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RECONSIDERING LOCALIZATION WITH NEUTROSOPHIC APPROACH

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Abstract In recent years, many algebraic structures have been re-examined through the neutrosophic approach, leading to new mathematical concepts.. Motivated by this, we aimed to bring a new dimension to localization, which has applications, particularly in geometry, through the neutrosophic approach. In this study, some fundamental properties of the localization of the real neutrosophic ring $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ are presented based on the increasingly popular neutrosophic approach. Additionally, the neutrosophic ideals of $\mathscr{R}$ are examined.

1 Introduction

Neutrosophic numbers, a concept with roots in neutrosophy, were first introduced by Florentin Smarandache in the early 21st century, [1]. Neutrosophic numbers expand classical number systems by incorporating ambiguity. They allow for a more comprehensive representation of uncertain data by combining degrees of accuracy, uncertainty and inaccuracy. This is especially useful when dealing with complex and uncertain data in fields such as artificial intelligence, decision-making, and quantum mechanics. Neutrosophic numbers have facilitated advances in these fields by providing a mathematical framework that can handle incomplete, inconsistent, and ambiguous information. Additionally, they are used to develop new algorithms for image processing and to increase the accuracy of medical diagnoses. Some studies conducted in medicine, artificial intelligence, data analysis, applied mathematics and psychology using the neutrosophic approach are as follows: [2, 3, 4, 5, 6, 7, 8, 9].

Definition 1.1. [10], Let $(R, +, .)$ be any ring. The set

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I under the operations of R.

Definition 1.2. [10], Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is said to be commutative if $\forall x, y \in \langle R \cup I \rangle$, $xy = yx$.

If in addition there exists $1 \in \langle R \cup I \rangle$ such that $1r = r1 = r$ for all $r \in \langle R \cup I \rangle$ then we call $\langle R \cup I \rangle$ a commutative neutrosophic ring with unity.

Definition 1.3. [10], Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring of $\langle R \cup I \rangle$ if $P = \langle S \cup nI \rangle$ where S is a subring of R and n an integer. P is said to be generated by S and nI under the operations of R.

Definition 1.4. [10], Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$ which is just a ring. Then P is called a subring.

Definition 1.5. [10], Let $\langle R \cup I \rangle$ be any neutrosophic ring. A non-empty subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic ideal of $\langle R \cup I \rangle$ if the following conditions hold:

i. P is a neutrosophic subring of $\langle R \cup I \rangle$,

ii. for every $p \in P$ and $r \in \langle R \cup I \rangle$, $rp \in P$ and $pr \in P$.

If only $rp \in P$, we call P a left neutrosophic ideal and if only $pr \in P$, we call P a right neutrosophic ideal. When $\langle R \cup I \rangle$ is commutative, there is no distinction between rp and pr and therefore P is called a left and right neutrosophic ideal or simply a neutrosophic ideal.

Definition 1.6. [10], An element x in a neutrosophic ring $\langle R \cup I \rangle$ is called a left zero divisor if there exists a nonzero element $y \in \langle R \cup I \rangle$ such that $xy = 0$.

A right zero divisor can be defined similarly. If an element $x \in \langle R \cup I \rangle$ is both a left and a right zero divisor, it is then called a zero divisor.

Definition 1.7. [10], Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is called a neutrosophic integral domain if $\langle R \cup I \rangle$ is commutative with no zero divisors.

Definition 1.8. [10], Let $\langle R \cup I \rangle$ be a neutrosophic ring and P be a neutrosophic ideal of $\langle R \cup I \rangle$. The set

$$\langle R \cup I \rangle / P = \{r + P : r \in \langle R \cup I \rangle\}$$

is called the neutrosophic quotient ring provided that $\langle R \cup I \rangle / P$ is a neutrosophic ring.

Definition 1.9. [10], Let $\langle R \cup I \rangle$ and $\langle S \cup I \rangle$ be any two neutrosophic rings. The mapping

$$\phi : \langle R \cup I \rangle \to \langle S \cup I \rangle$$

is called a neutrosophic ring homomorphism if the following conditions hold:

i. ϕ is a ring homomorphism,

ii. $\phi(I) = I$.

The set $\{x \in \langle R \cup I \rangle : \phi(x) = 0\}$ is called the kernel of ϕ and is denoted by $Ker\phi$.

Definition 1.10. [10], Let $\langle R \cup I \rangle$ be a commutative ring with unity and x be an indeterminate in $\langle R \cup I \rangle [x]$. By the neutrosophic polynomial ring in x denoted by $\langle R \cup I \rangle [x]$ we mean the set of all symbols $\sum_{i=1}^{n} a_i x_i$, where n can be any nonnegative integer and the coefficients a_i, $i = n, n-1, ...2, 1, 0$ are all in $\langle R \cup I \rangle$.

Example 1.11. [1], Let $\mathbb{R}$ be the field of real numbers. $\langle \mathbb{R} \cup I \rangle$ be the neutrosophic ring of real numbers, $\langle \mathbb{R} \cup I \rangle [x]$ is the neutrosophic polynomial ring over the neutrosophic ring of real numbers.

Localization and local rings are used not only in algebra and number theory but also, particularly, in geometry. Some mathematicians have utilized local rings to define modules, while others have examined the geometric properties of local rings and defined certain projective planes using them. Some of the studies conducted in these fields are as follows: [11, 12, 13, 14, 15, 16, 17, 18, 19].

Definition 1.12. [20], Let R be a commutative ring with identity and P a prime ideal of R. Then, $S = R - P$ is a multiplicative subset of R. The ring of quotients $S^{-1}R$ is called localization of R at P and is denoted R_P.

Definition 1.13. [20] A local ring is a commutative ring with an identity that has a unique maximal ideal.

For more information on localization, please refer to [21, 20].

The neutrosophic approach introduced by Smarandache has provided effective solutions to challenges in various scientific disciplines and has broadened the perspective of these fields. As research on this approach progresses, it is likely to pave the way for new discoveries and technological innovations. Motivated by this, we aim to bring a new dimension to localization and local rings, particularly in the field of geometry, through a neutrosophic approach.

