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MEROMORPHIC FUNCTIONS SHARING TWO MEROMORPHIC FUNCTIONS OF FINITE ORDER WITH THEIR K-TH DERIVATIVES

Jeet Sarkar

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Corresponding Author: J. Sarkar

Abstract. In this paper, we investigate the uniqueness problem of transcendental meromorphic functions having finitely many poles that share two meromorphic functions of finite order with their k-th derivatives. Our result improves and generalizes the results of Lü and Yi [8]. Also as an application of our result, we solve an unsolved problem of Lü, Li and Yang [9]. Moreover we exhibit some relevant examples to fortify our results.

1 Introduction, definitions and results

Let $\mathscr{M}(\mathbb{C})$ be the family of non-constant functions which are meromorphic in $\mathbb{C}$, whereas $\mathscr{E}(\mathbb{C})$ denotes the family of non-constant entire functions. On the other hand we denote by $\mathscr{M}_T(\mathbb{C})$ and $\mathscr{E}_T(\mathbb{C})$ the family of transcendental meromorphic and entire functions respectively.

In the paper, we assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [3, 17]). Also we denote by $\rho(f)$ the order of f. Again we recall that a meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$.

Let us denote by $E(a;f)$ the collection of all zeros of $f-a$, counted with multiplicities, where $a = a(z)$ is a small function relative to f, in particular, a may be a constant. Also denote by $\overline{E}(a;f)$ the collection of all distinct zeros of $f-a$.

Two meromorphic functions f and g are said to share a value or a small function a CM (with counting multiplicities) or IM (with ignoring multiplicities) if $E(a;f) = E(a;g)$ or $\overline{E}(a;f) = \overline{E}(a;g)$.

Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. We denote by $E_k(a;f)$ the set of all zeros of $f-a$, where a zero of $f-a$ of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

Let A and B be two sets of complex numbers. We denote by $A\Delta B$, called the symmetric difference of A and B, the set $(A \setminus B) \cup (B \setminus A)$.

Let $R = \frac{P}{Q} (\not\equiv 0)$ be a rational function, where P and Q are relatively prime polynomials. We define the degree of R as $\deg R = \deg P - \deg Q$. If $R \equiv 0$, then we define $\deg R = -\infty$. If R, R_1, R_2 are three rational functions, then following are obvious:

1. $\deg R_1 R_2 = \deg R_1 + \deg R_2$.
2. $\deg \frac{R_1}{R_2} = \deg R_1 - \deg R_2$.
3. $\deg R' \leq \deg R - 1$.
4. $\deg \frac{R'}{R} \leq -1$.
5. $\deg(R_1 + R_2) \leq \max\{\deg R_1, \deg R_2\}$.

Let $f \in \mathscr{E}(\mathbb{C})$ and we assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We denote by

$$\mu(r,f) = \max_{n\in\mathbb{N},\ |z|=r} \{|a_n z^n|\} \text{ and } \nu(r,f) = \sup\{n : |a_n| r^n = \mu(r,f)\}.$$

Clearly for a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $a_n \neq 0$, we have

$$\mu(r,P) = |a_n| r^n \text{ and } \nu(r,P) = n$$

for all r sufficiently large.

In the general case, $|a_n| r^n \leq \mu(r,f)$ for all $n \geq 0$ and $|a_n| r^n < \mu(r,f)$ for all $n > \nu(r,f)$.

Here it is enough to recall that (see [4])

(1) $\mu(r,f)$ is strictly increasing for all r sufficiently large, is continuous and tends to $+\infty$ as $r \to \infty$;

(2) $\nu(r,f)$ is increasing, piecewise constant, right-continuous and also tends to $+\infty$ as $r \to \infty$;

(3) $\log \nu(r,F) = O(\log r)$, if $\rho(f) < +\infty$.

Rubel and Yang [12] first studied the uniqueness problem of an entire function f when $E(a;f) = E(a;f')$ for two distinct finite values of a. We recall their result.

Theorem 1.1. *[12] Let $f \in \mathscr{E}(\mathbb{C})$ and $a, b \in \mathbb{C}$ be distinct. If $E(a;f) = E(a;f')$ and $E(b;f) = E(b;f')$, then $f \equiv f'$.*

In 1979, Mues and Steinmetz [11] further improved Theorem 1.1 and obtained the following result.

Theorem 1.2. *[11] Let $f \in \mathscr{E}(\mathbb{C})$ and $a, b \in \mathbb{C}$ be distinct. If $\overline{E}(a;f) = \overline{E}(a;f')$ and $\overline{E}(b;f) = \overline{E}(b;f')$, then $f \equiv f'$.*

Since then, shared value problems, especially the case of f and f' sharing two values, have undergone various extensions and improvements (see [17]).

Changing $E(a;f) = E(a;f')$ by $\overline{E}(a;f) \subseteq \overline{E}(a;f')$ in Theorem A, Li and Yi [5] obtained the following result.

Theorem 1.3. *[5] Let $f \in \mathscr{E}(\mathbb{C})$ and $a, b(\neq 0) \in \mathbb{C}$ be distinct. If $\overline{E}(a;f) \subseteq \overline{E}(a;f')$ and $E(b;f) = E(b;f')$, then one of the following cases must occur.*

(1) $f \equiv f'$,

(2) $f(z) = c\exp(\frac{b}{b-a}z) + a$, where $c \in \mathbb{C} \setminus \{0\}$.

With the idea of sharing polynomials, Lü, Xu and Chen [7] further improved Theorem 1.3 and obtained the following result.

Theorem 1.4. *[7] Let $f \in \mathscr{E}(\mathbb{C})$ and Q_1, $Q_2(\not\equiv 0)$ be distinct polynomials. If $\overline{E}(Q_1;f) \subseteq \overline{E}(Q_1;f')$ and $E(Q_2;f) = E(Q_2;f')$, then one of the following cases must occur.*

(1) $f \equiv f'$,

(2) $f(z) = Q_1(z) + A\exp(\lambda z)$ and $(\lambda - 1)Q_2 = \lambda Q_1 - Q_1'$, where $A, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda \neq 1$.

Replacing f' by $f^{(k)}$ in Theorem 1.4, Lü and Yi [8] obtained the following result, which is the improvement of Theorem 1.4.

Theorem 1.5. *[8] Let $f \in \mathscr{E}(\mathbb{C})$, $k \in \mathbb{N}$ and Q_1, $Q_2(\not\equiv 0)$ be distinct polynomials. If*

(i) all zeros of $f - Q_1$ have multiplicity at least k,

(ii) $\overline{E}(Q_1;f) \subseteq \overline{E}(Q_1;f^{(k)})$ and $E(Q_2;f) = E(Q_2;f^{(k)})$,

then one of the following cases must occur:

(1) $f \equiv f^{(k)}$,

(2) $f(z) = Q_1(z) + A\exp(\lambda z)$ *and* $(\mu - 1)Q_2 = \mu Q_1 - Q_1^{(k)}$, *where* $A, \lambda, \mu \in \mathbb{C} \setminus \{0\}$ *such that* $\lambda^k = \mu \neq 1$.

Remark 1.1. If a is a Picard exceptional value of f, then one can easily conclude that the zeros of $f - a$ have multiplicity ∞. Therefore, Theorem 1.5 holds even when $f - Q_1$ has no zeros. But it is to be noted that if $f - Q_1$ has a zero at z_0, say, then the multiplicity of z_0 must be at least k. On the other hand if we add the condition $f - Q_1$ has at least one zero in Theorem 1.5, then $f \equiv f^{(k)}$.

2 Main Result

In the paper, we have extended and improved Theorem 1.5 in the following directions:

(1) We replace sharing polynomials by sharing meromorphic functions of finite order.

(2) We relax the nature of sharing.

We now state our main result.

Theorem 2.1. *Let* $f \in \mathscr{M}_T(\mathbb{C})$ *with finitely many poles,* $\alpha_i = R_i \exp(Q)$ $(i = 1, 2, 3)$, *where* R_1, $R_2(\not\equiv 0)$ *and* $R_3(\not\equiv 0)$ *are rational functions such that* $R_1 \not\equiv R_2, R_3$ *and* Q *is a polynomial such that* $\deg(Q) < \rho(f)$. *If*

(i) all the zeros of $f - \alpha_1$ *have multiplicity at least* k,

(ii) $\overline{E}(\alpha_1; f) \subseteq \overline{E}(\alpha_1; f^{(k)})$,

(iii) $E_1(\alpha_2; f) \setminus A = E_1\left(\alpha_3; f^{(k)}\right) \setminus A$, *where* $A = \overline{E}(\alpha_2; f)\,\Delta\overline{E}\left(\alpha_3; f^{(k)}\right)$ *is finite,*

then one of the following cases must occur:

(1) $R_2 f^{(k)} \equiv R_3 f$,

(2) $(R_1 - R_2)f^{(k)} \equiv (R_1 - R_3)f + R_1(R_3 - R_2)\exp(Q)$,

(3) if $\deg(R_4 - R_3) = \deg(R_1 - R_2)$, *where* $\exp(-Q)\alpha_1^{(k)} = R_4$, *then* Q *reduces to a constant and* $f(z) = R_1(z) + \mathcal{P}(z)\exp(\lambda z)$, *where* $\mathcal{P}(\not\equiv 0)$ *is a rational function,* $\lambda \in \mathbb{C} \setminus \{0\}$ *such that* $\lambda^k = \lim\limits_{z\to\infty} \frac{R_4 - R_3}{R_1 - R_2}$ *and*

$$\frac{R_4 - R_3}{R_1 - R_2} = \lambda^k + \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} \frac{\mathcal{P}^{(i)}}{\mathcal{P}}, \tag{2.1}$$

(4) if $\deg(R_4 - R_3) > \deg(R_1 - R_2)$, *where* $\exp(-Q)\alpha_1^{(k)} = R_4$, *then* $f = R_1\exp(Q) + \mathcal{P}\exp(\mathcal{Q})$, *where* $\mathcal{P}(\not\equiv 0)$ *is a rational function and* $\mathcal{Q}$ *is a polynomial such that* $k\deg(Q') = \deg(R_4 - R_3) - \deg(R_1 - R_2)$ *and* $\frac{R_4 - R_3}{R_1 - R_2} = \frac{(\mathcal{P}\exp(\mathcal{Q}))^{(k)}}{\mathcal{P}\exp(\mathcal{Q})}$.

