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Hub Coloring of Graphs

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Abstract A subset H of vertices of a graph G is said to be hub set of G if between any two non-adjacent vertices in $V(G) - H$, there is a H-path (path in which all intermediate vertices are in H) between them. The minimum cardinality of a hub set of G is called as hub number of G, and is denoted by $h(G)$. A proper vertex coloring of a graph G is said to be a hub coloring of G if there is a hub set H of G such that no two vertices in H receive the same color. The minimum number of colors required for a hub coloring of G is called the hub chromatic number of G and, is denoted by $\chi_h(G)$. In this paper, we conduct a detailed study on the hub chromatic number of graphs (graph operations), establish some bounds for χ_h in terms of various graph parameters and characterize the graphs of order n with hub chromatic number equal to 2, $n-1$ and n.

1 Introduction

All graphs considered in this paper are finite, undirected, simple (without loops and multiple edges). The sets $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G, respectively, and the cardinality of $V(G)$ is the order of G. The degree of a vertex v of G is the number of edges incident to v in G, denoted by $\deg(v)$. A vertex with degree 1 is called a pendant vertex. The open neighborhood $N(u)$ of a vertex u is the set of all vertices of G adjacent to u, and the closed neighborhood is $N[u] = N(u) \cup \{u\}$. A subset of the vertex set is said to be a vertex independent set if no two vertices in it are adjacent. The maximum cardinality among all vertex independent sets of a graph G is called the independence number, denoted by $\alpha(G)$. An independent set with cardinality $\alpha(G)$ is called the α-set of G. A graph is said to be bipartite if its vertex set can be partitioned into two independent sets. In a bipartite graph G with bi-partition $V_1 \cup V_2$, a vertex v in V_1 said to be major vertex in V_1 if v is adjacent to all vertices in V_2 ($N(v) = V_2$). The diameter of a graph G is the maximum distance between any two vertices, denoted by $diam(G)$. A graph G is said to be split graph if its vertex set is partitioned into two sets K and I with $\langle K \rangle = K_{\omega(G)}$ and I is independent set. A tree is a connected acyclic (cycle-free) graph. The clique number of a graph G is the maximum order among all complete subgraphs of G, denoted by $\omega(G)$. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = (V, E)$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. For any positive integer k, kG denoted the graph of k disjoint copies of the graph G. The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph formed by connecting each vertex of G_1 to each vertex of G_2. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' \mid x \in V\}$ and V' is disjoint from V, and edge set $E' = E \cup \{xy' \mid xy \in E\} \cup \{x'u \mid x' \in V'\}$. For more details on basic definitions and terminology, refer [3]. A dominating set S of a graph G is a set of vertices of G such that each vertex in $V(G) \setminus S$ has an adjacent vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ-set of G.For a graph G without isolated vertex, a subset D of the vertex set of G is said to be total dominating set if each vertex of G has an

adjacent vertex in D. The minimum cardinality of a total dominating set is called the total domination number and is denoted by $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ is called the γ_t-set of G. A proper (vertex) coloring of a graph G is the assignment of colors to the vertices of G in which no two adjacent vertices receive the same color. The minimum number of colors required for a proper vertex coloring of G is called the chromatic number of G, and is denoted by $\chi(G)$. A χ-coloring of G is the proper vertex coloring of G using $\chi(G)$ colors. In k-coloring $\mathcal{C} = (V_1, \ldots V_k)$ of G, V_i denotes the set of vertices receiving color-i, called the color class-i and any set S is said to be colorful in $\mathcal{C}$ if no two vertices of S receive the same color in $\mathcal{C}$. A vertex v of G is said to be dominates the color class V_i (V_i dominated by v) under the coloring $\mathcal{C}$ if $V_i = \{v\}$ or v is adjacent to all the vertices in V_i. For more details on coloring of graphs refer [2, 4].

A subset H of the vertex set of a graph G is said to be a hub set if for any non-adjacent vertices $x, y \in V(G) - H$, there is a H-path (a path in which all intermediate vertices are in H) between x and y. The hub number of G is the minimum cardinality of a hub set of G and, is denoted by $h(G)$. A hub set of G with cardinality $h(G)$ is called a minimum hub set or an h-set of G. For more details on hub parameters of graphs, refer to [14, 19, 18].

