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INVERSE STRONG ROMAN DOMINATION IN GRAPHS

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Abstract For a graph G, a strong Roman dominating function (SRDF) is a function $f : V(G) \to \{0, 1..., \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\}$ such that each vertex v with $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) \geq 1 + \left\lceil \frac{1}{2} |N(u) \cap V_0| \right\rceil$ where $V_0 = \{v \in V : f(v) = 0\}$. Among all possible SRDF of a graph, the one with minimum weight is called a minimum SRDF and its weight is denoted by $\gamma_{sR}(G)$.

In this paper, we have introduced the inverse strong Roman dominating function (ISRDF) of a graph. Any SRDF with minimum weight $\gamma_{sR}(G)$ is also called a $\gamma_{sR}(G)$ -function. If f is a $\gamma_{sR}(G)$ -function, then its ISRDF g is also a SRDF on G, such that $g(v) = 0$, for all $v \in S$ where $S = \{v \in V(G) : f(v) > 0\}$. The minimum weight of such a function is called the inverse strong Roman domination number of G, denoted by $\gamma_{isR}(G)$.

We obtain the $\gamma_{isR}(G)$ of some classes of graphs and Cartesian products. We find a few bounds for the $\gamma_{isR}(G)$. We also derive a bound for $\gamma_{isR}(T)$ for any tree in terms of the order and the number of leaves in a tree T.

1 Introduction

To enhance the concept of Roman domination, a stronger version of this strategy has been proposed by MP Álvarez-Ruiz et al. in [1]. In Roman domination, every location is either defended by a Roman legion or adjacent to a location with two Roman legions protecting during an attack. However, if a location with two legions has multiple unsecured neighbouring locations, it may struggle to defend against simultaneous attacks, as it can only repel a single attack. To address this vulnerability, the strong Roman domination ensures that each unsecured location is protected by ensuring that at least one neighbouring location has a minimum of $\left\lceil \frac{k}{2} \right\rceil + 1$ legions, where k represents the number of its unsecured neighbours.

Chen and Sohn [3] and [4] discussed the SRDF of the Cartesian product of paths. They also characterized the trees with $\gamma_{sR}(G) = \gamma_R(G)$. Akram Mahmoodi et al., [12] presented some sharp bounds for the $\gamma_{sR}(G)$.

Henning [9] discussed a strategy for defending the Roman Empire against multiple attacks by introducing the concept of the k-Roman dominating function. For more details refer [2, 14]. This function is designed to protect the empire from consecutive attacks. In contrast, a strong Roman dominating function protects to at least half of its unsecured neighbours against multiple simultaneous attacks.

It has numerous applications in facility location problems, as it enables the provision of services to multiple locations simultaneously. For example, if there is an outbreak of fire at several places at the same time, a strong Roman dominating function can quickly and cost-effectively provide fire engines.

As an improvement to Roman domination, the strong Roman domination is introduced in [1]. In Roman domination, each place is either secured by placing a Roman legion or has a

neighbour where two Roman legions are placed. The neighbour will protect the unsecured place during the attack. What if a place where two Roman legions are placed has many unsecured neighbouring places? It can only defend a single attack and it fails in case of multiple attacks. The unsecured places are better protected by strong Roman domination. Each unsecured place will have a neighbour with at least $\lceil \frac{k}{2} \rceil + 1$ legions placed, where k is the number of its unsecured neighbours.

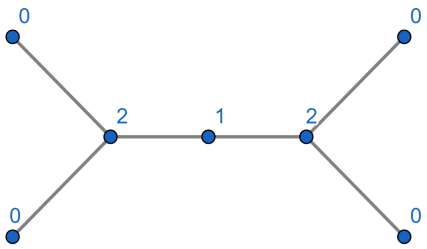


Figure 1. γ_{sR} -function of a graph.

What if the enemy discovers our strategy? We need a stronger alternative strategy. In this case, the concept of inverse strong Roman domination is relevant.

Every graph used here is simple, finite, and free of isolated vertices. The notation $\Delta(G)$ represents the highest degree of the graph. Any vertex in a tree that is adjacent to any leaf is called a support vertex, and a strong support vertex is adjacent to several leaves. The open neighbourhood $N(p)$ of a vertex p is defined as $N(p) = \{s \in V(G) : ps \in E(G)\}$ and $N[p] = N(p) \cup \{p\}$.

