

Analysis of a Fractional SEIRS Epidemic Model with Diffusion Using the Homotopy Perturbation Sumudu Transform Method

A. Gouri^{1,2}, R. S. Dubey¹, S. Sharma¹ and K. Modi¹

MSC 2010 Classifications: Primary 33C20; Secondary 33C65.

Keywords and phrases: SEIRS epidemic model, Sumudu transform method, Homotopy perturbation method.

Abstract, This work introduces a novel framework for analyzing a fractional SEIRS epidemic model that incorporates spatial diffusion and nonlinear incidence rates. The population is classified into four compartments: susceptible (S), exposed (E), infected (I), and recovered (R). In this setting, susceptible individuals become exposed after contact with infected individuals, progress to the infectious class following a latent period, and eventually move into the recovered class, with the possibility of losing immunity and returning to the susceptible pool. To obtain approximate analytical solutions, we apply the Homotopy Perturbation Sumudu Transform Method (HPSTM). Furthermore, the existence, uniqueness, and boundedness of solutions are rigorously established. Numerical simulations and graphical illustrations are provided to demonstrate the dynamical behavior of the system under different fractional orders. The findings emphasize the accuracy, robustness, and efficiency of the proposed methodology, confirming its suitability for analyzing fractional epidemic models with diffusion.

1 Introduction

Mathematical modeling has long served as a fundamental tool for describing complex phenomena in science and engineering, with calculus providing a precise framework for capturing dynamic processes. In particular, nonlinear fractional differential equations have emerged as an effective approach for modeling systems exhibiting memory and hereditary effects, which are often observed in epidemiological, physical, and biological systems ([1]-[3]). Solving such equations—especially nonlinear systems—poses significant analytical and numerical challenges. Over the years, several techniques have been developed to tackle these challenges, including the Adomian decomposition method [4], homotopy analysis method [5], generalized transform methods [6], homotopy perturbation method (HPM) [7], modified homotopy perturbation transform methods [8], and homotopy analysis transform approaches [9]. Among these, the HPM, introduced by Professor He [10], is especially notable for its rapid convergence and accuracy in approximating analytical solutions. Additionally, various fractional operators—such as Caputo, Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu derivatives—have been proposed to model memory effects in complex dynamical systems.

The continuous emergence of infectious diseases remains a significant global health challenge, despite advances in medical research and public health interventions. Many pathogens spread through direct contact, airborne transmission, or other mechanisms, often exhibiting complex and nonlinear patterns of transmission. Mathematical modeling plays a crucial role in understanding these dynamics, predicting outbreaks, and evaluating strategies for disease control. A critical aspect of epidemic modeling is the incidence rate, which describes how susceptible individuals become infected upon contact with infectives. Real-world disease transmission is rarely purely linear; behavioral changes, limited contact opportunities, or saturation effects often influence it. Nonlinear incidence functions, such as quadratic or saturating forms, provide a more realistic representation of transmission dynamics, capturing phenomena like threshold effects, backward bifurcations, and finite-amplitude outbreaks.

Significant progress has been made in developing and analyzing epidemiological models ([11]-[13]). The foundational work of McKendrick and Kermack laid the groundwork for compartmental models [14], which have since evolved into widely used structures such as SIR, SIS, SEIR, and SIRS models ([15]-[18]). These models classify the population into distinct compartments based on disease status and are instrumental in understanding the spread and control of infectious diseases.

In this study, we extend these classical models by proposing a fractional SEIRS epidemic model that incorporates spatial diffusion, an exposed compartment, and nonlinear incidence. The population is divided into four compartments: susceptible (S), exposed (E), infected (I), and recovered (R). Susceptible individuals become exposed upon contact with infectious individuals, representing the latent period during which the disease is present but not yet transmissible. Exposed individuals then progress to the infectious class at a rate determined by the latency period. Infectious individuals recover and acquire temporary immunity, while recovered individuals may lose immunity and return to the susceptible compartment. The generalized fractional SEIRS epidemic model can be defined as

$$\begin{aligned}
 D_{\vartheta}^h S &= -\beta SI + \gamma R, \\
 D_{\vartheta}^h E &= \beta SI - \sigma E, \\
 D_{\vartheta}^h I &= \sigma E - \mu I, \\
 D_{\vartheta}^h R &= \mu I - \gamma R.
 \end{aligned}
 \tag{1.1}$$

Here, β is the rate of transmission, σ is the exposure rate, γ is the rate of immunity loss, and μ is the recovery rate.