2 Neutrosophic Localization

In this section, we provided the fundamental definitions, theorems and properties related to neutrosophic localization. Then, we defined the neutrosophic local ring and provided a few examples of neutrosophic local rings.

Definition 2.1. Let $\langle \mathbb{R} \cup I \rangle$ be a neutrosophic ring and $\emptyset \neq S \subset \langle \mathbb{R} \cup I \rangle$. If $(a+cI).(b+dI) \in S$ for all $a+cI, b+dI \in S$, then S is called n-th multiplicative subset of $\langle \mathbb{R} \cup I \rangle$.

Example 2.2. Let $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ be a neutrosophic ring without zero divisors. The set $\langle \mathbb{R} \cup I \rangle \setminus \{0\}$ is a multiplicative subset.

Example 2.3. Let $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ be a unitary neutrosophic ring and S be the set of unitaries of $\mathscr{R}$. S is a multiplicative subset of $\mathscr{R}$.

Example 2.4. If $\mathscr{R}$ is a commutative neutrosophic ring and P is a neutrosophic prime ideal of $\mathscr{R}$, then the sets P and $S = \mathscr{R} - P$ are multiplicative subsets of $\mathscr{R}$.

Theorem 2.5. *Let $\mathscr{R}$ be a commutative neutrosophic ring and S be a multiplicative subset of $\mathscr{R}$. For some $t \in S$,*

$$(r_1 + r_2 I, s_1 + s_2 I) \sim \left(r_1^{'} + r_2^{'} I, s_1^{'} + s_2^{'} I\right) \Longleftrightarrow t\left(rs^{'} - r^{'}s\right) = 0$$

is an equivalence relation defined on $\mathscr{R} \times S$, where $r = r_1 + r_2 I$, $s = s_1 + s_2 I$, $r^{'} = r_1^{'} + r_2^{'} I$ and $s^{'} = s_1^{'} + s_2^{'} I$.

Furthermore if $\mathscr{R}$ has no zero divisor and $0_{\mathscr{R}} \notin S$, then

$$(r_1 + r_2 I, s_1 + s_2 I) \sim \left(r_1^{'} + r_2^{'} I, s_1^{'} + s_2^{'} I\right) \Longleftrightarrow \left(rs^{'} - r^{'}s\right) = 0$$

is an equivalence relation on $\mathscr{R} \times S$.

Proof. i. Reflexive: Since $\mathscr{R}$ is commutative, for all $(r,s) \in \mathscr{R} \times S$ there exists $t \in S$ such that

$$(r_1 + r_2 I, s_1 + s_2 I) \sim (r_1 + r_2 I, s_1 + s_2 I) \Longleftrightarrow t(rs - sr) = 0.$$

ii. Symmetric: For all $(r,s), \left(r^{'}, s^{'}\right) \in \mathscr{R} \times S$ there exists $t \in S$ such that

$$\begin{aligned} (r,s) \sim \left(r^{'}, s^{'}\right) &\Longleftrightarrow t\left(rs^{'} - r^{'}s\right) = 0 \\ &\Longleftrightarrow t\left(r^{'}s - rs^{'}\right) = 0 \\ &\Longleftrightarrow \left(r^{'}, s^{'}\right) \sim (r,s). \end{aligned}$$

iii. Transitive: For all $(r,s), \left(r^{'}, s^{'}\right), \left(r^{''}, s^{''}\right) \in \mathscr{R} \times S$ there exists $t, p \in S$ such that

$$\begin{aligned} &(r,s) \sim \left(r^{'}, s^{'}\right) \wedge \left(r^{'}, s^{'}\right) \sim \left(r^{''}, s^{''}\right) \\ &\Longleftrightarrow \; t\left(rs^{'} - r^{'}s\right) = 0 \wedge p\left(r^{'}s^{''} - r^{''}s^{'}\right) = 0 \\ &\Longleftrightarrow \; ps^{''}t\left(rs^{'} - r^{'}s\right) = 0 \wedge tsp\left(r^{'}s^{''} - r^{''}s^{'}\right) = 0 \\ &\Longleftrightarrow \; ps^{''}t\left(rs^{'} - r^{'}s\right) + tsp\left(r^{'}s^{''} - r^{''}s^{'}\right) = 0 \\ &\Longleftrightarrow \; pts^{'}\left(s^{''}r - sr^{''}\right) = 0 \\ &\Longleftrightarrow \; pts^{'}\left(s^{''}r - r^{''}s\right) = 0 \\ &\Longleftrightarrow \; (r,s) \sim \left(r^{''}, s^{''}\right). \end{aligned}$$

If $\mathscr{R}$ has no zero divisor and $0_{\mathscr{R}} \notin S$, then we obtained

$$\begin{aligned}(r,s) \sim \left(r^{'}, s^{'}\right) &\iff t\left(rs^{'} - r^{'}s\right) = 0 \\ &\iff \left(rs^{'} - r^{'}s\right) = 0.\end{aligned}$$

□

Remark 2.6. The equivalence class of $(r,s) \in \mathscr{R} \times S$ will be denoted $\frac{r}{s} = \frac{r_1 + r_2 I}{s_1 + s_2 I}$.

Definition 2.7. The set of all equivalence classes of $\mathscr{R} \times S$ under $\sim$ denoted by $S^{-1}\mathscr{R}$. It is defined as

$$S^{-1}\mathscr{R} = \left\{\frac{r}{s} : r \in \mathscr{R}, s \in S\right\}$$

and has the following properties:

i. For some $t \in S$ then

$$\frac{r}{s} = \frac{r^{'}}{s^{'}} \Longleftrightarrow t\left(rs^{'} - r^{'}s\right) = 0\,;$$

ii. For all $r \in \mathscr{R}$ and $s, t \in S$ then

$$\frac{tr}{ts} = \frac{r}{s}\,;$$

iii. If $0_{\mathscr{R}} \in S$ then $S^{-1}\mathscr{R}$ consist of a single equivalence class.