Many researchers have introduced various coloring concepts by combining vertex coloring with other graph parameters, such as domination (dominator coloring [9], dominated coloring [15], global dominator coloring [10], global dominated coloring [6] and many more [17, 1]) and irredundance (irredundance coloring [7]). Recently, I S Hamid et al. [8], and David A Kalarkop et al. [7] have introduced new coloring concepts, namely gamma coloring and irredundance coloring (a proper vertex coloring in which no two vertices in at least one irredundant set of G receive the same color), respectively. A gamma coloring of a graph G a proper vertex coloring in which no two vertices in at least one dominating set of G receive the same color. The minimum number of colors required for a gamma coloring of a graph G is called the gamma chromatic number and is denoted by $\chi_\gamma(G)$. Also, Srikantha R et al. [16] introduced the concept called total gamma coloring in graphs. A proper vertex coloring $\mathcal{C}$ of a graph G without isolated vertex, is said to be total gamma coloring if there is a total dominating set which is colorful under $\mathcal{C}$. The minimum number of colors used in a total gamma coloring of G is called the total gamma chromatic number of G, denoted by $\chi_{\gamma_t}(G)$. In above concepts, the authors aim to ensure that at least one specific set (dominating set, total dominating set and irredundant set) of the graph is colorful in a proper vertex coloring. Motivated by these works, we extend this concept to hub sets of graphs and introduce hub coloring of graphs in this paper.

The following results are used to prove some of our main results.

Corollary 1.1. *[14] G is a complete graph if and only if $h(G) = 0$.*

Theorem 1.2. *[14] $G = (V, E)$ is a graph with $h(G) = h_c(G) = 1$ if and only if G has the following structure:*

- *G is not a complete graph*
- *$V(G) = A \cup B \cup \{x\}$, where $\{x\}$ is a hub set.*
- *x is adjacent to every vertex in A and no vertex in B.*
- *Every vertex in A is adjacent to every vertex in B.*
- *$G[B]$ is a complete graph.*

Theorem 1.3. *[14] $h(C_n) = h_c(C_n) = n - 3$.*

Theorem 1.4. *[14] Let T be a tree with n vertices and l leaves. Then $h(G) = h_c(G) = n - l$.*

Theorem 1.5. *[19] $h(\mu(P_n)) = \lfloor \frac{n}{2} \rfloor + 2$, for $n \geq 4$ and $n \equiv 2$ or $3 (mod\ 4)$.*

Theorem 1.6. *[16] For the path P_n with $n \geq 3$, $\chi_{\gamma_t}(P_n) = \gamma_t(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

2 Definition and Example

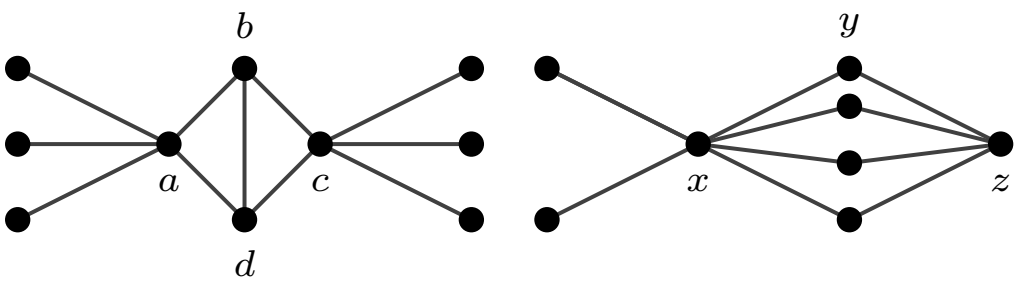
Definition 2.1. Hub coloring of a graph G is a proper vertex coloring of G in which there exists a hub set H of G such that no two vertices in H receive the same color. The minimum number

of colors used in a hub coloring of G is called the hub chromatic number of G and, is denoted by $\chi_h(G)$. A minimum hub coloring or χ_h-coloring of G is a hub coloring of G using $\chi_h(G)$ colors.

For any graph G, the coloring $\mathcal{C}$ in which each vertex receives a unique color (not used for any other vertex) is a hub coloring of G (every hub set of G is colorful in $\mathcal{C}$). Thus every graph admits a hub coloring of G and therefore, the hub chromatic number is well-defined.
For any graph G, every hub coloring of G is a proper vertex coloring of G and in every hub coloring of G at least one hub set is colorful, it follows that $\chi_h(G) \geq max\{\chi(G), h(G)\}$.