This study investigates the dynamics of spatially localized epidemics through a fractional SEIRS model incorporating spatial diffusion. Epidemic outbreaks can arise in systems where the infection transmission exhibits nonlinear behavior, such as quadratic incidence. While other nonlinear forms, like Holling-type functional responses, can also generate localized infection hotspots, for simplicity, we consider the nonlinear transmission function $f(S, I) = \chi SI^2$. Spatial diffusion is incorporated to represent the movement of individuals across the domain, leading to the following system:

$$\begin{aligned}
 D_{\vartheta}^h S &= \varphi_S S_{xx} - \chi SI^2 + \gamma R, \\
 D_{\vartheta}^h E &= \varphi_E E_{xx} + \chi SI^2 - \sigma E, \\
 D_{\vartheta}^h I &= \varphi_I I_{xx} + \sigma E - \mu I, \\
 D_{\vartheta}^h R &= \varphi_R R_{xx} + \mu I - \gamma R.
 \end{aligned}
 \tag{1.2}$$

Where, S, E, I, R represent the population densities in each compartment, and $\varphi_S, \varphi_E, \varphi_I, \varphi_R$ denote their respective diffusion coefficients. Neumann boundary conditions are imposed on the 1-D interval $[-L, L]$, ensuring no-flux at the domain boundaries: $S_x = E_x = I_x = R_x = 0$ at $x = -L$ and $x = L$. Given that the typical duration of infection and recovery is much shorter than the average lifespan of individuals, birth and natural death rates are neglected in this analysis.

Observations suggest that infectious individuals often exhibit reduced mobility compared to other classes. This phenomenon can arise physiologically, as infection may impair an individual’s ability to migrate or move freely. In human populations, similar effects can result from quarantine measures or social distancing policies, which deliberately reduce movement to control disease spread. To capture this effect, the diffusion coefficient of the infectious class is assumed to be small as $\varphi_I = \delta^2$ is written, where $\delta \ll 1$. The system can be simplified to the following perturbed reaction–diffusion model:

$$\begin{aligned}
 D_{\vartheta}^h S &= \varphi_S S_{xx} - \chi SI^2 + \gamma R, \\
 D_{\vartheta}^h E &= \varphi_E E_{xx} + \chi SI^2 - \sigma E, \\
 D_{\vartheta}^h I &= \delta^2 I_{xx} + \sigma E - \mu I, \\
 D_{\vartheta}^h R &= \varphi_R R_{xx} + \mu I - \gamma R.
 \end{aligned}
 \tag{1.3}$$

Under these assumptions, localized concentrations of infection can emerge in the system, corresponding to spike-type solutions or hotspots. These phenomena illustrate how spatial heterogeneity, nonlinear incidence, and reduced mobility of infectives contribute to the formation of localized epidemic outbreaks in a fractional SEIRS framework.

The model incorporates spatial diffusion to account for the movement of individuals, allowing the study of localized outbreaks, traveling waves, and complex spatio-temporal patterns. Additionally, fractional-order derivatives are included to capture memory effects in disease dynamics, reflecting how past states influence current transmission and progression rates. By combining these features with nonlinear incidence functions, the model provides a realistic and flexible framework for analyzing epidemic dynamics under heterogeneous and spatially structured populations.

To obtain analytical approximations, we employ the Homotopy Perturbation Transform Method (HPTM) in conjunction with the Sumudu transform, which offers high accuracy and computational efficiency for fractional-order systems. The existence, uniqueness, and boundedness of solutions are rigorously established, and numerical simulations are performed to explore the effects of fractional order, diffusion, and nonlinear incidence on disease dynamics. This integrated approach enables a comprehensive understanding of epidemic spread and offers valuable insights for public health planning and intervention strategies.

2 Basic Definitions

Definition 1 [19] The Caputo derivative for fractional order \hbar is denoted by ${}^C D_\tau^\hbar$ and defined by

$${}^C D_\tau^\hbar h(\tau) = \frac{1}{\Gamma(n - \hbar)} \int_a^\tau (\tau - u)^{n-\hbar-1} h^{(n)}(u) du, \quad (2.1)$$

where $h^{(n)}(\tau) = \frac{d^n h}{d\tau^n}$ and $0 < \hbar < 1$ with $n - 1 < \hbar < n$.