We now make the following observations on the conclusions of Theorem 2.1:

(i) If either $\alpha \equiv 0$ or $\alpha_2 \equiv \alpha_3$, then conclusions (1) and (2) are same.

(ii) Conclusions (3) and (4) do not occur if $f - \alpha_1$ has infinitely many zeros or $\deg(R_4 - R_3) < \deg(R_1 - R_2)$.

(iii) If $R_1 \equiv 0$ and $R_2 \equiv R_3$, then $\deg(R_4 - R_3) = \deg(R_1 - R_2)$ and so letting $|z| \to \infty$ in (2.1), we conclude that $\lambda^k = 1$ and $\mathcal{P}$ is a non-zero constant. In this case, we must have $f(z) = c\exp(\lambda z)$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$, which implies that $f \equiv f^{(k)}$. Consequently, from Theorem 2.1, we solely have $f \equiv f^{(k)}$.

(iv) If we assume that $f - \alpha_1$ has no zeros and $\deg(R_4 - R_3) > \deg(R_1 - R_2)$, then P is a non-zero constant, and so from (2.1), we deduce that $\frac{R_4 - R_3}{R_1 - R_2} = \frac{(\exp(\mathcal{Q}))^{(k)}}{\exp(\mathcal{Q})}$.

Remark 2.1. It is easy to see that the condition "$\deg(Q) < \rho(f)$" in Theorem 2.1 is sharp by the following examples.

Example 2.1. Let $f(z) = \exp(z^3)$, $Q(z) = z^3$, $\alpha_1(z) = \frac{3z^3-1}{z-1}\exp(z^3)$ and $\alpha_2(z) = \alpha_3(z) = \frac{6z^3-z-1}{z-1}\exp(z^3)$. Clearly $\deg(Q) = \rho(f)$ and since $f(z) - \alpha_1(z) = z(f'(z) - \alpha_1(z))$, it follows that $\overline{E}(\alpha_1; f) \subseteq \overline{E}(\alpha_1; f')$. By a simple calculation, we deduce that

$$z(6z^2 - 2)(f'(z) - \alpha_3(z)) \equiv (3z^3 + 3z^2 - z - 1)(f(z) - \alpha_2(z)). \tag{2.2}$$

Clearly $E_1(\alpha_2; f) \setminus A = E_1(\alpha_3; f') \setminus A$, where $A = \overline{E}(\alpha_2; f)\Delta\overline{E}(\alpha_3; f')$ is finite but f does not satisfy any case of Theorem 2.1.

Example 2.2. Let $f(z) = \exp(z^4)$, $Q(z) = z^4$, $\alpha_1 \equiv 0$ and $\alpha_2(z) = \alpha_3(z) = -\exp(z^4)$. Clearly $\deg(Q) = \rho(f)$ and since $f(z) - \alpha_1(z) = \exp(z^4)$ has no zero, we have $\overline{E}(\alpha_1; f) \subseteq \overline{E}(\alpha_1; f')$. Note that $f(z) - \alpha_2(z) = 2\exp(z^4)$ and $f'(z) - \alpha_3(z) = (4z^3 + 1)\exp(z^4)$. Clearly $E_1(\alpha_2; f) \setminus A = E_1(\alpha_3; f') \setminus A$, where $A = \overline{E}(\alpha_2; f)\Delta\overline{E}(\alpha_3; f')$ is finite, but f does not satisfy any case of Theorem 2.1.

Remark 2.2. It is easy to see that the condition "$\overline{E}(\alpha_1; f) \subseteq \overline{E}(\alpha_1; f^{(k)})$" in Theorem 2.1 is sharp by the following examples.

Example 2.3. Let $f(z) = \exp(\frac{1}{2}(z+1)^2) + 1$, $Q \equiv 0$, $\alpha_1 \equiv 0$ and $\alpha_2(z) = \alpha_3(z) = 1 + \frac{1}{z}$. Clearly $\overline{E}(\alpha_1; f) \not\subset \overline{E}(\alpha_1; f')$. Note that

$$f(z) - \alpha_2(z) = \exp(\frac{1}{2}(z+1)^2) - \frac{1}{z} \tag{2.3}$$

and

$$f'(z) - \alpha_3(z) = (z+1)\Big[\exp(\frac{1}{2}(z+1)^2) - \frac{1}{z}\Big]. \tag{2.4}$$

Clearly $E_1(\alpha_2; f) \setminus A = E_1(\alpha_3; f') \setminus A$, where $A = \overline{E}(\alpha_2; f)\Delta\overline{E}(\alpha_3; f') = \{-1\}$, but f does not satisfy any case of Theorem 2.1.

Example 2.4. Let $f(z) = \exp(-\frac{1}{2}(z-1)^2) + 1$, $Q \equiv 0$ $\alpha_1 \equiv 0$ and $\alpha_2(z) = \alpha_3(z) = 1 - \frac{1}{z}$. Clearly $\overline{E}(\alpha_1; f) \not\subset \overline{E}(\alpha_1; f')$. Note that

$$f(z) - \alpha_2(z) = \exp(-\frac{1}{2}(z-1)^2) + \frac{1}{z} \tag{2.5}$$

and

$$f'(z) - \alpha_3(z) = -(z-1)\Big[\exp(-\frac{1}{2}(z-1)^2) + \frac{1}{z}\Big]. \tag{2.6}$$

Clearly $E_1(\alpha_2; f) \setminus A = E_1(\alpha_3; f') \setminus A$, where $A = \overline{E}(\alpha_2; f)\Delta\overline{E}(\alpha_3; f') = \{1\}$, but f does not satisfy any case of Theorem 2.1.

Remark 2.3. By the following example, it is easy to see that Case (3) in Theorem 2.1 cannot be deleted.

Example 2.5. Let $f(z) = \frac{1}{2}\exp(\frac{1}{2}z) + z$, $Q \equiv 0$, $\alpha_1(z) = z$ and $\alpha_2(z) = \alpha_3(z) = 2 - z$. Clearly $\deg(R_4 - R_3) = \deg(R_1 - R_2)$ and $f(z) - \alpha_1(z) = \frac{1}{2}\exp(\frac{1}{2}z)$ has no zero and so $\overline{E}(\alpha_1; f) \subset \overline{E}(\alpha_1; f')$. Note that

$$f(z) - \alpha_2(z) = \frac{1}{2}\exp(\frac{1}{2}z) + 2z - 2 \text{ and } f'(z) - \alpha_3(z) = \frac{1}{4}\exp(\frac{1}{2}z) + z - 1. \tag{2.7}$$

Clearly $E_1(\alpha_2; f) \setminus A = E_1(\alpha_3; f') \setminus A$, where $A = \overline{E}(\alpha_2; f)\Delta\overline{E}(\alpha_3; f') = \phi$ and $\frac{R_4-R_3}{R_1-R_2} = \frac{1}{2}$, where $\mathcal{P}(z) = \frac{1}{2}$ and $\mathcal{Q}(z) = \frac{1}{2}z$.

Remark 2.4. By the following example, it is easy to see that Case (4) in Theorem 2.1 cannot be deleted.

Example 2.6. Let $f(z) = \exp(-\frac{1}{2}(z-1)^2)+z^2-2z-1$, $Q \equiv 0$, $\alpha_1(z) = z^2-2z-1$ and $\alpha_2(z) = \alpha_3(z) = \frac{(z-1)^3}{z}$. Clearly $\deg(R_4 - R_3) > \deg(R_1 - R_2)$ and $f(z) - \alpha_1(z) = \exp(-\frac{1}{2}(z-1)^2)$ has no zero and so $\overline{E}(\alpha_1; f) \subset \overline{E}(\alpha_1; f')$. Note that

$$f(z) - \alpha_2(z) = \frac{(z-1)^2}{z} - 2 + \exp(-\frac{1}{2}(z-1)^2)$$

and

$$f'(z) - \alpha_3(z) = -(z-1)\Big[\frac{(z-1)^2}{z} - 2 + \exp(-\frac{1}{2}(z-1)^2)\Big].$$

Clearly $E_1(\alpha_2; f) \setminus A = E_1(\alpha_3; f') \setminus A$, where $A = \overline{E}(\alpha_2; f)\Delta\overline{E}(\alpha_3; f') = \{1\}$ and $\frac{R_4-R_3}{R_1-R_2} = -(z-1)$, where $\mathcal{P}(z) = 1$ and $\mathcal{Q}(z) = -\frac{1}{2}(z-1)^2$.

Remark 2.5. By the following example, it is easy to see that the hypothesis of the transcendental of f in Theorem 2.1 is necessary.