Example 2.2. For the graph G_1 shown in Figure 1, $h(G_1) = \chi(G_1) = 3$. Clearly, $S_1 = \{a, b, c\}$ and $S_2 = \{a, d, c\}$ are the only h-sets of G_1. To make either of these sets colorful, at least four colors are needed in any proper vertex coloring of G_1. Moreover, any hub set of G_1 other than S_1 and S_2 contains at least four vertices, so at least four colors are required to make these hub sets colorful. This proves that any hub coloring of G_1 requires at least four colors. Now, consider the coloring $(\{a\}, \{b\}, \{c\}, V(G_1) - \{a, b, c\})$, which is a proper vertex coloring of G_1 with a colorful hub set $\{a, d, c\}$. Therefore, the minimum number of colors required for a hub coloring of G_1 is 4. Hence, $\chi_h(G_1) = 4$.

Example 2.3. For the graph G_2 shown in Figure 1, $\{x, y\}$ and $\{x, z\}$ are h-sets of G_2. Any proper coloring in which x and z receive different colors requires at least 3 colors, and in every χ-coloring of G_2, x and z receive the same color. Therefore, any χ-coloring of G_2 is a hub coloring with colorful hub set $\{x, y\}$, and it proves that $\chi_h(G_2) = 2$. For G_2, we have $\chi(G_2) = h(G_2) = \chi_h(G_2)$.



Graph : G_1 Graph: G_2

Figure 1.

Proposition 2.4. *For Petersen graph P, we have $\chi_h(P) = 4$.*

Proof. Clearly, $\{u_2, u_5, u_8, u_9\}$ is a h-set of the Petersen graph P as shown in Figure 2, which implies that $\chi_h(P) \geq 4$. The coloring $(\{u_5, u_7\}, \{u_2, u_6, u_{10}\}, \{u_1, u_4, u_8\}, \{u_3, u_9\})$ is a hub coloring of P with colorful hub set $\{u_2, u_5, u_8, u_9\}$. Thus $\chi_h(P) = 4$. □

3 Complexity result

This section explores the complexity of finding hub chromatic number of a graph. The decision problem of hub chromatic number is defined as follows.

Hub chromatic number
Instance: A graph G of order n and a positive integer $k \leq n$
Question: Is there a hub coloring of G with k colors?

Theorem 3.1. *Hub chromatic number is NP-complete.*

Proof. The decision problem chromatic number is NP-complete for arbitrary graphs. To prove the result, it is enough to construct a graph G' such that G has a proper vertex coloring with k

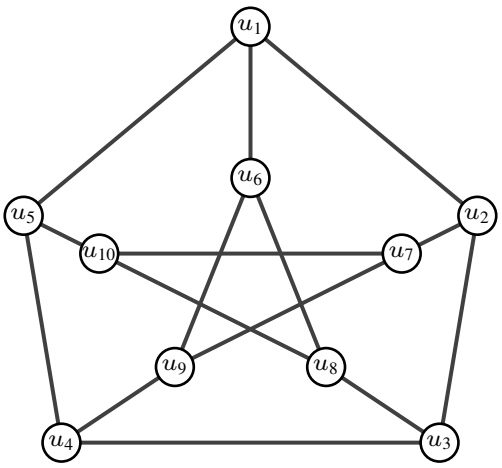


Figure 2. Petersen graph

colors if and only if G' has a hub coloring with $k+1$ colors.
For a given graph G, we construct a graph G' by adding a new vertex x to G , joining x to each vertex in G. Clearly, $\chi_h(G') = \chi(G') = \chi(G) + 1$.
It is easy to verify the coloring $(V_1, V_2, \ldots, V_k)$ is a proper vertex coloring of G with k colors if and only if the coloring $(V_1, V_2, \ldots, V_k, V_{k+1} = \{x\})$ is a hub coloring of G' with $k+1$ colors. Thus, for any graph G, G has a proper vertex coloring with k colors if and only if G' has a hub coloring with $k+1$ colors. Hence the decision problem hub chromatic number is NP-complete. □

4 Existence results

Proposition 4.1. *For any two integers a and b with $a \geq b \geq 2$, there is a graph G of order a with $\chi_h(G) = b$.*

Proof. Let G be a graph obtained from K_b by joining $a-b$ pendant vertices to exactly one vertex say u of K_b. Clearly, $\chi(G) = b$ and $\{u\}$ is a hub set of G. Thus, in any χ-coloring of G, $\{u\}$ is colorful hub set of G. Therefore, $\chi_h(G) = b$. □

Theorem 4.2. *For any integers $a \geq 2$, $b \geq 2$ and $c \geq 2$ with $max\{a, b\} \leq c \leq a + b - 2$, there is a connected graph G with $\chi(G) = a$, $h(G) = b$ and $\chi_h(G) = c$.*