Proof. i. For $\frac{r}{s}, \frac{r^{'}}{s^{'}} \in S^{-1}\mathscr{R}$ there exists $t \in S$ such that

$$\frac{r}{s} = \frac{r^{'}}{s^{'}} \Rightarrow (r,s) \sim \left(r^{'}, s^{'}\right) \Rightarrow t\left(rs^{'} - r^{'}s\right) = 0\,.$$

Conversely,

$$t\left(rs^{'} - r^{'}s\right) = 0 \Rightarrow (r,s) \sim \left(r^{'}, s^{'}\right) \Rightarrow \frac{r}{s} = \frac{r^{'}}{s^{'}}\,.$$

ii.

$$\begin{aligned}\frac{tr}{ts} = \frac{r}{s} &\iff (tr, ts) \sim (r,s) \\ &\iff p\left((tr)s - r(ts)\right) = 0 \\ &\iff pt\left(rs - rs\right) = 0 \\ &\iff (r,s) \sim (r,s)\,.\end{aligned}$$

iii. If $0_{\mathscr{R}} \in S$ then

$$0\left(rs^{'} - r^{'}s\right) = 0 \Rightarrow (r,s) \sim \left(r^{'}, s^{'}\right),$$

for all $(r,s) \sim \left(r^{'}, s^{'}\right) \in \mathscr{R} \times S$. Then $\frac{0}{0}$ is a single equivalence class.

□

Theorem 2.8. *Let S be a multiplicative subset of the commutative neutrosophic ring $\mathscr{R}$. Let $S^{-1}\mathscr{R}$ be the set of equivalence classes of $\mathscr{R} \times S$ under the equivalence relation defined in Theorem 2.5.*

i. *$S^{-1}\mathscr{R}$ is a commutative neutrosophic ring with identity. For $\frac{r}{s}, \frac{r^{'}}{s^{'}} \in S^{-1}\mathscr{R}$ addition and multiplication operations are defined by*

$$\begin{aligned}\frac{r}{s}+\frac{r^{'}}{s^{'}} &= \frac{rs^{'}+r^{'}s}{ss^{'}} \\ &= \frac{\left(r_1s_1^{'}+r_1^{'}s_1\right)+\left(r_1s_2^{'}+r_2s_1^{'}+r_2s_2^{'}+r_1^{'}s_2+r_2^{'}s_1+r_2^{'}s_2\right)I}{s_1s_1^{'}+\left(s_1s_2^{'}+s_2s_1^{'}+s_2s_2^{'}\right)I}\end{aligned}$$

and

$$\frac{r}{s}.\frac{r^{'}}{s^{'}}=\frac{rr^{'}}{ss^{'}}=\frac{r_1r_1^{'}+\left(r_1r_2^{'}+r_2r_1^{'}+r_2r_2^{'}\right)I}{s_1s_1^{'}+\left(s_1s_2^{'}+s_2s_1^{'}+s_2s_2^{'}\right)I}.$$

ii. *If $\mathscr{R}$ is a nonzero neutrosophic ring with no zero divisors and $0_{\mathscr{R}} \notin S$ then $S^{-1}\mathscr{R}$ is a neutrosophic integral domain.*

iii. *If $\mathscr{R}$ is a nonzero neutrosophic ring with no zero divisors and S is the set of all nonzero elements of $\mathscr{R}$, then $S^{-1}\mathscr{R}$ is a neutrosophic field.*

Proof. i. Let $\frac{r}{s}=\frac{r_1}{s_1}$ and $\frac{r^{'}}{s^{'}}=\frac{r_1^{'}}{s_1^{'}}$. There exists $t,p \in S$ such that $t\left(rs_1-r_1s\right)=0$ and $p\left(r^{'}s_1^{'}+r_1^{'}s^{'}\right)=0$. Then,

$$tps_1^{'}s^{'}\left(rs_1-r_1s\right)=0 \wedge pts_1s\left(r^{'}s_1^{'}-r_1^{'}s^{'}\right)=0$$

$$\begin{aligned}&\Rightarrow \quad tps_1^{'}s^{'}rs_1-tps_1^{'}s^{'}r_1s+pts_1sr^{'}s_1^{'}-pts_1sr_1^{'}s^{'}=0\\ &\Rightarrow \quad tp(s_1^{'}s^{'}rs_1-s_1^{'}s^{'}r_1s+s_1sr^{'}s_1^{'}-s_1sr_1^{'}s^{'})=0\\ &\Rightarrow \quad k\left[s_1s_1^{'}\left(rs^{'}+r^{'}s\right)-ss^{'}\left(r_1s_1^{'}+r_1^{'}s_1\right)\right]=0\\ &\Rightarrow \quad \frac{rs^{'}+r^{'}s}{ss^{'}}=\frac{r_1s_1^{'}+r_1^{'}s_1}{s_1s_1^{'}}.\end{aligned}$$

So, the addition operation is well-defined. Similarly, the multiplication operation is well-defined.

□

Definition 2.9. The neutrosophic ring $S^{-1}\mathscr{R}$ is called the neutrosophic quotient ring of the neutrosophic ring $\mathscr{R}$ with respect to S. If $\mathscr{R}$ is a neutrosophic integral domain and S is the set of all nonzero elements of $\mathscr{R}$, then $S^{-1}\mathscr{R}$ is a neutrosophic field. This is called the neutrosophic fraction field of $\mathscr{R}$. More generally, let $\mathscr{R} \neq \{0\}$ be a commutative neutrosophic ring and S be the subset of $\mathscr{R}$ consisting of all nonzero and non-zero divisor elements. In this case, $S^{-1}\mathscr{R}$ is called the neutrosophic total ring of fractions of neutrosophic ring $\mathscr{R}$.

Definition 2.10. Based on the operations defined above, $\left(S^{-1}\mathscr{R},+,.\right)$ is a neutrosophic ring. This neutrosophic ring is called the neutrosophic localization of $\mathscr{R}$ with respect to S. The zero element of this neutrosophic ring is $0_{S^{-1}\mathscr{R}}=\frac{0_{\mathscr{R}}}{1+0I}$ and its unity is $1_{S^{-1}\mathscr{R}}=\frac{1+0I}{1+0I}$.