Example 2.7. Let $f(z) = z^3$, $Q \equiv 0$, $\alpha_1 \equiv 0$ and $\alpha_2(z) = \alpha_3(z) = 2z^3 - 3z^2$. Clearly $\overline{E}(\alpha_1; f) \subset \overline{E}(\alpha_1; f')$. Note that $E(\alpha_2; f) = E(\alpha_3; f')$, but f does not satisfy any case of Theorem 2.1.

Remark 2.6. The following example shows that Theorem 2.1 does not hold when f has infinitely many poles.

Example 2.8. Let $f(z) = \frac{z}{1-\exp(-z)}$, $Q \equiv 0$, $\alpha_1 \equiv 0$ and $\alpha_2 \equiv \alpha_3 \equiv 1$. Clearly $f - \alpha_1$ has no zero and so $\overline{E}(\alpha_1; f) \subset \overline{E}(\alpha_1; f')$. Note that

$$f(z) - \alpha_2(z) = \frac{z - 1 + \exp(-z)}{1 - \exp(-z)} \quad \text{and} \quad f'(z) - \alpha_3(z) = -\exp(-z)\ \frac{z - 1 + \exp(-z)}{(1 - \exp(-z))^2}. \tag{2.8}$$

Clearly $E(\alpha_2; f) = E(\alpha_3; f')$, but f does not satisfy any case of Theorem 2.1.

3 Auxiliary lemmas

Lemma 3.1. [[4], Corollary 2.3.4] Let $f \in \mathscr{M}_T(\mathbb{C})$ such ,that $\rho(f) < +\infty$ and $k \in \mathbb{N}$. Then $m(r, f^{(k)}/f) = O(\log r)$ as $r \to \infty$.

Lemma 3.2. ([4], Theorem 3.2.) Let $f \in \mathscr{E}_T(\mathbb{C})$. Then there exists a set $E \subset (1, +\infty)$ with a finite logarithmic measure, we choose z that satisfies $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$, such that $f^{(j)}(z) = (\nu(r, f)/z)^j f(z)(1 + o(1))$ for $j \in \mathbb{N}$.

Lemma 3.3. ([4], Lemma 1.3.1.) Let $P(z) = \sum_{i=1}^{n} a_i z^i$, $a_n \neq 0$. Then $\forall\ \varepsilon > 0$, there exists $r_0 > 0$ such that $\forall\ r = |z| > r_0$ the inequalities $(1 - \varepsilon)|a_n|r^n \leq |P(z)| \leq (1 + \varepsilon)|a_n|r^n$ hold.

Lemma 3.4. ([3], Lemma 3.5.) Let F be meromorphic in a domain D and $n \in \mathbb{N}$. Then $F^{(n)} = (f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4}(f')^2 + P_{n-3}(f))F$, where $f = \frac{F'}{F}$, $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n - 3$ when $n > 3$.

Lemma 3.5. [10] Let $f \in \mathscr{E}_T(\mathbb{C})$ and $E \subset [1, +\infty)$ be a set having a finite logarithmic measure. Then there exists $\{z_n = r_n \exp(\dot{\mathrm{B}}\theta_n)\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to+\infty} \theta_n = \theta_0 \in [0, 2\pi)$, $r_n \notin E$ and if $0 < \rho(f) < +\infty$, then for any given $\varepsilon > 0$ and sufficiently large r_n, $r_n^{\rho(f)-\varepsilon} < \nu(r_n, f) < r_n^{\rho(f)+\varepsilon}$. If $\rho(f) = +\infty$, then for any given large $M > 0$ and sufficiently large r_n, $\nu(r_n, f) > r_n^M$.

Lemma 3.6. [3] Let $f \in \mathscr{M}(\mathbb{C})$ and let $a_1, a_2 (\not\equiv \infty)$ be two distinct small functions of f. Then $T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f)$.

Lemma 3.7. [14] Let $f \in \mathscr{M}(\mathbb{C})$ and $a_n (\not\equiv 0)$, a_{n-1}, ... , a_0 be small functions of f. Then $T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f)$.

Next we introduce some basic ideas about normal families.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [13]).

Now we introduce the notation of the spherical derivative. Let $h \in \mathscr{M}(\mathbb{C})$. The spherical derivative of h at $z \in \mathbb{C}$ is given as

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}.$$

We recall that h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \; \forall \; z \in \mathbb{C}$.

Here we introduce some other results related to Zalcman's lemma. We also use Zalcman's lemma to prove our Lemma 3.12 which plays an important role in the proof of the main result of the paper.

The following lemma is the famous Marty's Criterion.

Lemma 3.8. [13] A family $\mathcal{F}$ of meromorphic functions on a domain D is normal and only if for each compact subset $K \subseteq D$, there exists a constant M such that $f^{\#}(z) \leq M \; \forall \; f \in \mathcal{F}$ and $z \in K$.

Lemma 3.9. [6] Let $f \in \mathscr{M}(\mathbb{C})$ be of infinite order. Then there exist points $z_n \to \infty$ such that for every $N > 0$, $f^{\#}(z_n) > |z_n|^N$, if n is sufficiently large.

Lemma 3.10. [18] (**Zalcman's lemma**) Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in $\mathcal{F}$ have multiplicity greater than or equal to l and all poles of functions in $\mathcal{F}$ have multiplicity greater than or equal to j and α be a real number satisfying $-l < \alpha < j$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist

(i) points $z_n \in \Delta$, $z_n \to z_0$,

(ii) positive numbers ρ_n, $\rho_n \to 0^+$ and

(iii) functions $f_n \in \mathcal{F}$,

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.

Lemma 3.11. [2] Let $f \in \mathscr{M}(\mathbb{C})$ have finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most 1.

Lemma 3.12. Let $f \in \mathscr{M}(\mathbb{C})$ have finitely many poles, $k \in \mathbb{N}$ and let $\alpha_i = R_i \exp(Q)$ for $i = 1, 2, 3$, where R_i's are rational functions and Q is a polynomial such that $\alpha_1 \not\equiv \alpha_2, \alpha_3$. If all the zeros of $f - \alpha_1$ have multiplicities at least k, $\overline{E}(\alpha_1; f) \subseteq \overline{E}(\alpha_1; f^{(k)})$ and $\overline{E}(\alpha_2; f) \setminus A = \overline{E}(\alpha_3; f^{(k)}) \setminus A$, where $A = \overline{E}(\alpha_2; f) \Delta \overline{E}(\alpha_3; f^{(k)})$ is a finite set, then $\rho(f) < +\infty$.

Proof. Let $H = f - \alpha_1$. Clearly all the zeros of H have multiplicities at least k. Also

$$\overline{E}(0; H) \subseteq \overline{E}(\alpha_1 - \alpha_1^{(k)}; H^{(k)}) \quad \text{and} \quad \overline{E}(\alpha_2 - \alpha_1; H) = \overline{E}(\alpha_3 - \alpha_1^{(k)}; H^{(k)}).$$

Suppose $\alpha_1^{(k)} = R_4 \exp(Q)$, $R_1 - R_4 = \frac{P_{11}}{Q_{11}}$, $R_2 - R_1 = \frac{P_{12}}{Q_{12}}$ and $R_3 - R_4 = \frac{P_{13}}{Q_{13}}$, where P_{1i} and Q_{1i} are non-zero polynomials for $i = 1, 2, 3$. Let $\gamma = \alpha_2 - \alpha_1$ and $\theta = \alpha_3 - \alpha_1^{(k)}$. Also let $F = \frac{H}{\gamma}$.

First, we suppose $\rho(F) = +\infty$. Then by Lemma 3.9, there exist $w_j \to \infty$ such that for every $N > 0$

$$F^{\#}(w_j) > |w_j|^N, \tag{3.1}$$

if j is sufficiently large. Since f and α_1 have finitely many poles, then H has finitely many poles. Also γ has only finitely many zeros and poles. Therefore F has finitely many poles. So there

exists a $r > 0$ such that $H \neq \infty, \gamma \neq 0, \infty$ and $A = \phi$ in $D = \{z : |z| \geq r\}$. Also since $w_j \to \infty$ as $j \to \infty$, without loss of generality we may assume that $|w_j| \geq r + 1$ for all $j \in \mathbb{N}$. Let

$$F_j(z) = F(w_j + z) = \frac{H(w_j + z)}{\gamma(w_j + z)}.$$

Since $|w_j + z| \geq |w_j| - |z|$, it follows that $w_j + z \in D$ for all $z \in \Delta$. Also since F is analytic in D, it follows that F_j is analytic in Δ for all $j \in \mathbb{N}$. On the other hand zeros of F_j are of multiplicities at least k for all $j \in \mathbb{N}$. Note that $F_j^{\#}(0) = F^{\#}(w_j) \to \infty$ as $j \to \infty$ and so by Lemma 3.8, we conclude that $\{F_j\}$ is not normal at $z = 0$. Now if we apply Lemma 3.10, then there exist

(i) points $z_j \in \Delta$ such that $z_j \to 0$ as $j \to \infty$

(ii) positive numbers ρ_j such that $\rho_j \to 0^+$ as $j \to \infty$

(iii) a subsequence $\{g_j(\zeta) = F_j(z_j + \rho_j\zeta)\}$ of $\{F_j(z)\}$

such that

$$g_j(\zeta) = F_j(z_j + \rho_j\zeta) = \frac{H(w_j + z_j + \rho_j\zeta)}{\gamma(w_j + z_j + \rho_j\zeta)} \to g(\zeta) \tag{3.2}$$

converges locally and uniformly in $\mathbb{C}$, where $g \in \mathscr{E}(\mathbb{C})$. Using Lemma 3.11, we have $\rho(g) \leq 1$. By Hurwitz's theorem, we say that zeros of g are of multiplicities at least k. Also

$$\rho_j \leq \frac{M}{F_j^{\#}(a_j)} = \frac{M}{F^{\#}(w_j)}, \quad 0 < M < \infty. \tag{3.3}$$

Now from (3.1) and (3.3), we see that for every $N > 0$

$$\rho_j \leq M|w_j|^{-N}, \tag{3.4}$$

if j is sufficiently large. Also we see that

$$\frac{\gamma'(w_j + z_j + \rho_j\zeta)}{\gamma(w_j + z_j + \rho_j\zeta)} = \frac{(R_2 - R_1)'(w_j + z_j + \rho_j\zeta)}{(R_2 - R_1)(w_j + z_j + \rho_j\zeta)} + Q'(w_j + z_j + \rho_j\zeta). \tag{3.5}$$

Observe that $\frac{(R_2-R_1)'(w_j+z_j+\rho_j\zeta)}{(R_2-R_1)(w_j+z_j+\rho_j\zeta)} \to 0$ as $j \to \infty$. Suppose $N > s$, where $s = \deg(Q')$.