Proof. Let $a \geq 2$, $b \geq 2$ and $c \geq 2$ with $max\{a, b\} \leq c \leq a + b - 2$. Let us construct a required graph G as follows. Consider a path (u_1, u_2,. . .,u_b) on b vertices. Attach $c-b+1$ pendant vertices with each of the vertices u_1, u_2,. . .,u_b. Take a complete graph K_{a-1} and join u_1 with every vertex of K_{a-1} by an edge. Since $c \geq b$, we have $c - b \geq 0$. If $c - b > 0$, take a complete graph K_{c-b} and join u_i, where $i = 1, 2, \ldots b$ with every vertex of K_{c-b} by an edge. For illustration when $a = 4$, $b = 5$ and $c = 7$ see Figure 3.
Claim 1. $\chi(G) = a$.
Proof of Claim 1. Clearly, $\omega(G) = a$, it implies that $\chi(G) \geq a$. Let us define a coloring $\mathcal{C}_1$ as follows. Now, color the vertex u_1 and vertices of K_{a-1} with colors $1, 2, \ldots, a$ respectively. Color the vertices in $\{u_i \ / \ 3 \leq i \leq b, i \ is \ odd\}$ with color 1, color the vertices in $\{u_i \ / \ 2 \leq i \leq b, i \ is \ even\}$ with color 2, color the pendant vertices of $\{u_i \ / \ 3 \leq i \leq b, i \ is \ odd\}$ with color 2 and color the pendant vertices of $\{u_i \ / \ 2 \leq i \leq b, i \ is \ even\}$ with color 1. Since, $c - b \leq a - 2$, it is possible to color the vertices of K_{c-b} with the colors $3, \ldots, a$. Clearly, $\mathcal{C}_1$ is a proper coloring with a colors. Therefore, $\chi(G) = a$.
Claim 2. $h(G) = b$.
Proof of Claim 2. Clearly, $\{u_i / 1 \leq i \leq b\}$ is a hub set of G, and any minimum hub set contains u_i or a pendant vertex attached to it. Therefore, $h(G) = b$.
Claim 3. $\chi_h(G) = c$.

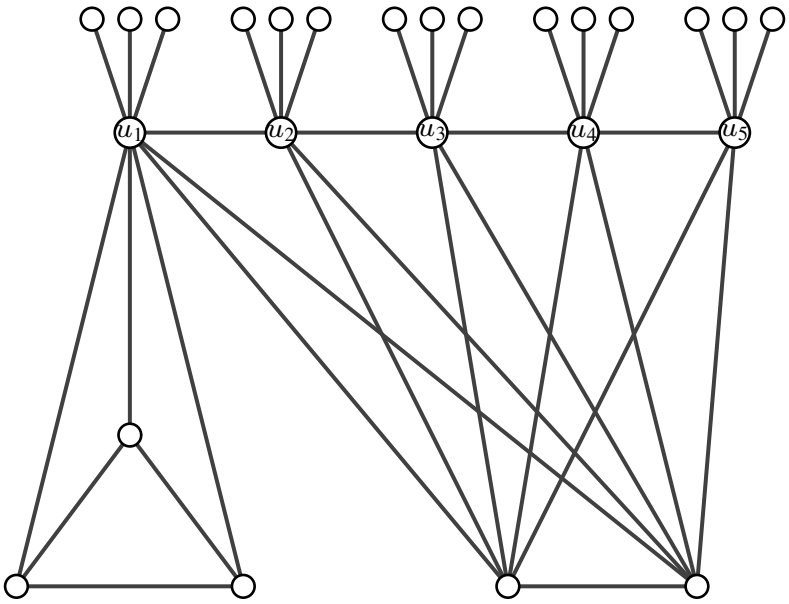


Figure 3.

Proof of Claim 3. Let us define a hub coloring of G using c colors. Color the vertices of $u_1, u_2, \ldots, u_b$ with colors $1, 2, \ldots b$ respectively, color the vertices of K_{c-b} with colors $b+1, \ldots c$. Color the vertices of K_{a-1} with colors $2, 3, \ldots c$ (this is possible since $a-1 \leq c-1$) and color all the pendant vertices with color $b+1$. Clearly, this is a proper vertex coloring of G with colorful hub set $\{u_i \mid 1 \leq i \leq b\}$. Therefore, $\chi_h(G) \leq c$.
Let $\mathcal{C}$ be any χ_h-coloring of G with colorful hub set S. Clearly, $|S| \leq c$, and all vertices in K_{c-b} receive different colors. Suppose $u_i, x \notin S$ for any pendant vertex x adjacent to u_i, implies that there is no S-path between x and any vertex $y \in V(G) - (S \cup \{x, y\})$ (since u_i is present in every $x - y$ path). This contradiction prove that S contains u_i or all pendant vertices adjacent to u_i ($1 \leq i \leq b$). Clearly, at least b colors (not used for vertices in K_{c-b}) are required to color the vertices in $V(G) - (V(K_{a-1}) \cup V(K_{c-b}))$. Therefore, $\chi_h(G) = c$. □