A Cartesian product of graphs H_1 and H_2 is denoted by $H_1 \Box H_2$. It has the vertex set $V(H_1)\Box V(H_2)$ and $(p_1, p_2)(q_1, q_2) \in E(H_1 \Box H_2)$ if $p_1 q_1 \in E(H_1)$ and $p_2 = q_2$ or $p_2 q_2 \in E(H_2)$ and $p_1 = q_1$. A double star $DS_{p,q}$ is a tree with exactly 2 non-pendant vertices u, v where $deg(u) = p$ and $deg(v) = q$. We have used the notation $K_{p,q}$ for a complete bipartite graph with partite sets of cardinality p and q.

A set $S \subseteq V$ in a graph G is called a dominating set if every vertex of G is either in S or adjacent to a vertex of S. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set in G. For more details refer [7] and [8].

"Let S be a minimum dominating set of G. If $V - S$ contains a dominating set S' of G, then S' is called an inverse dominating set of G with respect to S [10]."The inverse domination number of G is the minimum cardinality of an inverse dominating set of G, denoted by $\gamma_i(G)$. This concept was introduced by Kulli and Sigarkanti in [10].

"A Roman dominating function(RDF) is a function $f : V \to \{0, 1, 2\}$ such that, each vertex v with $f(v) = 0$ is adjacent to at least one vertex x with $f(x) = 2$. Adding all the labellings of the function f gives the weight of f denoted by $w(f)$. The Roman domination number of G is the minimum weight of a RDF on G, indicated by $\gamma_R(G)$ [5]. "

"A strong Roman dominating function (SRDF) is a function $f : V(G) \to \{0, 1..., \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\}$ such that each vertex v with $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) \geq 1 + \lceil \frac{1}{2} |N(u) \cap V_0| \rceil$ where $V_0 = \{v \in V : f(v) = 0\}$. Among all possible SRDF of a graph, the one with minimum weight is called a minimum SRDF and its weight is denoted by $\gamma_{sR}(G)$[1] ."

We introduce the study on Inverse strong Roman dominating functions(ISRDF) in graphs. Any SRDF of weight $\gamma_{sR}(G)$ is also known as γ_{sR} -function. Let $S = \{v \in V : f(v) > 0\}$ for some γ_{sR} -function f. It is also called the γ_{sR} -set of f. If it is possible to define another SRDF g such that $g(v) = 0$, for all $v \in S$, then g is called an ISRDF of f. A γ_{sR} -function can have multiple ISRDF or it may not have any ISRDF. The $\gamma_{isR}(G)$ is the minimum weight among all ISRDF of G. The inverse strong Roman dominating function of weight $\gamma_{isR}(G)$ is denoted by γ_{isR} -function. For more details refer [13].

The figure 1 shows a graph with its only possible SRDF. It is a γ_{sR} -function of weight 5. It

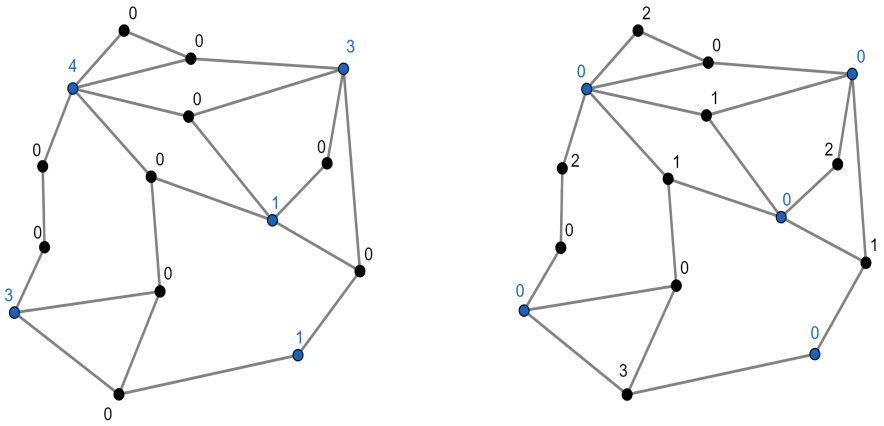


Figure 2. Illustration of γ_{sR} -function and an γ_{isR} -function of a graph

doesn't have an ISRDF, as both the neighbouring vertices of the vertex with label 1, have positive labels in its γ_{sR} -function. Hence in its inverse SRDF, this vertex will be labelled 0 and it will not have any neighbours with positive label.

The figure 2 shows a γ_{sR} -function and a γ_{isR} -function of a graph with $\gamma_{sR}(G) = \gamma_{isR}(G) = 12$.