Therefore from (3.4), we have

$$\lim_{j\to\infty} \rho_j|w_j|^s \leq \lim_{j\to\infty} M|w_j|^{s-N} = 0. \tag{3.6}$$

Note that $|Q'(w_j + z_j + \rho_j\zeta)| = O(|w_j|^s)$ and so from (3.6), we have

$$\rho_j|Q'(w_j + z_j + \rho_j\zeta)| = O(\rho_j|w_j|^s) \to 0 \ \ (\text{as} \ \ j \to \infty). \tag{3.7}$$

Now from (3.5) and (3.7), we have

$$\rho_j \frac{\gamma'(w_j + z_j + \rho_j\zeta)}{\gamma(w_j + z_j + \rho_j\zeta)} \to 0 \ \ (\text{as} \ \ j \to \infty). \tag{3.8}$$

Also we have

$$\frac{(\alpha_1 - \alpha_1^{(k)})(w_j + z_j + \rho_j\zeta)}{\gamma(w_j + z_j + \rho_j\zeta)} = \frac{(R_1 - R_4)(w_j + z_j + \rho_j\zeta)}{(R_2 - R_1)(w_j + z_j + \rho_j\zeta)}$$

$$= \frac{(P_{11}Q_{12})(w_j + z_j + \rho_j\zeta)}{(P_{12}Q_{11})(w_j + z_j + \rho_j\zeta)} = O(|w_j|^l),$$

where $l = \max\{\deg(P_{11}Q_{12}), \deg(P_{12}Q_{11})\}$. Since $N > 0$ is arbitrary, we may suppose that $kN > l$ and so from (3.4) we deduce that

$$\rho_j^k \frac{(\alpha_1 - \alpha_1^{(k)})(w_j + z_j + \rho_j\zeta)}{\gamma(w_j + z_j + \rho_j\zeta)} = O(|w_j|^{l-kN}) \to 0 \text{ as } j \to \infty. \tag{3.9}$$

Let $l_1 = \max\{\deg(P_{13}Q_{12}), \deg(P_{12}Q_{13})\}$. Since $N > 0$ is arbitrary, we may suppose that $kN > l_1$ and so from (3.4), we deduce that

$$\rho_j^k \frac{\theta(w_j + z_j + \rho_j\zeta)}{\gamma(w_j + z_j + \rho_j\zeta)} = O(|w_j|^{l_1-kN}) \to 0 \text{ as } j \to \infty. \tag{3.10}$$

Now proceeding in the same way as done in the proof of Theorem 1.3 [8], we arrive at a contradiction. Consequently F can not be of infinite order. Hence $\rho(F) < +\infty$ and so $\rho(f) = \rho(F\gamma + \alpha_1) \leq \max\{\rho(F), \rho(\gamma), \rho(\alpha_1)\} < +\infty$.

This completes the proof. □

4 Proof of Theorem 2.1

Proof of Theorem 2.1. Let $\alpha_1^{(k)} = R_4 \exp(Q)$, $R_1 - R_4 = \frac{P_{11}}{Q_{11}}$, $R_2 - R_1 = \frac{P_{12}}{Q_{12}}$ and $R_3 - R_4 = \frac{P_{13}}{Q_{13}}$, where P_{1i} and Q_{1i} are non-zero polynomials for $i = 1, 2, 3$.

Let $L(f) = (\alpha_1' - \alpha_2')(f - \alpha_1) - (\alpha_1 - \alpha_2)(f' - \alpha_1') = (\alpha_1' - \alpha_2')(f - \alpha_2) - (\alpha_1 - \alpha_2)(f' - \alpha_2')$. If possible, suppose $L(f) \equiv 0$. Then we have

$$\frac{f' - \alpha_1'}{f - \alpha_1} \equiv \frac{\alpha_1' - \alpha_2'}{\alpha_1 - \alpha_2}.$$

On integration, we have $f - \alpha_1 \equiv c_0(\alpha_1 - \alpha_2)$, where $c_0 \in \mathbb{C}$, i.e., $f \equiv \alpha_1 + c_0(\alpha_1 - \alpha_2)$. This shows that $\rho(f) = \deg(Q)$, which is a contradiction. Hence $L(f) \not\equiv 0$. Let

$$\Phi = \frac{L(f)((\alpha_1 - \alpha_3)f - (\alpha_1 - \alpha_2)f^{(k)} - \alpha_1(\alpha_2 - \alpha_3))}{(f - \alpha_1)(f - \alpha_2)}. \tag{4.1}$$

We now consider the following two cases.

Case 1. Let $\Phi \not\equiv 0$. Then clearly $(\alpha_1 - \alpha_3)f - (\alpha_1 - \alpha_2)f^{(k)} \not\equiv \alpha_1(\alpha_2 - \alpha_3)$. Also by Lemma 3.12, we have $\rho(f) < +\infty$. Note that

$$\begin{aligned}\frac{fL(f)}{(f - \alpha_1)(f - \alpha_2)} &= \frac{1}{\alpha_1 - \alpha_2}\left(\alpha_1 \frac{L(f)}{f - \alpha_1} - \alpha_2 \frac{L(f)}{f - \alpha_2}\right) \\ &= \alpha_2 \frac{f' - \alpha_2'}{f - \alpha_2} - \alpha_1 \frac{f' - \alpha_1'}{f - \alpha_1} + (\alpha_1' - \alpha_2')\end{aligned}$$

and

$$\frac{L(f)}{(f - \alpha_1)(f - \alpha_2)} = \frac{1}{\alpha_1 - \alpha_2}\left(\frac{L(f)}{f - \alpha_1} - \frac{L(f)}{f - \alpha_2}\right) = \frac{f' - \alpha_2'}{f - \alpha_2} - \frac{f' - \alpha_1'}{f - \alpha_1}.$$

Now from (4.1), we get

$$\Phi = \left((\alpha_1 - \alpha_3) - (\alpha_1 - \alpha_2)\frac{f^{(k)}}{f}\right)\frac{fL(f)}{(f - \alpha_1)(f - \alpha_2)} - \alpha_1(\alpha_2 - \alpha_3)\frac{L(f)}{(f - \alpha_1)(f - \alpha_2)}$$

$$= \left((\alpha_1 - \alpha_3) - (\alpha_1 - \alpha_2)\frac{f^{(k)}}{f}\right)\left(\alpha_2 \frac{f' - \alpha_2'}{f - \alpha_2} - \alpha_1 \frac{f' - \alpha_1'}{f - \alpha_1} + (\alpha_1' - \alpha_2')\right)$$

$$-\alpha_1(\alpha_2 - \alpha_3)\left(\frac{f' - \alpha_2'}{f - \alpha_2} - \frac{f' - \alpha_1'}{f - \alpha_1}\right)$$

and so

$$\exp(-2Q)\Phi = \left((R_1 - R_3) - (R_1 - R_2)\frac{f^{(k)}}{f}\right) \times$$

$$\left(R_2\frac{f'-\alpha_2'}{f-\alpha_2}-R_1\frac{f'-\alpha_1'}{f-\alpha_1}+(R_1'-R_2')+(R_1-R_2)Q'\right)-R_1(R_2-R_3)\left(\frac{f'-\alpha_2'}{f-\alpha_2}-\frac{f'-\alpha_1'}{f-\alpha_1}\right)$$

Since R_i for $i=1,2,3$ are rational functions and $\rho(f)<+\infty$, then by Lemma 3.7, we conclude that $m\left(r,\exp(-2Q)\Phi\right)=O(\log r)$ as $r\to\infty$.

Let z_0 be a zero of $f-\alpha_1$ of multiplicity $p_0(\geq k)$ such that $\alpha_i(z_0)\neq 0,\infty$, where $i=1,2,3$. Since $\overline{E}(\alpha_1;f)\subseteq\overline{E}(\alpha_1;f^{(k)})$, it follows that z_0 must be a zero of $f^{(k)}-\alpha_1$ of multiplicity q_0. Clearly z_0 will be a zero of $L(f)$ and $(\alpha_1-\alpha_3)f-(\alpha_1-\alpha_2)f^{(k)}-\alpha_1(\alpha_2-\alpha_3)$ of multiplicities p_0-1 and $t_0=\min\{p_0,q_0\}(\geq 1)$ respectively and so from (4.1), we conclude that Φ is holomorphic at z_0.