Indeed, for all $\frac{a_1+a_2I}{s_1+s_2I} \in S^{-1}\mathscr{R}$, it is clear that

$$\begin{aligned}\frac{a_1+a_2I}{s_1+s_2I}+\frac{0+0I}{1+0I} &= \frac{a_1+a_2I+0I+0+0I}{s_1+s_2I+0I}\\ &= \frac{a_1+a_2I}{s_1+s_2I}\end{aligned}$$

and

$$\frac{a_1+a_2I}{s_1+s_2I}.\frac{1+0I}{1+0I}=\frac{a_1+a_2I}{s_1+s_2I}.$$

So,

$$0_{S^{-1}\mathscr{R}} = \frac{0+0I}{1+0I}, 1_{S^{-1}\mathscr{R}} = \frac{1+0I}{1+0I}.$$

Corollary 2.11. *If $0_{\mathscr{R}} \in S$, then every pair in $\mathscr{R} \times S$ is equivalent, $\frac{0_{\mathscr{R}}}{1+0I} = 0_{S^{-1}\mathscr{R}}$ and $S^{-1}\mathscr{R} = \{0\}$.*

Proposition 2.12. *The function π, defined by*

$$\begin{aligned} \pi &: R \to S^{-1}\mathscr{R} \\ r &= r_1 + r_2 I \to \pi(r) = \frac{r_1 + r_2 I}{1+0I} \end{aligned}$$

is a neutrosophic ring homomorphism. π is called the neutrosophic natural homomorphism.

Theorem 2.13. *Let S be a multiplicative subset of a commutative neutrosophic ring $\mathscr{R}$.*

i. For any $s \in S$, the map φ_s defined as

$$\begin{aligned} \varphi_s : \mathscr{R} &\to S^{-1}\mathscr{R} \\ a = a_1 + a_2 I &\to \varphi(a_1 + a_2 I) = \frac{as}{s} \end{aligned}$$

is well-defined and it's a neutrosophic homomorphism. For every $s \in S$, $\varphi_s(s) \in S^{-1}\mathscr{R}$ is a unit.

ii. If $0_{\mathscr{R}} \notin S$ and S contains no zero divisor, then φ_s is a neutrosophic monomorphism. In particular, any integral domain may be embedded in its neutrosophic quotient field.

iii. If $\mathscr{R}$ is a neutrosophic ring with unit and S contains of units, then φ_s is a neutrosophic isomorphism.

Proof. i. For $s, s' \in S$, we can write $\frac{rs}{s} = \frac{rs'}{s'}$. Since the definition is independent of $s \in S$, φ_s is well-defined. For all $r, r' \in \mathscr{R}$,

$$\begin{aligned} \varphi_s\left(r + r'\right) &= \frac{\left(r + r'\right)s}{s} \\ &= \frac{\left(r + r'\right)s.s}{s.s} \\ &= \frac{rs^2 + r's^2}{s^2} \\ &= \frac{rs}{s} + \frac{r's}{s} \\ &= \varphi_s(r) + \varphi_s\left(r'\right) \end{aligned}$$

and

$$\varphi_s\left(r.r'\right) = \frac{\left(r.r'\right)s}{s} = \frac{\left(r.r'\right)s.s}{s.s} = \frac{rs}{s}.\frac{r's}{s} = \varphi_s(r).\varphi_s\left(r'\right).$$

So, φ_s is a homomorphism.

Let $s \in S$. For $\varphi_s(s) = \frac{s^2}{s}$ and $\frac{s}{s^2} \in S^{-1}\mathscr{R}$, we have $\frac{s^2}{s}.\frac{s}{s^2} = 1_{S^{-1}\mathscr{R}}$. So, the multiplicative inverse of the $\frac{s^2}{s}$ is $\frac{s}{s^2}$.

ii. For $r \in \mathscr{R}$

$$\begin{aligned} \varphi_s(r) = \frac{rs}{s} = 0_{S^{-1}\mathscr{R}} &\Rightarrow \frac{rs}{s} = \frac{0_{S^{-1}\mathscr{R}}}{s} \\ &\Rightarrow rs^2 s_1 = 0_{S^{-1}\mathscr{R}} \text{ , for some } s_1 \in S \\ &\Rightarrow r = 0_{S^{-1}\mathscr{R}} . \end{aligned}$$

So, φ_s is one-to-one.

Let $\mathscr{R}$ be the neutrosophic integral domain and $S = \mathscr{R} \setminus \{0\}$.

$$\begin{aligned} \varphi_s : \mathscr{R} &\rightarrow S^{-1}\mathscr{R} \\ r = r_1 + r_2 I &\rightarrow \varphi_s(r) = \frac{rs}{s} \end{aligned}$$

is a monomorphism. Therefore φ_s is embeddable.

iii. Since every element of S is a unit, each element has an inverse. Since $0_{\mathscr{R}} \notin S$ and there are no zero divisors in S, φ_s is a monomorphism. Given that

$$\frac{r}{s} \in S^{-1}\mathscr{R}, \frac{r}{s} = \varphi_s\left(rs^{-1}\right) = \frac{rs^{-1}s}{s},$$

it follows that φ_s is surjective. So, $\mathscr{R} \cong S^{-1}\mathscr{R}$.

□

The following theorem shows that neutrosophic quotient rings may be completely characterized by a universal mapping property, for $\mathscr{R} = \mathbb{R}$. This theorem will sometimes be used to define the neutrosophic quotient rings.