2 Bounds of $\gamma_{isR}(G)$

Proposition 2.1. *For any graph G,*

$$\gamma_{sR}(G) \leq \gamma_{isR}(G).$$

Proof. Since any ISRDF is also an SRDF of a graph G, the desired result follows. □

Theorem 2.2. *[1] For any graph G,*

$$\gamma_{sR}(G) \leq \gamma(G)\left(\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\right).$$

Theorem 2.3. *For any graph G of order $n \geq 2$,*

$$\gamma_{sR}(G) + \gamma_{isR}(G) \leq \gamma(G)\left(\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\right) + n,$$

and this bound is sharp for stars.

Proof. Let S be any γ - set of G and let f be a SRDF. By using Theorem 2.2, we define f such that $f(v) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ for all $v \in S$ and 0 to the remaining vertices. Let us define an inverse f' that assigns 2 to one neighbour of every vertex in S, 0 to the vertices of S and 1 to the remaining vertices. Then $w(f') = n$. The sum $w(f) + w(f') = \gamma(G)\left(\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\right) + n$. Since f is some SRDF, we get $\gamma_{sR}(G) + \gamma_{isR}(G) \leq \gamma(G)\left(\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\right) + n$. This bound is sharp for stars as $\gamma_{sR}(K_{1,n-1}) + \gamma_{isR}(K_{1,n-1}) = 1 + \lceil \frac{n-1}{2} \rceil + n$. □

Theorem 2.4. *[12] for any tree T of order $n \geq 2$ and l leaves,*

$$\gamma_{sR}(T) \geq \left\lceil \frac{2(n+2-l)}{3} \right\rceil.$$

Theorem 2.5. *For any tree T of order $n \geq 2$ and l leaves,*

$$\gamma_{isR}(T) \leq \gamma(G)\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\right)+\left\lceil\frac{n-4+2l}{3}\right\rceil.$$

Proof. Let T be a tree of order $n \geq 2$ and l leaves. By theorem 2.4 we have,

$$\gamma_{sR}(T) \geq \left\lceil\frac{2(n+2-l)}{3}\right\rceil$$

$$\Rightarrow \gamma_{isR}(T)+\gamma_{sR}(T) \geq \gamma_{isR}(T)+\left\lceil\frac{2(n+2-l)}{3}\right\rceil$$

$$\Rightarrow \gamma_{isR}(T) \leq \gamma(G)\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\right)+n-\left\lceil\frac{2(n+2-l)}{3}\right\rceil$$

$$\Rightarrow \gamma_{isR}(T) \leq \gamma(G)\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\right)+\left\lceil\frac{n-4+2l}{3}\right\rceil \text{ by Theorem 2.3} \qquad \square$$

Theorem 2.6. *[1] If G is any graph of order n, then*

$$\gamma_{sR}(G) \geq \left\lceil\frac{n+1}{2}\right\rceil.$$

Theorem 2.7. *For any graph G of order n,*

$$\gamma_{isR}(G) \leq \gamma(G)\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\right)+\left\lceil\frac{n-1}{2}\right\rceil.$$

Proof. Let G be any graph of order n. By Theorem 2.6 we have,

$$\gamma_{sR}(G) \geq \left\lceil\frac{n+1}{2}\right\rceil$$

$$\Rightarrow \gamma_{isR}(G)+\gamma_{sR}(G) \geq \gamma_{isR}(G)+\left\lceil\frac{n+1}{2}\right\rceil$$

$$\Rightarrow \gamma_{isR}(G) \leq \gamma(G)\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\right)+n-\left\lceil\frac{n+1}{2}\right\rceil$$

$$\Rightarrow \gamma_{isR}(G) \leq \gamma(G)\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\right)+\left\lceil\frac{n-1}{2}\right\rceil \text{ by Theorem 2.3.} \qquad \square$$

3 Specific families of graphs

Remark 3.1. [12] For any connected graph G with $\Delta(G) \leq 2$,

$$\gamma_{sR}(G) = \gamma_R(G).$$

Proposition 3.2. *[5] For paths, P_n and cycles C_n,*

$$\gamma_{sR}(P_n) = \gamma_{sR}(C_n) = \lceil\tfrac{2n}{3}\rceil.$$

Proposition 3.3. *For any path P_n of order $n \geq 2$,*

$$\gamma_{isR}(P_n) = \lceil\tfrac{2n+1}{3}\rceil.$$

Proof. From Proposition 3.2 we have, $\gamma_{sR}(P_n) = \lceil\frac{2n}{3}\rceil$. Consider a path P_n of vertices u_i, $1 \leq i \leq n$. We can define a γ_{sR} -function g as follows: $g(u_i) = 2$ if $i \equiv 2\ (mod\ 3)$, 0 otherwise. Reassign $g(u_n) = 1$ if $n \equiv 1\ (mod\ 3)$.
Case 1: $n \not\equiv 0\ (mod\ 3)$