5 χ_h for some standard classes of graphs

Proposition 5.1. $\chi_h(C_n) = \begin{cases} 3, & n = 3, 5 \\ 2, & n = 4 \\ n-3, & n \geq 6 \end{cases}$.

Proof. Let $C_n : v_1, v_2, \ldots, v_n$ be the cycle of order n. It is easy to verify that for $n \in \{3, 4, 5\}$, any χ-coloring is a hub coloring. Since $\chi(C_3) = \chi(C_5) = 3$ and $\chi(C_4) = 2$, it follows that $\chi_h(C_3) = \chi_h(C_5) = 3$ and $\chi_h(C_4) = 2$.
for $n \geq 6$, by Theorem 1.3 we have $h(C_n) = n - 3$, which implies that $\chi_h(C_n) \geq n - 3$. In order to prove the result it is enough to obtain a hub coloring using $n - 3$ colors. Let us define a coloring $\mathcal{C}$ as follows. Color the vertices in $V(G) - \{v_1, v_2, v_3\}$ with different colors, color the vertex v_1 and v_3 with the color of v_5 and color the vertex v_2 with color of v_4. Clearly, $\mathcal{C}$ is a hub coloring of C_n ($n \geq 6$) with colorful hub set $\{v_4, v_5, \ldots, v_n\}$ using $n - 3$ colors. Therefore, $\chi_h(C_n) = n - 3$.

□

Proposition 5.2. *For a split graph G, $\chi_h(G) = \chi(G) = \omega(G)$.*

Proof. Let G be a split graph with split partition (K, I), where K is a maximum clique in G. Clearly, K is a hub set of G. Since $\langle K \rangle$ is complete graph of order $\omega(G)$, it follows that in any χ-coloring of G, no two vertices in K receive the same color. Hence every χ-coloring of G is a hub coloring of G with colorful hub set K. Therefore, $\chi_h(G) = \chi(G) = \omega(G)$. □

6 Bounds

Theorem 6.1. *For any graph G ($\neq K_n$), $max\{\ \chi(G),\ h(G)\ \} \leq \chi_h(G) \leq \chi(G) + h(G) - 1$.*

Proof. Every hub coloring is a proper vertex coloring, and in any hub coloring of G, the vertices in at least one hub set of G receive different colors. This proves the lower bound.
Let $\mathcal{C}$ be any χ-coloring of G and S be any minimum hub set of G. Let $a \in S$ be arbitrary. Let us define the coloring $\mathcal{C}'$ as follows. Assign colors to the vertices in $V(G) - (S - \{a\})$ as in $\mathcal{C}$ and assign new different colors (not used in $\mathcal{C}$) to the different vertices in $S - \{a\}$. Clearly, the coloring $\mathcal{C}$ is a hub coloring of G with colorful hub set S. Therefore, $\chi_h(G) \leq \chi(G) + h(G) - 1$.
This bound is sharp for all the graphs $K_{1,n-1}$ with $n \geq 2$. □

Theorem 6.2. *For any bipartite graph $G = (\ V_1 \cup V_2,\ E)$ of order $n \geq 3$, $\chi_h(G) \leq n - \Delta(G) + 1$. Further, if both V_1 and V_2 have no major vertices then $\chi_h(G) \leq n - \Delta(G)$.*

Proof. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph and $a \in V_1$ such that $\deg(a) = \Delta(G)$. Clearly, $V(G) - N(a)$ is a hub set of G. Let us define a coloring as follows. Assign different colors to the different vertices in $V(G) - N(a)$ and assign a new color to all vertices in $N(a)$. This coloring is a hub coloring of G with colorful hub set $V(G) - N(a)$. Therefore, $\chi_h(G) \leq n - \Delta(G) + 1$.
This bound is sharp for all star graphs $K_{1,n-1}$.
If $N(a) \neq V_2$, then the coloring in which all different vertices in $V(G) - N(a)$ receive different colors and all vertices in $N(a)$ receive a color used for any vertex $b \in V_2 - N(a)$ gives a hub coloring of G with colorful hub set $V(G) - N(a)$. Therefore, $\chi_h(G) \leq n - \Delta(G)$. This bound is sharp for paths of order $n \geq 6$. □