Let z_1 be a common zero of $f-\alpha_2$ and $f^{(k)}-\alpha_3$ such that $\alpha_i(z_1)\neq 0,\infty$ for $i=1,2,3$. Let z_1 be a zero of both $f-\alpha_2$ and $f^{(k)}-\alpha_3$ of multiplicities p_1 and q_1 respectively. Then by some routine calculations, we can easily conclude from (4.1) that

$$\Phi(z)=O((z-z_1)^{t_1-1}), \tag{4.2}$$

where $t_1\geq\min\{p_1,q_1\}\geq 1$ and so Φ is holomorphic at z_1. Therefore it is easy to conclude that $N(r,\Phi)=O(\log r)$ as $r\to\infty$ and so $N\left(r,\exp(-2Q)\Phi\right)=O(\log r)$ as $r\to\infty$. Thus $T(r,\exp(-2Q)\Phi)=O(\log r)$ as $r\to\infty$, which means that $\exp(-2Q)\Phi$ is a rational function and so Φ has finitely many zeros and poles.

We use $\overline{N}_{(k}(r,a;f)$ to denote the reduced counting function of those a-points of f whose multiplicities are not less than k.

If $p_1\geq 2$ and $q_1\geq 2$, then from (4.1) and (4.2), we deduce that z_2 is a zero of Φ. Since Φ has finitely many zeros and poles, it follows that $\overline{N}_{(2}(r,\alpha_2;f)=O(\log r)$ and $\overline{N}_{(2}(r,\alpha_3;f^{(k)})=O(\log r)$ as $r\to\infty$. Consequently $f-\alpha_2$ and $f^{(k)}-\alpha_3$ have only finitely many multiple zeros. Let

$$\alpha=\frac{f^{(k)}-\alpha_3}{f-\alpha_2}.$$

By the given conditions, we can deduce that α has only finite many zeros and poles. Also $\rho(\alpha)<+\infty$. Therefore we can assume that $\alpha=\beta e^{\gamma}$, where $\beta(\not\equiv 0)$ is a rational function and γ is a polynomial. Hence

$$\frac{f^{(k)}-\alpha_3}{f-\alpha_2}=\beta\exp(\gamma). \tag{4.3}$$

Now we consider following two sub-cases.

Sub-case 1.1. Let $\rho(f)<1$. Since $\deg(Q)<\rho(f)$, we get $Q\in\mathbb{C}$. Now (4.3) gives

$$\rho(\exp(\gamma))=\rho(\beta\exp(\gamma))=\rho\left(\frac{f^{(k)}-\alpha_3}{f-\alpha_2}\right)\leq\max\{\rho\left(f^{(k)}\right),\rho(f)\}=\rho(f)<1$$

and so γ is a constant. Without loss of generality we assume that

$$f^{(k)}-\alpha_3\equiv\beta(f-\alpha_2),\ \text{ i.e., }\ f^{(k)}\equiv\beta f+\alpha_3-\beta\alpha_2. \tag{4.4}$$

If $\alpha_3-\beta\alpha_2\equiv 0$, then from (4.4), we have $f^{(k)}\equiv\frac{\alpha_3}{\alpha_2}f$. Next we suppose $\alpha_3-\beta\alpha_2\not\equiv 0$. We now consider following two sub-cases.

Sub-case 1.1.1. Suppose $f-\alpha_1$ has infinitely many zeros. Let z_1 be a zero of $f-\alpha_1$ such that $\beta(z_1),\alpha_i(z_1)\neq 0,\infty$ for $i=1,2$. Then $f(z_1)=\alpha_1(z_1)$. Since $\overline{E}(\alpha_1;f)\subseteq\overline{E}(\alpha_1;f^{(k)})$, we get $f^{(k)}(z_1)=\alpha_1(z_1)$. Putting z_1 into (4.4), we get $\alpha_1(z_1)-\alpha_3(z_1)=\beta(z_1)(\alpha_1(z_1)-\alpha_2(z_1))$. If $\alpha_1-\alpha_3\not\equiv\beta(\alpha_1-\alpha_2)$, then one can easily conclude that $f-\alpha_1$ has only finitely many zeros, which is impossible. Hence $\alpha_1-\alpha_3\equiv\beta(\alpha_1-\alpha_2)$. In this case, we have $f^{(k)}\equiv\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}f+\alpha_1\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}$, which is impossible.

Sub-case 1.1.2. Suppose $f-\alpha_1$ has finitely many zeros. Consequently $f-\alpha_1$ has finitely many zeros and poles. Since $\rho(f)<1$, we have $\rho(f-\alpha_1)<1$ and so we can take $f-\alpha_1=\mathcal{P}$, where $\mathcal{P}(\not\equiv 0)$ is a rational function. Therefore $f=\alpha_1+\mathcal{P}$ is a rational function, which is a contradiction.

Sub-case 1.2. Let $\rho(f)\geq 1$. We now prove that γ is a constant. If not, suppose $\deg(\gamma)=m\geq 1$. Let $\gamma(z)=c_mz^m+c_{m-1}z^{m-1}+\ldots+c_0$ where $c_i\in\mathbb{C}$ for $i=0,1,\ldots,m$ and $c_m\neq 0$.

Then from (4.3), we have

$$\beta \exp(\gamma) = \frac{\frac{f^{(k)}}{f} - \frac{R_3}{\exp(-Q)f}}{1 - \frac{R_2}{\exp(-Q)f}}, \quad \text{i.e.,} \quad \gamma = \log \frac{1}{\beta} \frac{\frac{f^{(k)}}{f} - \frac{R_3}{\exp(-Q)f}}{1 - \frac{R_2}{\exp(-Q)f}},$$

where $\log h$ is the principle branch of the logarithm. Therefore by Lemma 3.3, we have

$$|c_m| r^m (1 + o(1)) = |\gamma(z)| = \left| \log \frac{1}{\beta(z)} \frac{\frac{f^{(k)}(z)}{f(z)} - \frac{R_3(z)}{\exp(-Q)f(z)}}{1 - \frac{R_2(z)}{\exp(-Q)f(z)}} \right|. \tag{4.5}$$

Now by Hadamard factorization theorem, we obtain $f = \frac{g}{\delta}$, where $g \in \mathscr{E}(\mathbb{C})$ and $\delta(\not\equiv 0)$ is a polynomial. Let $F = \frac{f'}{f}$. Then $F = \frac{g'}{g} - \frac{\delta'}{\delta}$ and so by Lemma 3.4, we have

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2} F^{k-2} F' + a_k F^{k-3} F'' + b_k F^{k-4} (F')^2 + \tilde{P}_{k-3}(F). \tag{4.6}$$

Note that

$$\left(\frac{g'}{g}\right)' = \frac{g''}{g} - \left(\frac{g'}{g}\right)^2, \quad \left(\frac{g'}{g}\right)'' = \frac{g'''}{g} - 3\,\frac{g''}{g}\,\frac{g'}{g} + 2\left(\frac{g'}{g}\right)^3,$$

$$\left(\frac{g'}{g}\right)''' = \frac{g^{(iv)}}{g} - 4\,\frac{g'''}{g}\,\frac{g'}{g} - 3\left(\frac{g''}{g}\right)^2 + 12\,\frac{g''}{g}\left(\frac{g'}{g}\right)^2 - 6\left(\frac{g'}{g}\right)^4$$

and so on. Thus in general we have

$$\left(\frac{g'}{g}\right)^{(i)} = A^i_{i+1}\left(\frac{g'}{g}\right)^{i+1} + \sum_{\lambda} A^i_\lambda M^i_\lambda \left(\frac{g'}{g}\right), \tag{4.7}$$

where $M^i_\lambda\left(\frac{g'}{g}\right) = \left(\frac{g'}{g}\right)^{q_1^{\lambda_i}} \cdots \left(\frac{g^{(i+1)}}{g}\right)^{q_{i+1}^{\lambda_i}}$ and $q_1^{\lambda_i}, \ldots, q_{i+1}^{\lambda_i}$ are non-negative integers satisfying $\sum_{j=1}^{i+1} q_j^{\lambda_i} \le i$ and $A^i_\lambda \in \mathbb{R}$. Similarly we have

$$\left(\frac{\delta'}{\delta}\right)^{(i)} = A^i_{i+1}\left(\frac{\delta'}{\delta}\right)^{i+1} + \sum_{\lambda} A^i_\lambda M^i_\lambda \left(\frac{\delta'}{\delta}\right). \tag{4.8}$$

Now from (4.6), (4.7) and (4.8), we have

$$\begin{aligned}\frac{f^{(k)}(z)}{f(z)} = B^k_k \left(\frac{g'(z)}{g(z)}\right)^k + \sum_{\lambda} B^k_\lambda \left(\frac{\delta'(z)}{\delta(z)}\right)^{s_1^{\lambda_k}} \cdots \left(\frac{\delta^{(k)}(z)}{\delta(z)}\right)^{s_k^{\lambda_k}} \left(\frac{g'(z)}{g(z)}\right)^{r_1^{\lambda_k}} \cdots \left(\frac{g^{(k)}(z)}{g(z)}\right)^{r_k^{\lambda_k}} \\ + C^k_k \left(\frac{\delta'(z)}{\delta(z)}\right)^k,\end{aligned} \tag{4.9}$$

where $r_1^{\lambda_k}, \ldots, r_k^{\lambda_k} \in \mathbb{N} \cup \{0\}$ and $s_1^{\lambda_k}, \ldots, s_k^{\lambda_k} \in \mathbb{N} \cup \{0\}$ satisfying $\sum_{j=1}^k r_j^{\lambda_i} \le k-1$, $\sum_{j=1}^k s_j^{\lambda_i} \le k-1$ and $B^k_\lambda, C^k_k \in \mathbb{R}$.