Theorem 2.14. *Let $\mathscr{R}$ be a commutative neutrosophic ring, S be a multiplicative subset of $\mathscr{R}$ and $\mathcal{T} = \langle T \cup I \rangle$ be a commutative neutrosophic ring with the unit. If $f : \mathscr{R} \rightarrow \mathcal{T}$ is a neutrosophic homomorphism and $f(s)$ is a unit in $\mathcal{T}$ for all $s \in S$, then there exists a unique neutrosophic homomorphism $\overline{f} : S^{-1}\mathscr{R} \rightarrow \mathcal{T}$ such that $\overline{f}\varphi_s = f$. The neutrosophic ring $S^{-1}\mathscr{R}$ is completely determined up to neutrosophic isomorphism with respect to this property.*

Proof.

$$\begin{array}{ccc} \mathscr{R} & \underset{\rightarrow}{f} & \mathcal{T} \\ \varphi_s \searrow & & \nearrow \overline{f} \\ & S^{-1}\mathscr{R} & \end{array}$$

Let $\frac{r}{s} \in S^{-1}\mathscr{R}$ and $\overline{f}\left(\frac{r}{s}\right) = \frac{f(r)}{f(s)}$. Since there exists $f(s)^{-1}$, $\overline{f}$ is well-defined. For $\frac{r}{s}, \frac{r'}{s'} \in S^{-1}\mathscr{R}$,

$$\begin{aligned} \overline{f}\left(\frac{r}{s} + \frac{r'}{s'}\right) &= \overline{f}\left(\frac{rs' + r's}{ss'}\right) \\ &= \frac{f\left(rs' + r's\right)}{f(ss')} \\ &= \frac{f(r) f\left(s'\right) + f\left(r'\right) + f(s)}{f(s) f(s')} \\ &= \left[f(r) f\left(s'\right) + f\left(r'\right) f(s)\right] f(s)^{-1} f\left(s'\right)^{-1} \\ &= f(r) f\left(s'\right) f(s)^{-1} f\left(s'\right)^{-1} + f\left(r'\right) f(s) f(s)^{-1} f\left(s'\right)^{-1} \end{aligned}$$

$$\begin{aligned} &= \frac{f(r)}{f(s)} + \frac{f\left(r^{'}\right)}{f(s^{'})} \\ &= \overline{f}\left(\frac{r}{s}\right) + \overline{f}\left(\frac{r^{'}}{s^{'}}\right) \end{aligned}$$

and

$$\overline{f}\left(\frac{r}{s}.\frac{r^{'}}{s^{'}}\right) = \overline{f}\left(\frac{rr'}{ss^{'}}\right) = \frac{f(rr')}{f(ss^{'})} = \frac{f(r)\,f\left(r^{'}\right)}{f(s)\,f(s^{'})} = \overline{f}\left(\frac{r}{s}\right).\overline{f}\left(\frac{r^{'}}{s'}\right).$$

So, $\overline{f}$ is a neutrosophic ring homomorphism.

If $r \in \mathscr{R}$, then

$$\left(\overline{f}\varphi_s\right)(r) = \overline{f}\left(\varphi_s(r)\right) = \overline{f}\left(\frac{rs}{s}\right) = \frac{f(rs)}{f(s)} = \frac{f(r)\,f(s)}{f(s)} = f(r).$$

Therefore, $\overline{f}\varphi_s = f$ is obtained.

Let $g : S^{-1}\mathscr{R} \to \mathcal{T}$ be a neutrosophic homomorphism. For $\frac{r}{s} \in S^{-1}\mathscr{R}$, since $g\varphi_s = f$, it follows that $\left(g\varphi_s\right)(r) = g\left(\frac{rs}{s}\right) = f(r)$. From this,

$$g\left(\frac{r}{s}.\frac{s^{2}}{s}\right) = g\left(\frac{r}{s}\right).g\left(\frac{s^{2}}{s}\right) = g\left(\frac{r}{s}\right).f(s) = f(r).$$

So, $g\left(\frac{r}{s}\right) = f(r).f(s)^{-1} = \overline{f}\left(\frac{r}{s}\right)$ is obtained. Therefore, $g = \overline{f}$. □

Theorem 2.15. *Let S be a multiplicative subset of the commutative neutrosophic ring $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$.*

i. If K is the neutrosophic ideal of $\mathscr{R}$, then $S^{-1}K = \left\{\frac{k}{s} : k \in K, s \in S\right\}$ is the neutrosophic ideal of $S^{-1}\mathscr{R}$.

ii. If L is another neutrosophic ideal of $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$, then

$$\begin{aligned} S^{-1}(K+L) &= S^{-1}K + S^{-1}L\,; \\ S^{-1}(KL) &= \left(S^{-1}K\right)\left(S^{-1}L\right)\,; \\ S^{-1}(K \cap L) &= S^{-1}K \cap S^{-1}L\,. \end{aligned}$$

Proof. i. $\emptyset \neq K \subset \mathscr{R}$, $S^{-1}K \subset S^{-1}\mathscr{R}$. Since $\frac{0}{s} \in S^{-1}K$, $S^{-1}K \neq \emptyset$.

$S^{-1}K = \left\{\frac{a}{s} : a \in K, s \in S\right\}$.

$$\begin{aligned} \text{For any } \frac{a}{s}, \frac{b}{s^{'}} \in S^{-1}K \quad &\Rightarrow \quad \frac{a}{s} + \left(\frac{b}{s^{'}}\right) = \frac{as^{'} - bs}{ss^{'}} \in S^{-1}K \\ &\Rightarrow \quad as^{'} - bs \in K, ss^{'} \in S. \end{aligned}$$

Let any $\frac{r}{s^{'}} \in S^{-1}\mathscr{R}$ and $\forall \frac{a}{s} \in S^{-1}K$.

$$\frac{r}{s^{'}}.\frac{a}{s} = \frac{ra}{s^{'}s} \in S^{-1}K, \left(\frac{a}{s}\right)\left(\frac{r}{s^{'}}\right) = \frac{ar}{ss^{'}} \in S^{-1}K\,.$$

Therefore, $S^{-1}K$ is the neutrosophic ideal of $S^{-1}\mathscr{R}$.

ii. Let's show $S^{-1}(KL) = \left(S^{-1}K\right)\left(S^{-1}L\right)$. Let $\sum\left(\frac{a_ib_i}{s}\right) \in S^{-1}(KL)$, where $a_i \in K, b_i \in L$.

