}_1}(\nu) + d_{\mathcal{G}_1}(w) + n_2 - 2]^2 \\
& + \sum_{x \in V(\mathcal{G}_1)} \sum_{e \in I(\widehat{T}_p(\mathcal{G}_1))} [2d_{\mathcal{G}_1}(x) + d_{I(\mathcal{G}_1)}(e) - 2]^2 \\
& + \sum_{xv \in E(\mathcal{G}_1)} \sum_{w \in V(\mathcal{G}_2)} [d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v) + d_{\mathcal{G}_2}(w) + (m_1 + n_2 - 2)]^2 \\
& + \sum_{xv \in E(\mathcal{G}_1)} [2(d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v)) - 2]^2 \\
& + \sum_{xv \in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(x) + d_{\mathcal{G}_2}(v) + 2(m_1 - 1)]^2 \\
= & \sum_{\mathcal{P}_1} + \sum_{\mathcal{P}_2} + \sum_{\mathcal{P}_3} + \sum_{\mathcal{P}_4} + \sum_{\mathcal{P}_5} + \sum_{\mathcal{P}_6} (say)
\end{aligned}
$$

$$
\begin{aligned}
Firstly, \sum_{\mathcal{P}_1} = & \sum_{\mu\nu \in E(I(\widehat{T}_p(\mathcal{G}_1)))} 4[(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + n_2 - 1]^2 \\
= & \, 4(p-1) \sum_{\mu\nu \in E(\mathcal{G}_1)} [(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu) + n_2 - 1]^2 \\
= & \, 4(p-1) \sum_{\mu\nu \in E(\mathcal{G}_1)} [(d^2_{\mathcal{G}_1}(\mu) + d^2_{\mathcal{G}_1}(\nu)) \\
& + 2d_{\mathcal{G}_1}(\mu)d_{\mathcal{G}_1}(\nu) + 2(n_2-1)(d_{\mathcal{G}_1}(\mu) + d_{\mathcal{G}_1}(\nu)) + (n_2-1)^2] \\
= & \, 4(p-1)[F(\mathcal{G}_1) + 2M_2(\mathcal{G}_1) + 2(n_2-1)M_1(\mathcal{G}_1) + m_1(n_2-1)^2]
\end{aligned}
$$

$$
\begin{aligned}
Likewise, \sum_{\mathcal{P}_2} = & \sum_{e=\mu\nu, f=\nu w \in E(L(\mathcal{G}_1))} p[d_{L(\mathcal{G}_1)}(e) + d_{L(\mathcal{G}_1)}(f) + 2(n_2+1)]^2 \\
= & \sum_{xv, vw \in E(\mathcal{G}_1)} p[(d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v) + n_2) + (d_{\mathcal{G}_1}(v) + d_{\mathcal{G}_1}(w) + n_2) - 2]^2 \\
= & \sum_{ef \in E(L(\mathcal{G}_1)) e=xv, f=vw} [d_{L(\mathcal{G}_1)}(e) + 2 + n_2 + d_{L(\mathcal{G}_1)}(f) + n_2]^2
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{ef \in E(L(\mathcal{G}_1))} p[(d^2_{L(\mathcal{G}_1)}(e) + d^2_{L(\mathcal{G}_1)}(f) + 2d_{L(\mathcal{G}_1)}(e)d_{L(\mathcal{G}_1)}(f) \\
& + 4(n_2+1)^2 + 4(n_2+1)\Big(d_{L(\mathcal{G}_1)}(e) + d_{L(\mathcal{G}_1)}(f)\Big)] \\
= & \, p[F(L(\mathcal{G}_1)) + 2M_2(L(\mathcal{G}_1)) + 4(n_2+1)^2(\frac{M_1(\mathcal{G}_1)}{2} - m_1) + 4(n_2+1)M_1(L(\mathcal{G}_1))]
\end{aligned}
$$