An ISRDF g' can be defined as follows: $g'(u_i) = g(u_{n-i+1})\ \forall\ 1 \leq i \leq n$. So we get $w(g') = w(g) = \lceil\frac{2n}{3}\rceil = \lceil\frac{2n+1}{3}\rceil$.
Case 2: $n \equiv 0\ (mod\ 3)$

An ISRDF g' can be defined as follows: $g'(u_{i+1}) = g(u_i)\ \forall\ 1 \leq i \leq n-1$, $g'(u_1) = 1$. Hence $w(g') = w(g)+1 = \lceil\frac{2n+1}{3}\rceil$. $\square$

Proposition 3.4. *For any cycle C_n of order $n \geq 3$,*

$$\gamma_{isR}(C_n) = \lceil \tfrac{2n}{3} \rceil.$$

Proof. From Proposition 3.2 we have, $\gamma_{sR}(C_n) = \lceil \frac{2n}{3} \rceil$. Consider a cycle C_n of vertices u_i, $1 \leq i \leq n$. We can define a γ_{sR} -function g as follows: $g(u_i) = 2$ if $i \equiv 2 \ (mod\ 3)$ and 0 otherwise. Reassign $g(u_n) = 1$ if $n \equiv 1 \ (mod\ 3)$. We can define an ISRDF $g^{'}$ as follows: $g^{'}(u_{i+1}) = g(u_i)$ for all $1 \leq i \leq n-1$ and $g^{'}(u_1) = g(u_n)$. Hence $w(g^{'}) = w(g) = \lceil \frac{2n}{3} \rceil$. □

Proposition 3.5. *For any star $K_{1,n-1}$ $n \geq 2$,*

$$\gamma_{isR}(K_{1,n-1}) = n.$$

Proof. Let g be a γ_{sR} -function on $V(K_{1,n-1})$. Let u_n be the universal vertex in $K_{1,n-1}$ and let $u_1, u_2 ... u_{n-1}$ be the remaining vertices. Then $g(u_n) = 1 + \lceil \frac{n-1}{2} \rceil$ and $g(u_i) = 0$ where $1 \leq i \leq n-1$.

The only possible ISRDF $g^{'}$ on $V(K_{1,n-1})$ is defined as $g^{'}(u_n) = 0, g^{'}(u_1) = 2, g^{'}(u_i) = 1$ where $2 \leq i \leq n-1$. This implies $\gamma_{isR}(K_{1,n-1}) = n$. □

Proposition 3.6. *For any double star $DS_{p,q}$,*

$$\gamma_{isR}(DS_{p,q}) = \begin{cases} n & \text{if } p, q \geq 2 \\ n-1 & \text{otherwise} \end{cases}$$

Proof. Let $r_1, r_2, \ldots, r_p$ be the pendant vertices adjacent to the vertex r and let $s_1, s_2, \ldots, s_q$ be the pendant vertices adjacent to the vertex s of $DS_{p,q}$. Let g be a γ_{sR} -function on $DS_{p,q}$.
Case 1: When $p = q = 1$.
When $p = q = 1$, $DS_{p,q} = P_4$ and by Proposition 3.3, $\gamma_{isR}(P_4) = 3 = n - 1$.
Case 2: When $p = 1, q \geq 2$ or $q = 1, p \geq 2$.
Suppose $p = 1$ and $q \geq 2$. Then $g(r) = 0$, $g(r_1) = 1$ and $g(s) = 1 + \left\lceil \frac{q+1}{2} \right\rceil$. The only ISRDF $g^{'}$ assigns $g^{'}(r) = 2, g^{'}(r_1) = g^{'}(s) = 0$ and 1 to the remaining vertices. This implies $\gamma_{isR}(DS_{p,q}) = q + 2 = n - 1$.
Case 3: When $p, q \geq 2$.
In this case $g(r) = 1 + \lceil \frac{p}{2} \rceil$ and $g(s) = 1 + \lceil \frac{q}{2} \rceil$, whereas the remaining vertices are assigned 0 under g. The only ISRDF $g^{'}$ assigns $g^{'}(r) = g^{'}(s) = 0, g^{'}(r_1) = g^{'}(s_1) = 2$ and $g(r_i) = g(s_j) = 1$, where $2 \leq i \leq p-1, 2 \leq i \leq q-1$. This implies $\gamma_{isR}(DS_{p,q}) = n$. □