$$\begin{aligned} \sum\left(\frac{a_ib_i}{s}\right) &= \frac{a_1b_1 + ... + a_nb_n}{s} \\ &= \frac{a_1}{s}.\frac{b_1s}{s} + ... + \frac{a_n}{s}.\frac{b_ns}{s} \\ &\Rightarrow \quad S^{-1}(KL) \subseteq \left(S^{-1}K\right)\left(S^{-1}L\right). \end{aligned}$$

Let's take $a \in \left(S^{-1}K\right)\left(S^{-1}L\right)$. $a = \sum (x_i y_i)$, where $x_i \in S^{-1}K$, $y_i \in S^{-1}L$, $x_i = \frac{a_i}{s_i}$ and $y_i = \frac{b_i}{s_i}$.

$$a = \sum \frac{a_i}{s_i}\frac{b_i}{s_i} = \sum \left(\frac{a_i b_i}{s_i^2}\right) \in S^{-1}(KL)$$
$$\Rightarrow \left(S^{-1}K\right)\left(S^{-1}L\right) \subseteq S^{-1}(KL).$$

So, we can easily see that $S^{-1}(K.L) = \left(S^{-1}K\right)\left(S^{-1}L\right)$.
Other equalities can be shown similarly.

□

Theorem 2.16. *Let $\mathscr{R}$ be a neutrosophic ring with unit and S be a multiplicative subset of $\mathscr{R}$. Let K be a neutrosophic ideal of $\mathscr{R}$. Then*

$$S^{-1}K = S^{-1}\mathscr{R} \Leftrightarrow S \cap K \neq \emptyset.$$

Proof. ($\Leftarrow$:) Let $S \cap K \neq \emptyset$. $s \in S \cap K \Rightarrow s \in S \wedge s \in K$. $S^{-1}\mathscr{R}$ unitary and commutative neutrosophic ring. So, unit of $S^{-1}\mathscr{R}$

$$1_{S^{-1}\mathscr{R}} = \frac{s}{s} \in S^{-1}K.$$

Since the neutrosophic ideal includes the unit, $S^{-1}K = S^{-1}\mathscr{R}$ is found.
($\Rightarrow$:) Let $S^{-1}K = S^{-1}\mathscr{R}$.

$$\begin{array}{rcl} \varphi_s : \mathscr{R} & \rightarrow & S^{-1}\mathscr{R} = S^{-1}K \\ r & \rightarrow & \varphi_s(r) = \dfrac{rs}{s} \end{array}$$

and $\varphi_s^{-1}\left(S^{-1}K\right) = \mathscr{R}$. So, $\varphi_s(1_{\mathscr{R}}) = \frac{1_{\mathscr{R}}.s}{s}$ and $\varphi_s(1_{\mathscr{R}}) = \frac{a}{s}$, where $a \in K$, $s \in S$. Then there is $t \in S$ such that

$$\frac{1_{\mathscr{R}}s}{s} = \frac{a}{s} \Leftrightarrow t(1_{\mathscr{R}}.s.s - as) = 0.$$

From here

$$\begin{array}{rcl} ts^2 - tas = 0 & \Rightarrow & ts^2 = tas \\ & \Rightarrow & S \cap I \neq \emptyset \end{array}$$

is found.

If J is a neutrosophic ideal of $S^{-1}\mathscr{R}$, $\varphi_s^{-1}(J)$ is a neutrosophic ideal of $\mathscr{R}$. (The neutrosophic ideal $\varphi_s^{-1}(J)$ is called the neutrosophic contraction of J in $\mathscr{R}$.) □

Lemma 2.17. *Let $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ commutative neutrosophic ring with identity, S be a multiplicative subset of $\mathscr{R}$ and let K be neutrosophic ideal of $\mathscr{R}$. Then the following properties are true:*

i. $K \subset \varphi_s^{-1}\left(S^{-1}K\right)$.

ii. Let J is a neutrosohic ideal of a neutrosophic ring $S^{-1}\mathscr{R}$ and $K = \varphi_s^{-1}(J)$. Then $S^{-1}K = J$.

iii. Let P is a neutrosophic prime ideal of a neutrosophic ring $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ and $S \cap P = \emptyset$. Then $S^{-1}P$ is a neutrosophic prime ideal of $S^{-1}\mathscr{R}$ and $\varphi_s^{-1}\left(S^{-1}P\right) = P$.

Proof. i. $as \in K$, for $a \in K$ and $s \in S$. Since $\varphi_s(a) = \frac{as}{s} \in S^{-1}K$, $a \in \varphi_s^{-1}\left(S^{-1}K\right)$ is found. Therefore $K \subset \varphi_s^{-1}\left(S^{-1}K\right)$.

ii. Let $\frac{r}{s} \in J$. $\varphi_s(r) = \frac{rs}{s} = \frac{r}{s}.\frac{s^2}{s} \in J$ so $r \in \varphi_s^{-1}(J) = K$ is obtained. Since $\frac{r}{s} \in S^{-1}K$, $J \subseteq S^{-1}K$ is found. Let $\frac{r}{s} \in S^{-1}K$. When $K = \varphi_s^{-1}(J)$ then $r \in \varphi_s^{-1}(J)$, $\varphi_s(r) \in J$ is obtained. If $\frac{r}{s} = \frac{1_{\mathscr{R}}}{s}.\frac{rs}{s} = \frac{1_{\mathscr{R}}}{s}.\varphi_s(r) \in J$ is taken into account $S^{-1}K \subseteq J$ is found. Then $S^{-1}K = J$.