Since $g \in \mathscr{E}(\mathbb{C})$, it follows that $M(r, g) \to \infty$ as $r \to \infty$. Again we let

$$M(r, g) = |g(z_r)|, \quad \text{where} \quad z_r = re^{i\theta} \quad \text{and} \quad \theta \in [0, 2\pi). \tag{4.10}$$

Then from (4.10) and Lemma 3.2, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for some point $z_r = re^{i\theta}(\theta \in [0, 2\pi))$ satisfying $|z_r| = r \notin E$ and $M(r, g) = |g(z_r)|$, we have

$$\frac{g^{(j)}(z_r)}{g(z_r)} = \left(\frac{\nu(r, g)}{z_r}\right)^j (1 + o(1)) \text{ as } r \to \infty \ \ (1 \le j \le k). \tag{4.11}$$

Therefore from (4.9) and (4.11), we have

$$\frac{f^{(k)}(z)}{f(z)} = B_k^k\left(\frac{\nu(r,g)}{z}\right)^k(1+o(1))$$

$$+\sum_{\lambda} B_\lambda^k\left(\frac{\delta'(z)}{\delta(z)}\right)^{s_1^{\lambda_k}}\cdots\left(\frac{\delta^{(k)}(z)}{\delta(z)}\right)^{s_k^{\lambda_k}}\left(\frac{\nu(r,g)}{z}\right)^{n_\lambda}(1+o(1))+C_k^k\left(\frac{\delta'(z)}{\delta(z)}\right)^k$$

$$=\frac{1+o(1)}{z^k}\left[\nu(r,g)^k+\sum_{\lambda} B_\lambda^k\left(\frac{z\delta'(z)}{\delta(z)}\right)^{s_1^{\lambda_k}}\cdots\left(\frac{z\delta^{(k)}(z)}{\delta(z)}\right)^{s_k^{\lambda_k}} z^{k-n_\lambda-s_\lambda}\nu(r,g)^{n_\lambda}\right.$$

$$\left.+C_k^k\left(\frac{z\delta'(z)}{\delta(z)}\right)^k(1+o(1))\right], \tag{4.12}$$

where $1\le s_\lambda=\sum_{j=1}^{k}s_j^{\lambda_i}\le k-1$ and $1\le n_\lambda=\sum_{j=1}^{k}r_j^{\lambda_i}\le k-1$.

Let $\delta R_i=\frac{a_{1i}}{a_{2i}}$, where a_{1i} and $a_{2i}(\not\equiv 0)$ are polynomials for $i=2,3$. Let $a_{im_i}z^{m_i}$ and $b_{in_i}z^{n_i}$ denote the leading terms in the polynomials $a_{1i}(z)$ and $a_{2i}(z)$ respectively for $i=2,3$. Taking $\varepsilon=\frac{1}{2}$, we get from Lemma 3.3 that

$$\frac{1}{2}|a_{im_i}|r^{m_i}\le|a_{1i}(z_r)|\le\frac{3}{2}|a_{im_i}|r^{m_i}\ \text{ and }\ \frac{1}{2}|b_{in_i}|r^{n_i}\le|a_{2i}(z_r)|\le\frac{3}{2}|b_{in_i}|r^{n_i}$$

for $i=2,3$. Therefore

$$|\delta(z_r)R_i(z_r)|\le 3\frac{|a_{im_i}|r^{m_i}}{|b_{in_i}|r^{n_i}}$$

for $i=2,3$. Since $g\in\mathscr{E}(\mathbb{C})$, we know that $M(r,g)$ increases faster than the maximum modulus of any polynomial and hence faster than any power of r.

First, we suppose Q is a constant polynomial. Then from (4.10), we have

$$\lim_{r\to+\infty}\left|\frac{\delta(z_r)R_i(z_r)}{e^{-Q(z_r)}g(z_r)}\right|\le\lim_{r\to+\infty}3\frac{|a_{im_i}|r^{m_i}}{|b_{in_i}|r^{n_i}M(r,g)}=0\ (i=2,3).$$

Next, we suppose Q is a non-constant polynomial. We claim that $e^{-Q}g\in\mathscr{E}_T(\mathbb{C})$. If possible, suppose that $e^{-Q}g=p$, where $p(\not\equiv 0)$ is a polynomial. Therefore $g=pe^{Q}$ and so by Lemma 3.7, we have $T(r,g)=T(r,e^{Q})+S(r,e^{Q})$. This shows that $\rho(g)=\rho\left(e^{Q}\right)$. On the other hand we have $H=\frac{g}{\delta}$, i.e., $P(f)=\frac{g}{\delta}$ and so by Lemma 3.7, we have $n\,T(r,f)+S(r,f)=T(r,g)+S(r,g)$. This shows that $\rho(f)=\rho(g)$ and so $\rho(f)=\rho\left(e^{Q}\right)=\deg(Q)$, which is a contradiction. Hence $e^{-Q}g\in\mathscr{E}_T(\mathbb{C})$. Again since $e^{-Q}\in\mathscr{E}_T(\mathbb{C})$, it follows that $|e^{-Q(z)}|>C|z|^{k_1}$ as $|z|\to\infty$, where $C\in\mathbb{R}^+$ and $k_1\in\mathbb{N}$. Then from (4.10), we have

$$\lim_{r\to+\infty}\left|\frac{\delta(z_r)R_i(z_r)}{e^{-Q(z_r)}g(z_r)}\right|\le\lim_{r\to+\infty}\frac{|\delta(z_r)R_i(z_r)|}{C|z_r|^{k_1}|g(z_r)|}\le\lim_{r\to+\infty}\frac{3}{C}\frac{|a_{im_i}|r^{m_i}}{|b_{in_i}|r^{n_i}r^{k_1}M(r,g)}=0$$

for $i=2,3$. Therefore, in either case, one may conclude that

$$\lim_{r\to+\infty}\left|\frac{R_i(z_r)}{e^{-Q(z_r)}H(z_r)}\right|=\lim_{r\to+\infty}\left|\frac{\delta(z_r)R_i(z_r)}{e^{-Q(z_r)}g(z_r)}\right|\le 0\ (i=2,3). \tag{4.13}$$

Also we have

$$\left|\frac{z_r\delta^{(i)}(z_r)}{\delta(z_r)}\right|\le C_0\ \text{ as }\ |z_r|=r\to\infty\ \ (i=1,2,\ldots,k). \tag{4.14}$$

Now from Lemma 3.5, there exists $\{z_j=r_je^{i\theta_j}\}$ such that $|g(z_j)|=M(r_j,g)$, $\theta_j\in[0,2\pi)$, $\lim_{j\to\infty}\theta_j=\theta_0\in[0,2\pi)$, $r_j\notin E$. Then for any given ε satisfying

$$0<\varepsilon<\min_{\lambda}\frac{(k-n_\lambda)(\rho(g)-1)+s_\lambda}{n_\lambda+k}$$

and sufficiently large r_j, we have

$$r_j^{\rho(g)-\varepsilon} < \nu(r_j, g) < r_j^{\rho(g)+\varepsilon}. \tag{4.15}$$

Then from (4.14) and (4.15), we have

$$\begin{aligned}&\left|B_\lambda^k\left(\frac{z_j\delta'(z_j)}{\delta(z_j)}\right)^{s_1^{\lambda_k}}\cdots\left(\frac{z_j\delta^{(k)}(z_j)}{\delta(z_j)}\right)^{s_k^{\lambda_k}} z_j^{k-n_\lambda-s_\lambda}\nu(r,g)^{n_\lambda}(1+o(1))\right|\\ &\leq |B_\lambda^k|C_0^{s_\lambda} r_j^{k-n_\lambda-s_\lambda}\times r_j^{(\rho(g)+\varepsilon)n_\lambda} = |B_\lambda^k|C_0^{s_\lambda} r_j^{n_\lambda\rho(g)+n_\lambda\varepsilon+k-n_\lambda-s_\lambda}.\end{aligned} \tag{4.16}$$

Since $n_\lambda\rho(g)+n_\lambda\varepsilon+k-n_\lambda-s_\lambda < k(\rho(g)-\varepsilon)$, it follows from (4.15) and (4.16) that

$$\begin{aligned}&\left|B_\lambda^k\left(\frac{z_j\delta'(z_j)}{\delta(z_j)}\right)^{s_1^{\lambda_k}}\cdots\left(z_j\frac{\delta^{(k)}(z_j)}{\delta(z_j)}\right)^{s_k^{\lambda_k}} z_j^{k-n_\lambda-s_\lambda}\nu(r,g)^{n_\lambda}(1+o(1))\right|\\ &< C_1 r_j^{k(\rho(g)-2\varepsilon)} = O(\nu(r_j,g)^k)\end{aligned} \tag{4.17}$$

as $r_j\to+\infty$, $r_j\notin E$, where $C_1>0$. Also from (4.14) and (4.15), we have

$$\left|C_k^k\left(\frac{z_j\delta'(z_j)}{\delta(z_j)}\right)^k\right| \leq C_2 < C_2 r_j^{k(\rho(g)-\varepsilon)} = O(\nu(r_j,g)^k) \tag{4.18}$$

as $r_j\to+\infty$, $r_j\notin E$, where $C_2>0$. Since $\rho(g)<+\infty$, we have

$$\log\nu(r,g) = O(\log r). \tag{4.19}$$

Therefore from (4.5), (4.12), (4.13), (4.17), (4.18) and (4.19), we get

$$|c_m|r_j^m(1+o(1)) = \gamma(z_j)| = \left|\log\frac{1}{\beta(z_j)}\frac{\frac{H^{(k)}(z_j)}{H(z_j)}-\frac{R_2(z_j)}{e^{-Q(z_j)}H(z_j)}}{1-\frac{R_2(z_j)}{e^{-Q(z_j)}H(z_j)}}\right| = O(\log r_j),$$

for $|z_j| = r_j\to+\infty$, $r_j\notin E$, which is impossible. Hence $\gamma\in\mathbb{C}$. Therefore (4.4) holds. If $\alpha_3-\beta\alpha_2\equiv 0$, then from (4.4), we have $f^{(k)}\equiv\frac{\alpha_3}{\alpha_2}f$. Next we suppose $\alpha_3-\beta\alpha_2\not\equiv 0$. We now consider following two sub-cases.