Proposition 3.7. *[12] For any complete bipartite graph $K_{p,q}$ where $q \geq p \geq 2$,*

$$\gamma_{sR}(K_{p,q}) = \begin{cases} \left\lceil \frac{p+1}{2} \right\rceil + \lceil \frac{q}{2} \rceil & \text{if } q \text{ is odd} \\ \left\lceil \frac{p+1}{2} \right\rceil + \left\lceil \frac{q+1}{2} \right\rceil & \text{if } q \text{ is even and } p \geq 3 \\ 2 + \frac{q}{2} & \text{if } q \text{ is even and } p = 2 \end{cases}$$

Proposition 3.8. *For any complete bipartite graph $G \cong K_{p,q}$ where $q \geq p \geq 2$,*

$$\gamma_{isR}(K_{p,q}) = \begin{cases} n-1 & \text{if } p = 2 \text{ and } q \text{ is even} \\ \gamma_{sR}(K_{p,q}) & \text{otherwise} \end{cases}$$

Proof. Let X and Y be the partite sets of $K_{p,q}$. Let $r_1, r_2, r_3, \ldots, r_p$ be the vertices of X and let $s_1, s_2, s_3, \ldots, s_q$ be the vertices of Y.
Case 1: When q is odd.
In this case we have, $\gamma_{sR}(K_{p,q}) = \left\lceil \frac{p+1}{2} \right\rceil + \lceil \frac{q}{2} \rceil$. We can define a γ_{sR} -function g as follows: $g(r_2) = g(r_3) = ... = g(r_p) = g(s_2) = g(s_3) = ... = g(s_q) = 0$. Since r_1 is adjacent to $q-1$ zeros and since $q-1$ is even, $g(r_1) = 1 + \frac{q-1}{2} = \frac{q+1}{2} = \lceil \frac{q}{2} \rceil$.
If p is even, then s_1 will be adjacent to $p-1$ zeros where $p-1$ is odd. Hence, $g(s_1) = 1 + \left\lceil \frac{p-1}{2} \right\rceil = \frac{p+2}{2} = \left\lceil \frac{p+1}{2} \right\rceil$.

If p is odd, then s_1 will be adjacent to $p-1$ zeros where $p-1$ is even. Hence, $g(s_1) = 1+\left\lceil \frac{p-1}{2} \right\rceil = \frac{p+1}{2} = \left\lceil \frac{p+1}{2} \right\rceil$.

We can define a γ_{isR} -function $g^{'}$ of weight $\left\lceil \frac{p+1}{2} \right\rceil + \lceil \frac{q}{2} \rceil$ as follows: $g^{'}(r_2) = \lceil \frac{q}{2} \rceil$, $g^{'}(s_2) = \left\lceil \frac{p+1}{2} \right\rceil$, zero otherwise. Hence $\gamma_{isR}(K_{p,q}) = \left\lceil \frac{p+1}{2} \right\rceil + \lceil \frac{q}{2} \rceil$.

Case 2: When q is even and $p \geq 3$.

In this case we have, $\gamma_{sR}(K_{p,q}) = \left\lceil \frac{p+1}{2} \right\rceil + \left\lceil \frac{q+1}{2} \right\rceil$. We can define a γ_{sR} -function g as follows: $g(r_2) = g(r_3) = ... = g(r_p) = g(s_2) = g(s_3) = ... = f(s_q) = 0$. Since r_1 is adjacent to $q-1$ zeros and since $q-1$ is odd, $g(r_1) = 1 + \left\lceil \frac{q-1}{2} \right\rceil = \frac{q+2}{2} = \left\lceil \frac{q+1}{2} \right\rceil$.

If p is even, then s_1 will be adjacent to $p-1$ zeros where $p-1$ is odd. Hence, $g(s_1) = 1+\left\lceil \frac{p-1}{2} \right\rceil = \frac{p+2}{2} = \left\lceil \frac{p+1}{2} \right\rceil$. If p is odd, then s_1 will be adjacent to $p-1$ zeros where $p-1$ is even. Hence, $g(s_1) = 1 + \left\lceil \frac{p-1}{2} \right\rceil = \frac{p+1}{2} = \left\lceil \frac{p+1}{2} \right\rceil$.

We can define a γ_{isR} -function $g^{'}$ of weight $\left\lceil \frac{p+1}{2} \right\rceil + \left\lceil \frac{q+1}{2} \right\rceil$ as follows: $g^{'}(r_2) = \left\lceil \frac{q+1}{2} \right\rceil$, $g^{'}(s_2) = \left\lceil \frac{p+1}{2} \right\rceil$, zero otherwise. Hence $\gamma_{isR}(K_{p,q}) = \left\lceil \frac{p+1}{2} \right\rceil + \left\lceil \frac{q+1}{2} \right\rceil$.