$$Now, \sum_{\mathcal{P}_3} = \sum_{x\in V(\mathcal{G}_1)} \sum_{e\in I(\widehat{T}_p(\mathcal{G}_1))} [2d_{\mathcal{G}_1}(x) + d_{I(\mathcal{G}_1)}(e) - 2]^2$$
$$= \sum_{xv\in E(\mathcal{G}_1)} [(2d_{\mathcal{G}_1}(x) + d_{I(\mathcal{G}_1)}(e) - 2)^2 + (2d_{\mathcal{G}_1}(v) + d_{I(\mathcal{G}_1)}(f) - 2)^2]$$
$$= 10 \sum_{xv\in E(\mathcal{G}_1)} (d^2_{\mathcal{G}_1}(x) + d^2_{\mathcal{G}_1}(v)) + 12 \sum_{xv\in E(\mathcal{G}_1)} d_{\mathcal{G}_1}(x)d_{\mathcal{G}_1}(v)$$
$$+ 8(n_2 - 2) \sum_{xv\in E(\mathcal{G}_1)} (d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v)) + \sum_{xv\in E(\mathcal{G}_1)} 2(n_2 - 2)^2$$
$$= 10F(\mathcal{G}_1) + 12M_2(\mathcal{G}_1) + 8(n_2 - 2)M_1(\mathcal{G}_1) + 2m_1(n_2 - 2)^2$$

$$So, \sum_{\mathcal{P}_4} = \sum_{xv\in E(\mathcal{G}_1)} \sum_{w\in V(\mathcal{G}_2)} [d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v) + d_{\mathcal{G}_2}(w) + (m_1 + n_2 - 2)]^2$$
$$= \sum_{w\in V(\mathcal{G}_2)} \sum_{xv\in E(\mathcal{G}_1)} (d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v))^2 + \sum_{xv\in E(\mathcal{G}_1)} \sum_{w\in V(\mathcal{G}_2)} d^2_{\mathcal{G}_1}(w)$$
$$+ \sum_{w\in V(\mathcal{G}_2)} \sum_{xv\in E(\mathcal{G}_1)} (m_1 + n_2 - 2)^2$$

$$+ 2 \sum_{w\in V(\mathcal{G}_2)} d_{\mathcal{G}_2}(w) \sum_{xv\in E(\mathcal{G}_1)} (d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v))$$
$$+ 2(m_1 + n_2 - 2) \sum_{w\in V(\mathcal{G}_2)} \sum_{xv\in E(\mathcal{G}_1)} (d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v))$$
$$+ 2(m_1 + n_2 - 2) \sum_{w\in V(\mathcal{G}_2)} \sum_{\mu\nu\in E(\mathcal{G}_1)} d_{\mathcal{G}_2}(w)$$
$$= n_2HM(\mathcal{G}_1) + m_1M_1(\mathcal{G}_2) + m_1n_2(m_1 + n_2 - 2)^2 + 4m_2M_1(\mathcal{G}_1)$$
$$+ 2n_2(m_1 + n_2 - 2)M_1(\mathcal{G}_1) + 4m_1m_2(m_1 + n_2 - 2)$$

$$Similarly, \sum_{\mathcal{P}_5} = \sum_{xv\in E(\mathcal{G}_1)} [2(d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v)) - 2]^2$$
$$= 4 \sum_{xv\in E(\mathcal{G}_1)} [(d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v))^2 - 2(d_{\mathcal{G}_1}(x) + d_{\mathcal{G}_1}(v)) + 1]$$
$$= 4[HM(\mathcal{G}_1) - 2M_1(\mathcal{G}_1) + m_1]$$

$$Lastly, \sum_{\mathcal{P}_6} = \sum_{xv\in E(\mathcal{G}_2)} [d_{\mathcal{G}_2}(x) + d_{\mathcal{G}_2}(v) + 2(m_1 - 1)]^2$$
$$= \sum_{xv\in E(\mathcal{G}_2)} [(d_{\mathcal{G}_2}(x) + d_{\mathcal{G}_2}(v))^2 + 4(m_1 - 1)(d_{\mathcal{G}_2}(x) + d_{\mathcal{G}_2}(v)) + 4(m_1 - 1)^2]$$
$$= HM(\mathcal{G}_2) + 4(m_1 - 1)M_1(\mathcal{G}_2) + 4m_2(m_1 - 1)^2$$

After simplifying $\sum_{i=1}^{6} \sum_{\mathcal{P}_i}$, we get the requisite result. □

The edge T-join of graphs [24] or T-edge join graphs TEJG of $\mathcal{G}_1$ with $\mathcal{G}_2$, indicated by $\mathcal{G}_1 \uplus_{\widehat{T}} \mathcal{G}_2$, is created by adding each and every vertex of $T(\mathcal{G}_1)$ to one and all vertices of $\mathcal{G}_2$(see Fig. 4).