Theorem 6.3. *For any connected graph G (not a star) of order $n \geq 4$ with l pendant vertices, $\chi_h(G) \leq n - l$.*

Proof. Let K be the set of all pendant vertices in G and x be any pendant vertex adjacent to y. Clearly, $y \in V(G) - K$ (since G is connected) and $V(G) - K$ is the hub set of G. Let us define a coloring as follows. Assign different colors to the different vertices in $V(G) - K$, assign color used for y to all the pendant vertices in $K - \{x\}$ and assign color used for any non-pendant vertex (different from y) to x. This coloring is a hub coloring of G with colorful hub set $V(G) - K$. Therefore, $\chi_h(G) \leq n - l$.
This bound is sharp for all graphs $K_a \circ K_1$ with $a \geq 2$. □

Corollary 6.4. *Let T be a tree of order $n \geq 4$ and l pendant vertices, then*

$$\chi_h(T) = \begin{cases} 2, & \textit{if } T \textit{ is a star} \\ n - l, & \textit{otherwise} \end{cases}$$

Proof. Let T be the tree of order $n \geq 4$ and l pendant vertices.
Case 1. T is a star.
Let T be a star with x is universal vertex in T. Then every χ-coloring of G is a hub coloring of G with colorful hub set x. Therefore, $\chi_h(T) = 2$.
Case 2. $T \neq K_{1,n-1}$.
By Theorem 1.4 and Theorem 6.3, it follows that $\chi_h(T) = n - l$. □

Theorem 6.5. *For any connected graph G, $\chi_h(G) \geq \chi_\gamma(G) - 1$.*

Proof. Let $\mathcal{C}$ be a χ_h-coloring of G with colorful hub set H. If H is a dominating set, then $\mathcal{C}$ is the gamma coloring of G, it follows that $\chi_\gamma(G) \leq \chi_h(G)$. Suppose H is not a dominating set of G, proves that there is a vertex x in $V(G) - H$ such that x is not adjacent to any vertex in H. If x is not adjacent to some $y \in V(G) - H$, then there is no H-path between x and y, a contradiction (H is not hub set of G). Therefore, $N[x] = V(G) - H$, which implies that $H \cup \{x\}$ is a dominating set of G. Let us define the coloring $\mathcal{C}'$ as follows. Assign a new color to x and assign colors to the all remaining vertices in G as in $\mathcal{C}$. The coloring $\mathcal{C}'$ is a gamma coloring of G with colorful dominating set $H \cup \{x\}$. Therefore, $\chi_\gamma(G) \leq \chi_h(G) + 1$. □

7 Characterizations

Theorem 7.1. *For any connected graph G of order $n \geq 2$, $\chi_h(G) = 2$ if and only if G is bipartite and each part contains a major vertex.*

Proof. Let G be a connected graph with $\chi_h(G) = 2$. Let $\mathcal{C} = (V_1, V_2)$ is a hub coloring of G using 2 colors. Clearly, G is bipartite and $h(G) \leq 2$. It is easy to verify that if $h(G) = 0$, then $G = K_2$. Also if $h(G) = 1$, then $G = K_{1,r}$ or $G = K_{2,r}$ with $r \geq 2$.
Suppose $h(G) = 2$. Clearly, $|V_1| \geq 2$ and $|V_2| \geq 2$.
Case 1. It is easy to verify that if $| V_1 |=| V_2 |= 2$, then $G = P_4$.
Case 2. $| V_1 |= 2, | V_2 |\geq 3$.
Let $V_1 = \{x, x_1\}$ and $H = \{x, y\}$ be a hub set of G for some $y \in V_2$. Since $|V_2| \geq 3$, every vertex in $V_2 - \{y\}$ is adjacent to x. If x is adjacent to y, then x is a major vertex in V_1, and since G is connected, any vertex in V_2 which is adjacent to x_1 is a major vertex in V_2. If x is not adjacent to y, then x_1 must be adjacent to y. Suppose x_1 is not adjacent to some $y_1 \in V_2$; then there is no H-path between x_1 and y_1. This contradiction proves that x_1 is a major vertex in V_1 and any vertex in V_2 which is adjacent to x, is a major vertex in V_2.
Case 3. $| V_1 |\geq 3, | V_2 |\geq 3$.
Let $H = \{x, y\}$ be a hub set of G for some $x \in V_1$ and $y \in V_2$. Clearly, every vertex in $V_2 - \{y\}$ is adjacent to x and every vertex in $V_1 - \{x\}$ is adjacent to y. If x and y are adjacent, then we are done. Suppose x is not adjacent to y. If there exist $x_1 \in V_1 - \{x\}$ and $y_1 \in V_2 - \{y\}$ that are non-adjacent, then there is no H-path between x_1 and y_1. So every vertex in $V_1 - \{x\}$ is adjacent to every vertex in $V_2 - \{y\}$. Therefore, any vertex in $V_1 - \{x\}$ which is adjacent to y and any vertex in $V_2 - \{y\}$ which is adjacent to x are major vertices in V_1 and V_2, respectively.
Conversely, let x and y be major vertices in V_1 and V_2, respectively. Clearly, $\{x, y\}$ is a hub set of G, it follows that every χ-coloring of G is a hub coloring of G with colorful hub set $\{x, y\}$. Therefore, $\chi_h(G) = \chi(G) = 2$. □