Case 3: When q is even and $p = 2$.

In this case we have, $\gamma_{sR}(K_{p,q}) = 2 + \frac{q}{2}$. The only possible γ_{sR} -function g is defined as follows: $g(r_1) = \frac{q}{2}+1$, $g(r_2) = 1$, zero otherwise. The only possible γ_{isR} -function $g^{'}$ is defined as follows: $g^{'}(r_1) = g^{'}(r_2) = 0$, $g^{'}(s_1) = 2$ and 1 otherwise. In this case $w(g) = 2 + \frac{q}{2}$ and $w(g^{'}) = q + 1 = n - 1$. □

4 Caretesian products

Proposition 4.1. *[12] For $n \geq 1$,*

$$\gamma_{sR}(P_2 \Box P_n) = \begin{cases} \frac{4n}{3} & if \quad n \equiv 0 \ (mod\ 3) \\ \frac{4n+2}{3} & if \quad n \equiv 1 \ (mod\ 3) \\ \frac{4n+1}{3} & if \quad n \equiv 2 \ (mod\ 3) \end{cases}$$

Proposition 4.2. *For $n \geq 1$,*

$$\gamma_{isR}(P_2 \Box P_n) = \gamma_{sR}(P_2 \Box P_n).$$

Proof. By Proposition 2.1, we know that $\gamma_{isR}(G) \geq \gamma_{sR}(G)$ for any graph G. We show that, there exists a γ_{isR} -function such that $\gamma_{isR}(P_2 \Box P_n) = \gamma_{sR}(P_2 \Box P_n)$. Let g be a γ_{sR} -function on $P_2 \Box P_n$ and let $g^{'}$ be the corresponding γ_{isR} -function. Let p_j^1 and p_j^2 be the vertices in the j^{th} copy of P_2.

Case 1: When $n \equiv 0 \ (mod\ 3)$

We define g as follows: For $1 \leq i \leq n$, $g(p_i^1) = g(p_i^2) = 2$ if $i \equiv 2 \ (mod\ 3)$ and 0 otherwise. So $w(g) = \frac{4n}{3}$. We can define a γ_{isR} -function $g^{'}$ as follows: For $1 \leq i \leq n$, $g^{'}(p_i^1) = 2$ if $i \equiv 0, 1 \ (mod\ 6)$ and $g^{'}(p_i^2) = 2$ if $i \equiv 3, 4 \ (mod\ 6)$ and 0 otherwise. Then $w(g) = w(g^{'})$.

Case 2: When $n \equiv 1 \ (mod\ 3)$

We define g as follows: For $1 \leq i \leq n-1$, $g(p_i^1) = g(p_i^2) = 2$ if $i \equiv 2 \ (mod\ 3)$, $g(p_n^1) = g(p_n^2) = 1$ and 0 otherwise. So $w(g) = \frac{4n+2}{3}$. We can define a γ_{isR} -function $g^{'}$ as follows: For $2 \leq i \leq n$, $g^{'}(p_i^1) = g^{'}(p_{i-1}^1)$, $g^{'}(p_i^2) = g^{'}(p_{i-1}^2)$, $g(p_1^1) = g(p_1^2) = 1$ and 0 otherwise. Then $w(g) = w(g^{'})$.

Case 3: When $n \equiv 2 \ (mod\ 3)$

We define g as follows: For $1 \leq i \leq n-1$, $g(p_i^1) = g(p_i^2) = 2$ if $i \equiv 2 \ (mod\ 3)$, $g(p_n^1) = 2$, $g(p_{n-1}^2) = 1$ and 0 otherwise. So $w(g) = \frac{4n+1}{3}$. We can define a γ_{isR} -function $g^{'}$ as follows: For $1 \leq i \leq n-1$, $g^{'}(p_i^1) = 2$ if $i \equiv 0, 1 \ (mod\ 6)$ and $g^{'}(p_i^2) = 2$ if $i \equiv 3, 4 \ (mod\ 6)$, $g^{'}(p_n^2) = 1$ and 0 otherwise. Then $w(g) = w(g^{'})$. □

The Figure 3 shows the γ_{sR} -function and γ_{isR} -function of $(P_2 \Box P_n)$ for $n = 6, 7$ and 8.