Sub-case 1.2.1. Suppose $f-\alpha_1$ has infinitely many zeros. Proceeding in the same way as done in Sub-case 1.1.1, we arrive at a contradiction.

Sub-case 1.2.2. Suppose $f-\alpha_1$ has finitely many zeros. Since $\rho(f)<+\infty$, we can take $f=\alpha_1+\mathcal{P}\exp(\mathcal{Q})$, where $\mathcal{P}(\not\equiv 0)$ is a rational function and $\mathcal{Q}$ is a polynomial such that $\deg(Q)<\deg(\mathcal{Q})$. Then $f^{(k)} = R_4\exp(Q)+(\mathcal{P}(\mathcal{Q}')^k+\mathcal{P}_1)\exp(\mathcal{Q})$, with $\mathcal{P}_1 = k\mathcal{Q}^{(k-1)}\mathcal{P}'+\hat{P}(\mathcal{P}'',\mathcal{Q}')$, where $\hat{P}(\mathcal{P}'',\mathcal{Q}')$ is differential polynomial in $\mathcal{P}''$ and $\mathcal{Q}'$. Also we see that $\deg(\mathcal{P}_1)<\deg(\mathcal{P}(\mathcal{Q}')^k)$. If possible, suppose $f-\alpha_2$ has finitely many zeros. Since $\deg(Q)<\rho(f)$, it follows that $\exp(-Q)f\in\mathscr{M}_T(\mathbb{C})$. Then by Lemma 3.6, we have

$$\begin{aligned}T(r,\exp(-Q)f) &\leq \overline{N}(r,\alpha_1;\exp(-Q)f)+\overline{N}(r,\alpha_2;\exp(-Q)f)\\ &\quad+\overline{N}(r,\infty;\exp(-Q)f+S(r,\exp(-Q)f) = S(r,\exp(-Q)f),\end{aligned}$$

Which is impossible. Hence $f-\alpha_2$ has infinitely many zeros. Let z_2 be a zero of $f-\alpha_2$ such that $R_i(z_2)\neq 0,\infty$ for $i=1,2,3,4$. Therefore we have

$$R_1(z_2)\exp(Q(z_2))-R_2(z_2)\exp(Q(z_2))+\mathcal{P}(z_2)\exp(\mathcal{Q}(z_2)) = 0$$

and

$$R_4(z_2)\exp(Q(z_2))-R_3(z_2)\exp(Q(z_2))+(\mathcal{P}(z_2)(\mathcal{Q}'(z_2))^k+\mathcal{P}_1(z_2))\exp(\mathcal{Q}(z_2)) = 0.$$

Suppose $\alpha_1^{(k)}\equiv\alpha_3$, i.e., $R_4\equiv R_3$. Then, from above, we have $\mathcal{P}(z_2)(\mathcal{Q}'(z_2))^k+\mathcal{P}_1(z_2)=0$. This shows that zeros of $f-\alpha_2$ are the zeros of $\mathcal{P}(\mathcal{Q}')^k+\mathcal{P}_1$. Since $\mathcal{P}(\mathcal{Q}')^k+\mathcal{P}_1$ is a rational

function, it follows that $f-\alpha_2$ has only finitely many zeros, which is a contradiction. Hence $\alpha_1^{(k)} \not\equiv \alpha_3$, i.e., $R_4 \not\equiv R_3$. Now by eliminating $e^{\mathcal{Q}(z_2)}$ and $e^{\mathcal{Q}(z_2)}$ from above, we obtain

$$\frac{\mathcal{P}(z_2)}{R_1(z_2)-R_2(z_2)} = \frac{\mathcal{P}(z_2)(\mathcal{Q}'(z_2))^k+\mathcal{P}_1(z_2)}{R_4(z_2)-R_3(z_2)}.$$

If possible, suppose $\mathcal{P}(R_4-R_3) \not\equiv (\mathcal{P}(\mathcal{Q}')^k+\mathcal{P}_1)(R_1-R_2)$. This shows that z_2 is a zero of $\mathcal{P}(R_4-R_3)-(\mathcal{P}(\mathcal{Q}')^k+\mathcal{P}_1)(R_1-R_2)$ and so $f-\alpha_2$ has finitely many zeros, which is a contradiction. Hence $\mathcal{P}(R_4-R_3) \equiv (\mathcal{P}(\mathcal{Q}')^k+\mathcal{P}_1)(R_1-R_2)$, i.e.,

$$\frac{R_4-R_3}{R_1-R_2} = \frac{(\mathcal{P}\exp(\mathcal{Q}))^{(k)}}{\mathcal{P}\exp(\mathcal{Q})}. \tag{4.20}$$

We now consider the following sub-cases.

Sub-case 1.2.2.1. Let $\deg(R_4-R_3) < \deg(R_1-R_2)$. Set $F = \frac{(\mathcal{P}\exp(\mathcal{Q}))'}{\mathcal{P}\exp(\mathcal{Q})}$. Then $F = \mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}$ and so by Lemma 3.4, we have

$$\frac{(\mathcal{P}\exp(\mathcal{Q}))^{(k)}}{\mathcal{P}\exp(\mathcal{Q})} = \left(\mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}\right)^k + \tilde{P}_{k-1}\left(\mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}\right). \tag{4.21}$$

Now from (4.20) and (4.21), we obtain

$$\frac{R_4-R_3}{R_1-R_2} = \left(\mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}\right)^k + \tilde{P}_{k-1}\left(\mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}\right). \tag{4.22}$$

Letting $|z| \to \infty$, from (4.22), we see that $k\deg(Q') = 0$, i.e., $Q' \in \mathbb{C}$. We claim that $Q' \equiv 0$. If there suppose that $Q' = c \in \mathbb{C}\setminus\{0\}$. Note that

$$\frac{(\mathcal{P}\exp(\mathcal{Q}))^{(k)}}{\mathcal{P}\exp(\mathcal{Q})} = c^k + \sum_{i=1}^{k}\binom{k}{i}c^{k-i}\frac{\mathcal{P}^{(i)}}{\mathcal{P}}.$$

Then from above, we have

$$\frac{R_4-R_3}{R_1-R_2} = c^k + \sum_{i=1}^{k}\binom{k}{i}c^{k-i}\frac{\mathcal{P}^{(i)}}{\mathcal{P}}. \tag{4.23}$$

Letting $|z| \to \infty$, we arrive at a contradiction from (4.23). Hence $Q' \equiv 0$ and so $Q \in \mathbb{C}$.

Thus, it follows from $f = \alpha_1 + \mathcal{P}\exp(\mathcal{Q})$ that $\rho(f) = \deg(Q)$, a contradiction.

Sub-case 1.2.2.2. Let $\deg(R_4-R_3) = \deg(R_1-R_2)$. Set $\lim_{z\to\infty}\frac{R_4-R_3}{R_1-R_2} = \mu$. Clearly $\mu \neq 0, \infty$. Then from (4.22), we have

$$\mu + \frac{R_4-R_3-\mu R_1+\mu R_2}{R_1-R_2} = \left(\mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}\right)^k + \tilde{P}_{k-1}\left(\mathcal{Q}'+\frac{\mathcal{P}'}{\mathcal{P}}\right), \tag{4.24}$$

where $\deg(R_4-R_3-\mu R_1+\mu R_2) < \deg(R_1-R_2)$. Now letting $|z| \to \infty$, from (4.24) we conclude that $(Q')^k = \mu$. Let $Q' = \lambda$. Clearly $\lambda^k = \mu$. Since $\deg(Q) < \rho(f)$, it follows that $Q \in \mathbb{C}$. In this case we may assume that $f(z) = R_1(z) + \mathcal{P}(z)\exp(\lambda z)$not,there $\mathcal{P}(\not\equiv 0)$ is a rational function, $\lambda \in \mathbb{C}\setminus\{0\}$ such that $\lambda^k = \lim_{z\to\infty}\frac{R_4-R_3}{R_1-R_2}$ and

$$\frac{R_4-R_3}{R_1-R_2} = \lambda^k + \sum_{i=1}^{k}\binom{k}{i}\lambda^{k-i}\frac{\mathcal{P}^{(i)}}{\mathcal{P}}.$$

Sub-case 1.2.2.3. Let $\deg(R_4-R_3) > \deg(R_1-R_2)$. Now letting $|z| \to \infty$, from (4.22) we conclude that $k\deg(Q') = \deg(R_4-R_3) - \deg(R_1-R_2)$ and so $\deg(Q) \geq 2$. In this case, we have $f = \alpha_1 + \mathcal{P}\exp(\mathcal{Q})$, where $\mathcal{P}(\not\equiv 0)$ is a rational function and $\mathcal{Q}$ is a polynomial such that $k\deg(Q') = \deg(R_4-R_3) - \deg(R_1-R_2)$ and (4.20) holds.