Theorem 7.2. *For any connected graph G of order $n \geq 2$, $\chi_h(G) = n$ if and only if G is complete.*

Proof. For $n = 2$, the proof is obvious. Let G be a connected graph of order $n \geq 3$ with $\chi_h(G) = n$. Suppose G is not complete, implies that there are vertices x and y in G which are not adjacent . Clearly, $V(G) - \{x, y\}$ is a hub set of G. Color the different vertices in $V(G) - \{x, y\}$ with different colors and color x and y with a new color gives rise to a hub coloring of G with colorful hub set $V(G) - \{x, y\}$ using $n - 1$ colors, a contradiction. Therefore G is complete. The converse is obvious. □

Theorem 7.3. *For any connected graph G of order $n \geq 3$, $\chi_h(G) = n - 1$ if and only if $G \neq K_n$ with K_{n-1} is a subgraph of G.*

Proof. For $n = 3$, the result is obvious. Let G be a connected graph of order $n \geq 4$ with $\chi_h(G) = n - 1$. By Theorem 7.2, $G \neq K_n$. Let x and y be any two non-adjacent vertices in G. Clearly, $V(G) - \{x, y\}$ is a hub set of G. If $z \notin N[x] \cup N[y]$, then the coloring in which vertices x, y and z receive color 1 and all the remaining vertices receive colors $2, 3, \ldots, n - 3$ gives a hub coloring of G with colorful hub set $V(G) - \{x, y, z\}$ using $n - 2$ colors, a contradiction. Also, if there exists $z_1 \notin N[x]$ and $z_2 \notin N[y]$, then the coloring in which x and z_1 receive color 1, y and z_2 receive color 2 and all the remaining vertices receive colors $3, 4, \ldots, n - 4$ gives a hub coloring of G with colorful hub set $V(G) - \{x, y, z\}$ using $n - 2$ colors, a contradiction. It follows that x or y adjacent to all the vertices in $V(G) - \{x, y\}$. Let x is adjacent to all the vertices in $V(G) - \{x, y\}$. Since $\deg(y) \geq 1$, there exists $b \in V(G) - \{x, y\}$ adjacent to y. Clearly, $\{x, b\}$ is a hub set of G. Since x is adjacent to b, it follows that in any χ-coloring of G, $\{x, b\}$ is colorful hub set. Therefore, $\chi_h(G) = \chi(G) = n - 1$, it follows that $G \neq K_n$ and K_{n-1} is a subgraph of G. The converse is obvious. □

8 χ_h for some graph operations

Proposition 8.1. *For join $G_1 + G_2$ of two graphs G_1 and G_2, $\chi_h(G_1 + G_2) = \chi(G_1) + \chi(G_2)$.*

Proof. For any $a \in V(G_1)$ and $b \in V(G_2)$, the set $\{a, b\}$ is a hub set of $G_1 + G_2$. In any proper vertex coloring of $G_1 + G_2$, the set $\{a, b\}$ is colorful. Therefore, $\chi_h(G_1 + G_2) = \chi(G_1 + G_2) = \chi(G_1) + \chi(G_2)$. □

Theorem 8.2. *Let G be any connected graph of order n and F be any graph, then $\chi_h(G \circ F) = max\{n,\ \chi(F) + 1\}$.*