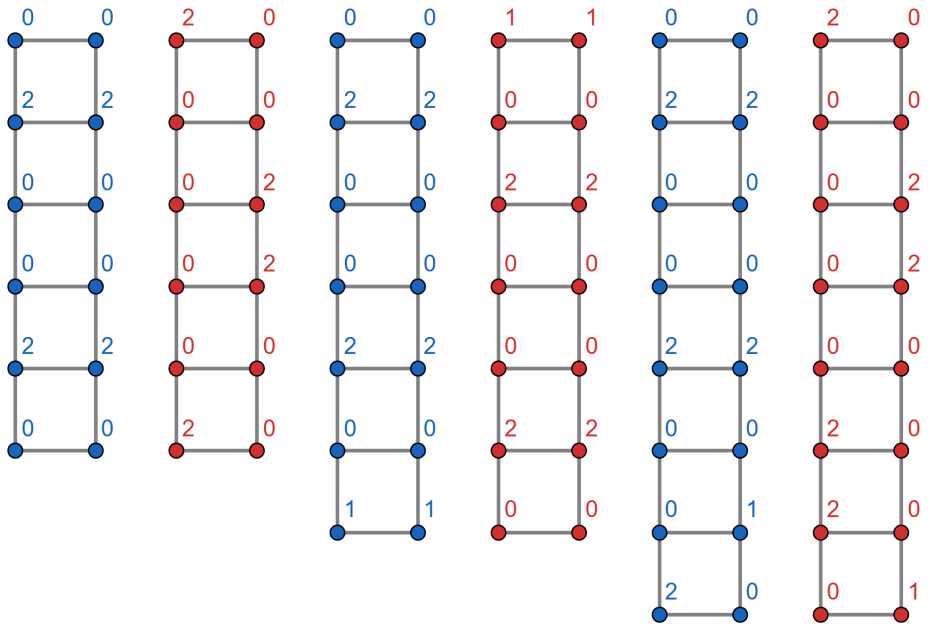


Figure 3. Illustration of γ_{sR} -function and γ_{isR} -function of $(P_2 \square P_n)$ where $n = 6, 7, 8$.

Theorem 4.3. *If* $G = K_1 + G_1$, *then*

$$\gamma_{isR}(G) = \gamma_{sR}(G_1)$$

for any graph G_1.

Proof. Let r be the universal vertex in G. The γ_{sR} -function h is defined as follows: $h(r) = 1 + \lceil \frac{n-1}{2} \rceil$ and it assigns 0 to the rest of the vertices. The only possible ISRDF h' assigns $h'(r) = 0$, which proves that $\gamma_{isR}(G) = \gamma_{sR}(G_1)$. □

5 Trees

In this section we will find a lower bound for the $\gamma_{isR}(G)$ of trees in terms of the order of trees and the number of leaves in it. We also show that this bound is sharp.

Theorem 5.1. *For any tree* T *of order* $n > 2$ *and* l *leaves,*

$$\gamma_{isR}(T) \geq \left\lceil \frac{2}{3}\left(n + \frac{5}{2} - l\right)\right\rceil$$

and the bound is sharp for paths.

Proof. We will prove this statement using mathematical induction on the order n. The statement holds true for all the trees of order $n \leq 4$. Hence, we consider only the trees with $n \geq 5$. Assume the result is true for all trees of order less than n. Let T be a tree of order n.

If $diam(T) = 2$, then T is a star and by Proposition 3.5, we have $\gamma_{isR}(T) = n$ and $n \geq 5$. Since $n - l = 1$ we get, $\lceil \frac{2}{3}\left(n + \frac{5}{2} - l\right)\rceil = \lceil \frac{2}{3}\left(\frac{5}{2} + 1\right)\rceil = \lceil \frac{7}{3} \rceil = 3$. If $diam(T) = 3$, then T is a double star. By Proposition 3.6, $\gamma_{isR}(T) = n$ or $n - 1$ and $n - 1 \geq 4$. Since $n - l = 2$ we get, $\frac{2}{3}\left(n + \frac{5}{2} - l\right) = \frac{2}{3}\left(\frac{5}{2} + 2\right) = 3$. So we prove the theorem for $diam(T) \geq 4$. Let $r_1, r_2, r_3 ... r_k$ be a diametrical path of T and we root the tree at r_k. The notation $d_T(r)$ denotes the degree of the vertex r in T. Let $d_T(r_2) = s$. Let h be a γ_{isR} -function on T.