Case 2. Let $\Phi \equiv 0$. Since $L(f) \not\equiv 0$, we get $(\alpha_1-\alpha_3)f - (\alpha_1-\alpha_2)f^{(k)} \equiv \alpha_1(\alpha_2-\alpha_3)$ and so $f^{(k)} \equiv \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}f + \alpha_1\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}$. This completes the proof. □

5 Applications

In 1996, Brück [1] first discussed the possible relation between f and f' when $E(a;f) = E(a;f')$, where $f \in \mathscr{E}(\mathbb{C})$ and $a \in \mathbb{C}$. In this direction, an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture 5.1. Let $f \in \mathscr{E}(\mathbb{C})$ such that $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$, where $\rho_2(f)$ is the hyperorder of f. If $E(a;f) = E(a;f')$, where $a \in \mathbb{C}$, then $f' - a \equiv c(f-a)$, where $c \in \mathbb{C} \setminus \{0\}$.

Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with their derivatives. Specifically, it was observed by L. Z. Yang and J. L. Zhang [15] that Brück's conjecture holds if instead of an entire function one considers its suitable power. They proved the following theorem.

Theorem 5.1. [15] Let $f \in \mathscr{E}(\mathbb{C})$ and $n \in \mathbb{N}$ such that $n \geq 7$. If $E(1;f^n) = E(1;(f^n)')$, then $f^n \equiv (f^n)'$ and $f(z) = c\exp(\frac{z}{n})$, where $c \in \mathbb{C} \setminus \{0\}$.

In 2010, J. L. Zhang and L. Z. Yang [16] improved and generalized Theorem 5.1 by considering higher order derivatives and by lowering the power of the entire function.

Theorem 5.2. [16] Let $f \in \mathscr{E}(\mathbb{C})$ and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If $E(1;f^n) = E(1;(f^n)^{(k)})$, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp(\frac{\lambda}{n}z)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

In 2011, Lü and Yi [8] replaced the sharing value 1 by sharing a polynomial in Theorem 5.2 and obtained the following result.

Theorem 5.3. [8] Let $f \in \mathscr{E}_T(\mathbb{C})$, $k, n \in \mathbb{N}$ such that $n \geq k+1$ and let $Q \not\equiv 0$ be a polynomial. If $E(Q;f^n) = E(Q;(f^n)^{(k)})$, then conclusion of Theorem 5.2 holds.

Generally speaking, solving any non-linear differential equation is a very difficult task. As an application of our result, we now consider the following non-linear differential equation:

$$(f^n)^{(k)} - R_3\exp(Q) = R\exp(\gamma)\left(f^n - R_2\exp(Q)\right), \tag{5.1}$$

where $k, n \in \mathbb{N}$, Q is a polynomial, γ is an entire function and R, R_2 and R_3 are non-zero rational functions.

For the very special case for $k = 1$ and $Q \in \mathbb{C}$, Lü, Li and Yang [9] obtained the following result.

Theorem 5.4. [9] Let $f \in \mathscr{E}_T(\mathbb{C})$, $n \in \mathbb{N} \setminus \{1\}$ and R_2, R_3 are polynomials. If f is a solution of equation (5.1), then $\frac{R_3}{R_2}$ is a polynomial and $f' \equiv \frac{R_3}{nR_2}f$.

6 An open problem

Certainly, the case for $k \geq 2$ remains as an open problem. In the paper, we settle the problem for $k \geq 2$ at the cost of considering the fact that $\deg(Q) < \rho(f)$. Therefore as a solution of the non-linear differential equation (5.1), we present the following result.

7 Result

Theorem 7.1. Let $k, n \in \mathbb{N}$ such that $n \geq k+1$ and $\deg(Q) < \rho(f)$. If f is a transcendental meromorphic solution of the non-linear differential equation (5.1), then $(f^n)^{(k)} \equiv \frac{R_3}{R_2}f^n$. In particular, if $R_2 \equiv R_3$, then $f(z) = c\exp(\frac{\lambda}{n}z)$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

Next, we present the following theorem, which is an improvement to Theorem 5.3.

Theorem 7.2. Let $f \in \mathscr{M}_T(\mathbb{C})$ having finitely many poles, $k, n \in \mathbb{N}$ such that $n \geq k+1$, Q be a polynomial such that $\deg(Q) < \rho(f)$, R_2 and R_3 be non-zero rational functions. If $\overline{E}_1(R_2\exp(Q);f) \setminus A = E_1(R_3\exp(Q);(f^n)^{(k)}) \setminus A$, where the value of A is given by $A = \overline{E}(R_2\exp(Q);f)\Delta\overline{E}(R_3\exp(Q);(f^n)^{(k)})$ is a finite set, then $(f^n)^{(k)} \equiv \frac{R_3}{R_2}f^n$. In particular, if $R_2 \equiv R_3$, then $f(z) = c\exp(\frac{\lambda}{n}z)$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

Proof of Theorem 7.1. Note that if $f \in \mathscr{M}(\mathbb{C})$ is a solution of the non-linear differential equation (5.1), then one can easily conclude from (5.1) that f has only finitely many poles. We prove the result, based on the proof of Theorem 2.1. We consider Theorem 2.1 for f^n instead of f. Taking $R_1 \equiv 0$, we see that $\overline{E}(0; f^n) \subseteq E(0; (f^n)^{(k)})$. Also from (5.1), one can easily conclude that $E_1(R_2 \exp(Q); f) \setminus A = E_1(R_3 \exp(Q); (f^n)^{(k)}) \setminus A$, where $A = \overline{E}(R_2 \exp(Q); f) \Delta \overline{E}(R_3 \exp(Q); (f^n)^{(k)})$ is a finite set. Then from Theorem 2.1, we conclude that $(f^n)^{(k)} \equiv \frac{R_3}{R_2} f^n$. In particular if $R_2 \equiv R_3$, then we have $(f^n)^{(k)} \equiv f^n$ and so $f(z) = c \exp(\frac{\lambda}{n} z)$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

This completes the proof. □

Proof of Theorem 7.2. We use f^n instead of f in Theorem 2.1 and taking $R_1 \equiv 0$, we see that $\overline{E}(0; f^n) \subseteq E(0; (f^n)^{(k)})$ and so the proof of Theorem 7.2 follows directly from the proof of Theorem 2.1. So we omit the details. □

References

[1] R. Brück, On entire functions which share one value CM with their first derivative, Results Math., 30 (1996), 21-24.

[2] J. M. Chang and L. Zalcman, Meromorphic functions that share a set with their derivatives, J. Math. Anal. Appl., 338 (2008), 1191-1205.

[3] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford (1964).

[4] I. Laine, Nevanlinna theory and complex differential equations, Walter de Gruyter, Berlin, 1993.

[5] J. T. Li and H. X. Yi, Normal families and uniqueness of entire functions and their derivatives, Arch. Math. (Basel), 87 (1) (2006), 52-59.

[6] X. J. Liu, S. Nevo and X. C. Pang, On the k-th derivative of meromorphic functions with zeros of multiplicity at least $k + 1$, J. Math. Anal. Appl., 348 (2008), 516-529.

[7] F. Lü, J. F. Xu and A. Chen, Entire functions sharing polynomials with their first derivatives, Arch. Math., 92 (2009), 593-601.

[8] F. Lü and H. X. Yi, The Brück conjecture and entire functions sharing polynomials with their k-th derivatives, J. Korean Math. Soc., 48 (3) (2011), 499-512.

[9] W. Lü, Q. Li and C. Yang, On the transcendental entire solutions of a class of differential equations, Bull. Korean Math. Soc., 51 (5) (2014), 1281-1289.

[10] Z. Q. Mao, Uniqueness theorems on entire functions and their linear differential polynomials, Results Math., 55 (2009), 447-456.

[11] E. Mues and N. Steinmetz, Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen, Manuscripta Math., 29 (1979), 195-206.

[12] L. A. Rubel and C. C. Yang, Values shared by an entire function and its derivative, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 599 (1977), 101-103.

[13] J. Schiff, Normal families, Berlin, 1993.

[14] C. C. Yang, On Deficiencies of Differential Polynomials, II, Math. Z., 125 (1972), 107-112.

[15] L. Z. Yang and J. L. Zhang, Non-existence of meromorphic solutions of Fermat type functional equation, Aequations Math., 76 (1-2) (2008), 140-150.

[16] J. L. Zhang and L. Z. Yang, A power of an entire function sharing one value with its derivative, Comput. Math. Appl., 60 (2010), 2153-2160.

[17] H. X. Yi and C. C. Yang, Uniqueness theory of meromorphic functions, Science Press, Beijing, 1995.

[18] L. Zalcman, Normal families, new perspectives, Bull. Amer. Math. Soc., 35 (1998), 215-230.

Author information

Jeet Sarkar, Department of Mathematics, St. Thomas' College of Engineering and Technology, Khidirpur, Kolkata, West Bengal 700023, India.
E-mail: jeetsarkar.math@gmail.com, jeet.sarkar@stcet.ac.in

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