Proof. Since $V(G)$ is a minimum hub set of $G \circ F$ and $\chi(G \circ F) \geq \chi(F) + 1$, it follows that $\chi_h(G \circ F) \geq \max\{n, \chi(F) + 1\}$.
Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, and let $\mathcal{C} = (V_1, V_2, \cdots, V_k)$ be a χ-coloring of F. For each $i = 1, 2, \ldots, n$, let F_i be the i^{th} copy of F such that v_i is adjacent to all vertices of F_i in $G \circ F$.
Case 1. $n \geq \chi(F) + 1$.
Assign n different colors to the different vertices of G, and for each i $(1 \leq i \leq n)$, assign colors to the vertices of F_i as in $\mathcal{C}$ by using the colors used to the vertices in $V(G) - \{v_i\}$ (this is possible since $\chi(F) \leq n - 1$). Clearly, this coloring is a proper vertex coloring of $G \circ F$ with $V(G)$ as a colorful set. Therefore, $\chi_h(G \circ F) = n$.
Case 2. $\chi(F) + 1 > n$.
Clearly, $\chi(F) = k$. For each i $(1 \leq i \leq n)$, it is possible to assign $k + 1$ different colors to the different vertices in $V(F_i) \cup \{v_i\}$ such that no two vertices in G have the same color. This coloring is a hub coloring of $G \circ F$ with colorful hub set $V(G)$. Therefore, $\chi_h(G \circ F) = \chi(F) + 1$. □

Lemma 8.3. *For any connected graph G, $h(\mu(G)) \leq \gamma_t(G) + 1$.*

Proof. Let $\mu(G)$ be the Mycielskian of G with vertex set $V(\mu(G)) = V(G) \cup V'(G) \cup \{u\}$. Let S be any γ_t-set of G, and let S' be the set of twins of vertices in S and $D = S' \cup \{u\}$.
Claim. D is hub set of $\mu(G)$.
Let $x, y \in V(\mu(G)) - D$ be arbitrary.
Case 1. $x, y \in V(G)$.
Since S is a γ_t-set of G, there exist $a, b \in S$ such that $a \in N(x)$ and $b \in N(y)$. If $a \in N(x) \cap N(y)$, then $xa'y$ is a D-path between x and y; otherwise, $xa'ub'y$ is a D-path between x and y.
Case 2. $x, y \in V'(G)$
xuy is the D-path between x and y.
Case 3. $x \in V(G)$ and $y \in V'(G)$
$xa'uy$ is the D-path between x and y for $a \in N(x) \cap S$.
This proves that D is a hub set of $\mu(G)$. Therefore, $h(\mu(G)) \leq \gamma_t(G) + 1$. □

Theorem 8.4. *For any connected graph G, $\chi_h(\mu(G)) \leq \chi_{\gamma_t}(G) + 1$. This bound is sharp.*

Proof. Let G be a connected graph with the vertex set $V(G)$. Let $\mathcal{C}$ be a minimum total gamma coloring of G with colorful total dominating set S. Let S' be the set containing twins of vertices in S. By Lemma 8.3, $S' \cup \{u\}$ is a hub set of $\mu(G)$. Let us define the coloring $\mathcal{C}'$ of $\mu(G)$ as follows. Color the vertices in $V(G)$ as $\mathcal{C}$, color the vertices in $V'(G)$ with color of their twins and color u with a new color. Clearly, $\mathcal{C}'$ is a proper coloring of $\mu(G)$ with colorful hub set $S' \cup \{u\}$ using $\chi_{\gamma_t}(G)$ colors. Therefore, $\chi_h(\mu(G)) \leq \chi_{\gamma_t}(G) + 1$.
This bound is sharp for the paths of order $n \geq 4$ with $n \equiv 2$ *or* 3 $(mod\ 4)$. For $n \geq 4$ with $n \equiv 2$ *or* 3 $(mod\ 4)$, we have $\chi_{\gamma_t}(P_n) = \gamma_t(P_n) = \lfloor \frac{n}{2} \rfloor + 1$ and $h(\mu(P_n)) = \lfloor \frac{n}{2} \rfloor + 2$. Therefore, $\chi_h(\mu(P_n)) = \chi_{\gamma_t}(P_n) + 1$. □

9 Concluding Remarks and Scope for Further Work

In this chapter, we conduct a detailed study on the hub chromatic number of graphs, bounds for χ_h-number with graph parameters and characterize the graphs of order n with hub chromatic number equal to 2, $n - 1$ and n. In addition to this work, we pose the following open problems.

(i) Characterize the graphs for which $\chi_h(G) = h(G)$.

(ii) Characterize the graphs for which $\chi_h(G) = \chi(G)$.

(iii) Characterize the graphs for which $\chi_h(G) = \chi_\gamma(G) - 1$.

(iv) Characterize the graphs for which $\chi_h(\mu(G)) = \chi_{\gamma_t}(G) + 1$.

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