Case 1: $d_T(r_2) = s \geq 3$

We remove the vertex r_2 and its descendants from T. Let $T' = T - T_{r_2}$. Let n' and l' be the order of T' and the number of leaves in T' respectively. Then $n' = n - s$. If $d_T(r_3) = 2$, then $l' = l - (s-2)$ and if $d_T(r_3) \geq 3$, then $l' = l - (s-1)$. Hence, $l' \leq l - (s-2)$. Any γ_{isR} -function will assign some positive value to r_2 as it's degree is greater than or equal to 3. Hence any γ_{sR} -function on T must assign 0 to r_2 and 2 to any one leaf and 1 to the remaining $s-3$ leaves of r_2. Hence we get,

$$\begin{aligned}
\gamma_{isR}(T) &\geq \gamma_{isR}(T') + (s-1) \\
&\geq \left\lceil \frac{2}{3}(n' + \frac{5}{2} - l') \right\rceil + (s-1) \\
&\geq \left\lceil \frac{2}{3}(n - s + \frac{5}{2} - l + s - 2) \right\rceil + \lceil (s-1) \rceil \\
&= \left\lceil \frac{2}{3}(n + \frac{5}{2} - l) \right\rceil + \left\lceil (s - \frac{7}{3}) \right\rceil \\
&> \left\lceil \frac{2}{3}(n + \frac{5}{2} - l) \right\rceil
\end{aligned}$$

Case 2: $d_T(r_2) = 2$
If $d_T(r_3) \geq 3$, then $T' = T - \{r_1, r_2\}$. In this case,

$$\begin{aligned}
\gamma_{isR}(T) &\geq \gamma_{isR}(T') + 2 \\
&\geq \left\lceil \frac{2}{3}(n' + \frac{5}{2} - l') \right\rceil + 2 \\
&= \left\lceil \frac{2}{3}(n - 2 + \frac{5}{2} - l + 1) \right\rceil + 2 \\
&= \left\lceil \frac{2}{3}(n + \frac{5}{2} - l) \right\rceil + \left\lceil \frac{4}{3} \right\rceil \\
&> \left\lceil \frac{2}{3}(n + \frac{5}{2} - l) \right\rceil
\end{aligned}$$

If $d_T(r_3) = 2$, then let $T' = T - \{r_1, r_2, r_3\}$. In this case,

$$\begin{aligned}
\gamma_{isR}(T) &\geq \left\lceil \gamma_{isR}(T') \right\rceil + 2 \\
&\geq \left\lceil \frac{2}{3}(n' + \frac{5}{2} - l') \right\rceil + 2 \\
&= \left\lceil \frac{2}{3}(n - 3 + \frac{5}{2} - l) \right\rceil + 2 \\
&= \left\lceil \frac{2}{3}(n + \frac{5}{2} - l) \right\rceil
\end{aligned}$$

This completes the proof of inequality. By Proposition 3.3, for paths P_n, $\gamma_{isR}(P_n) = \lceil \frac{2n+1}{3} \rceil = \lceil \frac{2}{3}(n + \frac{5}{2} - 2) \rceil$. □

6 Conclusion

In this paper, we introduced the inverse strong Roman domination in graphs. The inverse strong Roman domination number of paths, cycles, stars, double stars, complete bipartite graphs and the Cartesian product of paths is discussed. We also got a bound for the $\gamma_{isR}(T)$ of a tree T in terms of the order of T and the number of leaves of T. Inspired by the Kulli-Sigarkanti conjecture [6], we strongly propose the following conjecture.

Conjecture 6.1. For any graph G with no isolated vertices,

$$\gamma_{isR}(G) \leq 2\beta_0(G)$$

where $\beta_0(G)$ is the vertex independence number.

We propose a few open problems based on our research.

1) Characterise the graphs with $\gamma_{sR}(G) = \gamma_{isR}(G)$.
2) Characterise the graphs with $\gamma_{sR}(G) + \gamma_{isR}(G) = 2n$.
3) Characterise the graphs with $\gamma_{sR}(G) + \gamma_{isR}(G) = k$ for any $k < 2n$.
4) Explore the Nordhaus Gaddum inequalities for $\gamma_{isR}(G)$.
5) Develop algorithms to find the $\gamma_{isR}(G)$ of any graph or specific graph families.
6) Find bounds for $\gamma_{isR}(G)$ in terms of different graph parameters for arbitrary graphs or some classes of graphs.
7) Explore the relation between $\gamma_{iR}(G)$ and $\gamma_{isR}(G)$ of graphs.
8) Explore the relation between $\gamma_{isR}(G)$ and $\gamma_{idR}(G)$ of graphs.
9) Characterise the graphs having a particular value of $\gamma_{isR}(G)$.
10) Explore the computational complexity of $\gamma_{isR}(G)$ in graphs.

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