Corollary 2.31. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two disjoint graphs containing n_1, n_2 vertices and edges m_1, m_2, separately. Thus* $EM_1(\mathcal{G}_1 \uplus_{\widehat{T}} \mathcal{G}_2) = F(L(\mathcal{G}_1)) + 2EM_2(\mathcal{G}_1) + 4(n_2+1)EM_1(\mathcal{G}_1) + (n_2+4)HM(\mathcal{G}_1) + HM(\mathcal{G}_2) + 10F(\mathcal{G}_1) + (4n_2^2 + 2m_1n_2 + 8n_2 + 4m_2 - 22)M_1(\mathcal{G}_2) + (5m_1 - 4)M_1(\mathcal{G}_2) + 12M_2(\mathcal{G}_1) + n_2m_1(m_1+n_2-2)^2 + 4n_2m_1(n_2+m_1-2) + 2m_1(n_2-2)^2 + 4m_2(m_1 - 1)^2 - 4m_1(n_2+1)^2 + 4m_1$.

Corollary 2.32. *Let $\mathcal{G}_1$, $\mathcal{G}_2$ be the graphs containing n_i vertices and m_i edges, where $i = \{1, 2\}$. Then* $EM_1(\mathcal{G}_1 \otimes_{\widehat{T}} \mathcal{G}_2) = 2EM_1(\mathcal{G}_1 \uplus_{\widehat{T}} \mathcal{G}_2) + 4(n_2+3)M_1(\mathcal{G}_2) + 2m_1n_2(4m_1 + 2n_2 - 3) + 32m_1m_2$.

As an application, we can apply the above theorems for the graphs $\mathcal{G}_1 = P_m$ and $\mathcal{G}_2 = C_n$. We have $n_1 = m; m_1 = (m-1); M_1(\mathcal{G}_1) = (4m-6); M_2(\mathcal{G}_1) = (4m-8); F(\mathcal{G}_1) = (8m - 14); F(L(\mathcal{G}_1)) = (8m-22); HM(\mathcal{G}_1) = (16m-30)$; and $n_2 = n = m_2; M_1(\mathcal{G}_2) = M_2(\mathcal{G}_2) = 4n; F(\mathcal{G}_2) = 8n; HM(\mathcal{G}_1) = 16n$ for $m \geq 2$ and $n \geq 3$. Therefore

Example 2.33. $EM_1(P_m \uplus_S C_n) = mn\Big((m+n)^2 + 8m + 6n + 16\Big) - 2n(3n+7) + 2(4m-7)$,
$EM_1(P_m \uplus_R C_n) = mn\Big((n+m)^2 + 12m + 14n + 56\Big) - 14n(n+6) + 4(17m - 33)$,
$EM_1(P_m \uplus_Q C_n) = mn\Big((m+n)^2 + 8m + 6n + 24\Big) - 6n(n+5) + 4(17m - 41)$,
$EM_1(P_m \uplus_T C_n) = mn\Big((n+m)^2 + 12m + 14n + 64\Big) - 2n(7n+50) + 6(24m - 53)$.

Example 2.34. $EM_1(P_m \uplus_{\widehat{S}} C_n) = mn\Big((m+n)^2 + 5m + 2n + 7\Big) - n(n^2 + 4n + 13) + 2(4m-7)$,
$EM_1(P_m \uplus_{\widehat{R}} C_n) = mn\Big(n+m)^2 + 5m + 2n + 15\Big) - n(n^2 + 4n + 25) + 4(17m - 33)$,
$EM_1(P_m \uplus_{\widehat{Q}} C_n) = n(m-1)(m+n-3)(m+n+1) + 2mn(6m + 7n + 20) - 2n(11n + 53) + 4(17m - 41)$,
$EM_1(P_m \uplus_{\widehat{T}} C_n) = n(m-1)(n+m-3)(m+n+1) + 2nm(6m + 7n + 24) - 2n(11n + 59) + 6(24m - 53)$.

3 Conclusion

In this paper, we have investigated the topological graph index like $RFZI$ of graphs $\mathcal{G}_1 \uplus_{F_p} \mathcal{G}_2$ using the concept of several generalized subdivision graph. In addition, we have put in the results to calculate the $RFZI$ for some novel (chemical) graphs that can be established by applying these special types of graph operations. Also, we have followed up their special relationship with Indu-Bala Product. However, the open problem is to investigate for more topological indices in relation to certain generalized F-join graphs. In future, it will be more thoughtful to create new graphs by applying these results in chemical graph theory.

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