iii. From Theorem 2.16 it is seen that $S^{-1}P \neq S^{-1}\mathscr{R}$ and Theorem 2.15 (i) it is seen that $S^{-1}P$ is a neutrosophic ideal of $S^{-1}\mathscr{R}$. If $\frac{r}{s}.\frac{r_1}{s_1} \in S^{-1}P$, then $\frac{r}{s}.\frac{r_1}{s_1} = \frac{a}{t}$, where $a \in P$ and $t \in S$. From here

$$k\left(rr_1t - ass_1\right) = 0, \text{ for } \exists k \in S \Rightarrow krr_1t = kass_1$$

and since $S \cap P = \emptyset$, $rr_1 \in P$ is obtained. Since P is a neutrosophic ideal, $\frac{r}{s} \in S^{-1}P$ or $\frac{r_1}{s_1} \in S^{-1}P$. Therefore, $S^{-1}P$ is the neutrosophic ideal. Because of (i), let $P \subset \varphi_s^{-1}\left(S^{-1}P\right)$ and $r \in \varphi_s^{-1}\left(S^{-1}P\right)$. So, $\varphi_s(r) \in S^{-1}P$ and $\varphi_s(r) = \frac{rs}{s} = \frac{a}{t}$, where $a \in P$ and $s, t \in S$. Since $kstr = ksa$ for $\exists k \in S$ and $S \cap P = \emptyset$, we obtain $r \in P$. In this case $\varphi_s^{-1}\left(S^{-1}P\right) \subseteq P$. Thus $\varphi_s^{-1}\left(S^{-1}P\right) = P$.

□

Theorem 2.18. *Let $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ be a commutative and unitary neutrosophic ring and S be a multiplicative subset of $\mathscr{R}$. In this case there is a one-to-one correspondence between A and B given by $P \to S^{-1}P$, where A is the set of neutrosophic prime ideals of the neutrosophic ring $\mathscr{R}$ that are disjoint with S and B is the set of neutrosophic prime ideals of the neutrosophic ring $S^{-1}\mathscr{R}$.*

Proof. From Lemma 2.17 (iii), one-to-one mapping from A to B can be defined by the mappping $P \to S^{-1}P$. Let J be a neutrosophic prime ideal of $S^{-1}\mathscr{R}$ and $P = \varphi_s^{-1}(J).S^{-1}P = J$. From Lemma 2.17 (ii) $S^{-1}P = J$ is found, where $S \cap P = \emptyset$. If $S \cap P \neq \emptyset$ then $S^{-1}P = S^{-1}\mathscr{R}$ from Theorem 2.16, which contradicts J being a neutrosophic prime ideal.

Let $a.b \in P$ for $a, b \in \mathscr{R}$. Then $\varphi_s(ab) = \varphi_s(a)\varphi_s(b) \in J$. Since J is a neutrosophic prime in $S^{-1}\mathscr{R}$, either $\varphi_s(a) \in J$ or $\varphi_s(b) \in J$. Hence, its either $a \in \varphi_s^{-1}(J) = P$ or $b \in \varphi_s^{-1}(J) = P$. So, P is a neutrosophic prime ideal. Since $J = S^{-1}P$, the image of the neutrosophic ideal P is J and this mapping is surjective. □

Theorem 2.19. *Let $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$ be a commutative neutrosophic ring with the unit and P be a neutrosophic prime ideal of $\mathscr{R}$.*

i. There is a one-to-one correspondence between the neutrosophic prime ideals Q contained by P of the neutrosophic ring $\mathscr{R}$ and the neutrosophic prime ideals of the neutrosophic ring $\mathscr{R}_P$.

ii. The neutrosophic ideal P_P is the unique neutrosophic maximal ideal of the neutrosophic ring $\mathscr{R}_P$.

Proof. i. Since $S = \mathscr{R} - P \Rightarrow S \cap P = \emptyset$ and from Theorem 2.18, the desired result is seen.

ii. Let M be the neutrosophic maximal ideal of the neutrosophic ring $\mathscr{R}_P$. Since every neutrosophic maximal ideal is a neutrosophic prime ideal, M is a neutrosophic prime ideal. Form (i), there exists a neutrosophic prime ideal Q of the neutrosophic ring $\mathscr{R}$ such that $Q \subset P$ and $M = Q_P$. Since $Q \subset P$, $Q_p \subset P_P$ is obtained. On the other hand, since $S \cap P = \emptyset$, from Theorem 2.16, $\mathscr{R}_p \neq P_p$. Since Q_p is the neutrosophic maximal ideal of the neutrosophic ring $\mathscr{R}_P$ and $R_P \neq P_P$, we find $M = P_P$.

□

Definition 2.20. A commutative neutrosophic ring with unity and having a unique neutrosophic maximal ideal is called a neutrosophic local ring.

Theorem 2.21. *The following conditions are equivalent for a unitary and commutative neutrosophic ring $\mathscr{R} = \langle \mathbb{R} \cup I \rangle$.*

i. $\mathscr{R}$ is neutrosophic local ring.

ii. All non-unitary elements of the neutrosophic ring $\mathscr{R}$ are contained in some neutrosophic ideal $M \neq \mathscr{R}$.

iii. The set of non-unitary elements of $\mathscr{R}$ is a neutrosophic ideal.

Proof. $(i \Rightarrow ii)$ From the definition of a neutrosophic local ring, $\mathscr{R}$ has a unique neutrosophic maximal ideal. Let this be N. We assume that $a \in \mathscr{R}$ is not unitary. Then $(a) \neq \mathscr{R}$. In this case, there exists a neutrosophic maximal ideal N such that $(a) \subset N$. From here, $a \in N$ is found.

$(ii \Rightarrow iii)$ Let the neutrosophic ideal $M \neq \mathscr{R}$ contain all non-unitary elements of the neutrosophic ring $\mathscr{R}$. Since $M \neq \mathscr{R}$, no element of M is unitary.

$(iii \Rightarrow i)$ Suppose that M is a neutrosophic ideal consisting of non-unitary elements of the neutrosophic ring $\mathscr{R}$ and J is a neutrosophic ideal of the neutrosophic ring $\mathscr{R}$. For $a \in J$, $(a) \subset J$. If $a \in J$ were unitary then $(a) = J = \mathscr{R}$, otherwise since $(a) \subset J \subset M$, M is the only neutrosophic maximal ideal of the neutrosophic ring $\mathscr{R}$. So, $\mathscr{R}$ is a neutrosophic local ring. □

Remark 2.22. The set $\mathscr{R}$ is defined as $\mathscr{R} = \langle \mathbb{R} \cup I \rangle = \{a + bI : a, b \in \mathbb{R}\}$. It is easily observed from the definition of the set $\mathscr{R}$ that $\mathbb{R} \subset \mathscr{R}$. Therefore, if $b = 0$ is assumed in the definition and theorems provided in the paper, the properties of classical localization are obtained.

Example 2.23. If S is a multiplicative subset of the ring $\mathscr{R}$, we can think of S as a multiplicative subset of the polynomial ring $\mathscr{R}[x]$. In this case,

$$S^{-1}\left(\mathscr{R}[x]\right) \cong S^{-1}\mathscr{R}[x].$$

Indeed, we can extend the neutrosophic natural homomorphism $\pi : \mathscr{R} \to S^{-1}\mathscr{R}$ to the neutrosophic homomorphism $f : \mathscr{R}[x] \to \left(S^{-1}\mathscr{R}\right)[x]$ by identifying it with

$$f\left(a_0 + a_1 x + ... + a_n x_n\right) = \pi\left(a_0\right) + \pi\left(a_1\right) x + ... + \pi\left(a_n\right) x_n.$$

Since $f(s) = \pi(s) = \frac{s}{1} \in S^{-1}\mathscr{R}$ is unitary for all $s \in S$, one can find a neutrosophic homomorphism $g : S^{-1}\left(\mathscr{R}[x]\right) \to \left(S^{-1}\mathscr{R}\right)[x]$ defined uniformly from the universal property of neutrosophic localization. The neutrosophic homomorphism $g\left(\frac{a_0+a_1x+.....+a_nx_n}{s}\right) = \frac{a_0}{s} + \frac{a_1}{s}x + ... + \frac{a_n}{s}x^n$. A necessary and sufficient condition for $p(x) = a_0 + a_1 x + ... + a_n x_n$ to be $\frac{p(x)}{s} \in \text{Ker}g$ is that $\frac{a_i}{s} = 0 + 0I$ for $\forall i = 1, 2, ..., n$.

This expression is equivalent to finding a $u_i \in S$ such that $u_i a_i = 0 + 0I$ for all $i = 1, 2, ..., n$. If we say $u = u_1 ..... u_n \in S$, since a will be $u.p(x) = 0$, $\frac{p(x)}{s} \in \text{Ker}g \Leftrightarrow \frac{p(x)}{s} = 0$ will be found. Then g is one-to-one. By writing the coefficients of each polynomial in $\left(S^{-1}\mathscr{R}\right)[x]$ in the common denominator, it can also be shown that g is surjective.

Example 2.24. Let $S^{-1}\langle \mathbb{Z}_m \cup I \rangle$ be the set of unitary elements in $\langle \mathbb{Z}_m \cup I \rangle$ i.e the set of all non-zero divisors $S = \langle \mathbb{Z}_m^* \cup I \rangle$.

Indeed, for all $t \in S$, $\frac{a}{b} = \frac{at}{bt}$. Since the elments of S are unitary, we can express each $\frac{a}{b}$ as $\frac{a^*}{1}$. Then there is neutrosophic a natural homomorphism between $\langle \mathbb{Z}_m \cup I \rangle$ and $S^{-1}\langle \mathbb{Z}_m^* \cup I \rangle$.

Example 2.25. Let $\mathscr{R}$ be a neutrosophic ring and let S and T be multiplicative closed subsets. Let U and T denote the neutrosophic natural homomorphic image of in the neutrosophic ring $S^{-1}\mathscr{R}$.

Indeed, it is clear ST is a multiplicatively closed subset of the neutrosophic ring $\mathscr{R}$. The neutrosophic natural homomorhism of $\mathscr{R} \to (ST)^{-1}\mathscr{R}$ defined by $a \to \frac{a}{1}$ since the image of $s \in S$ is the invers $\frac{1}{s}$ of the element $\frac{s}{1}$, we can extend a neutrosophic homomorphism from neutrosophic ring $(ST)^{-1}\mathscr{R}$ to the neutrosophic ring $S^{-1}\mathscr{R}$ that takes $\frac{a}{s}$ to $\frac{a}{s}$. But this transformation also takes $\frac{t}{s}$ $(t \in T)$ to $\frac{t}{s}$ and $\frac{t}{s}$ has $\frac{s}{t}$ inverse in $(ST)^{-1}\mathscr{R}$. Then there exists a neutrosophic homomorphism $F : U^{-1}\left(S^{-1}\mathscr{R}\right) \to (ST)^{-1}\mathscr{R}$ satisfying $F\left(\frac{\frac{a}{s}}{\frac{t}{s^*}}\right) = \frac{as^*}{st}$.

Similarly with $a \to \frac{\frac{a}{1}}{\frac{1}{1}}$ let us define a transformation from $\mathscr{R}$ to $U^{-1}(S^{-1}\mathscr{R})$. The inverse of $\frac{\frac{st}{1}}{\frac{1}{1}}$, the image of st, is also $\frac{\frac{1}{s}}{\frac{1}{t}}$. Thus, it is seen that $G : (ST)^{-1}\mathscr{R} \to U^{-1}\left(S^{-1}\mathscr{R}\right)$ satisfies $G\left(\frac{a}{st}\right) = \frac{\frac{a}{s}}{\frac{t}{1}}$.

Example 2.26. Let $\mathscr{R}$ be a neutrosophic ring and S a multiplicatively closed subset. Let us show that if K and J are two neutrosophic ideals of the neutrosophic ring $\mathscr{R}$, then $K \neq J$ does not entail $S^{-1}K \neq S^{-1}J$.

3 Conclusion remarks

In this study, some fundamental results on the localization of real neutrosophic rings have been presented. Neutrosophic rings contain more elements compared to classical rings. The obtained results demonstrate that neutrosophic localization preserves the properties of classical localization. The results obtained are anticipated to facilitate further research in the field of neutrosophic algebra.

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