Definition 2 [20] The Sumudu transform, if $W = \{w(k) \mid \exists M, \vartheta_1 \text{ and } \vartheta_2 > 0, |f(k)| < M \exp(|k|/\vartheta_j)\}$, if $k \in (-1)^j \times [0, \infty)$ is the set of functions, hence the well-known Sumudu transform on W is

$$\mathbb{S}[h(k)](s) = \int_0^\infty e^{-k} f(sk) dk. \quad (2.2)$$

Definition 3 [21] The of Sumudu transform for Caputo fractional derivative of order $\hbar > 0$ is given as

$$\mathbb{S} [{}^C D_\tau^\hbar h(\tau)] (s) = \frac{\mathbb{S}[h] - h(0)}{s^\hbar}. \quad (2.3)$$

3 Basic idea of the Homotopy perturbation Sumudu transform method (HPSTM)

This section is devoted to the study of the HPSTM, employed as a means of addressing fractional differential equations with the aid of generalized fractional operators

Known as HPSTM, this approach integrates He's polynomials with the Sumudu transform. By employing the fundamental principles of HPSTM, nonlinear fractional differential equations can be effectively addressed as follows:

$$D_{\tau}^{\hbar}v(\tau) + Lv(\tau) + Nv(\tau) = g(\tau), \quad 0 < \hbar \leq 1, \quad \tau > 0, \tag{3.1}$$

in this context, D_{τ}^{\hbar} to represent the fractional derivative of v , the linear differential operator is labeled as L , and the generic non-linear differential operator is indicated by N .

Apply the Sumudu transform on (3.1),

$$\mathbb{S}[D_{\tau}^{\hbar}v(\tau)] + \mathbb{S}[Lv(\tau)] + \mathbb{S}[Nv(\tau)] = \mathbb{S}[g(\tau)],$$

we have from [22],

$$\mathbb{S}[D_{\tau}^{\hbar}v(\tau)](s) = \kappa^{-1} \left[\mathbb{S}[v(\tau)] - \sum_{k=0}^{n-1} s^k [v^k(\tau)]_{\tau=0} \right], \quad 0 < \hbar \leq 1,$$

where, $\kappa = s^{\hbar}$ for the Caputo fractional derivative.

Hence,

$$\mathbb{S}[v(\tau)] = \sum_{k=0}^{n-1} s^k [v^k(\tau)]_{\tau=0} - \kappa \{ \mathbb{S}[Lv(\tau)] + \mathbb{S}[Nv(\tau)] - \mathbb{S}[g(\tau)] \}.$$

Using the inverse Sumudu transform, we get

$$v(\tau) = \psi(\tau) - \mathbb{S}^{-1} \{ \kappa (\mathbb{S}[Lv(\tau)] + \mathbb{S}[Nv(\tau)]) \},$$

where

$$\psi(\tau) = \mathbb{S}^{-1} \left\{ \sum_{k=0}^{n-1} s^k [v^k(\tau)]_{\tau=0} + \kappa \mathbb{S}[g(\tau)] \right\}.$$

We have p as $v(\tau) = \sum_{n=0}^{\infty} p^n v_n(\tau)$ and consider $Nv(\tau) = \sum_{n=0}^{\infty} p^n H_n(v)$, where $H_n(v)$ represents the He's polynomial given by

$$H_n(v_0, v_1, v_2, \dots, v_n) = \frac{1}{n!} \frac{\partial}{\partial p^n} \left\{ N \left[\sum_{k=0}^{\infty} p^k v_k \right] \right\}_{p=0}, \quad n \geq 0,$$

and $p \in (0, 1)$ represents an embedding parameter.

Hence,

$$\sum_{n=0}^{\infty} p^n v_n(\tau) = \psi(\tau) - p \mathbb{S}^{-1} \left\{ \kappa \left(\mathbb{S} \left[L \sum_{n=0}^{\infty} p^n v_n(\tau) \right] + \mathbb{S} \left[\sum_{n=0}^{\infty} p^n H_n(v) \right] \right) \right\}. \tag{3.2}$$

Presently, by gathering the coefficients of different powers of p , we obtain:

$$\begin{aligned} p^0 : v_0(\tau) &= \psi(\tau), \\ p^1 : v_1(\tau) &= -\mathbb{S}^{-1} \{ \kappa (\mathbb{S}[Lv_0(\tau)] + \mathbb{S}[H_0(v)]) \}, \\ p^2 : v_2(\tau) &= -\mathbb{S}^{-1} \{ \kappa (\mathbb{S}[Lv_1(\tau)] + \mathbb{S}[H_1(v)]) \}, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

The remaining components of $v_n(\tau)$, $n \geq 0$ can be determined using a comparable methodology. Consequently, the HPSTM solutions are fully computed.

Hence, the solution of (3.1) is

$$v(\tau) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n v_n(\tau) = \sum_{n=0}^{\infty} v_n(\tau).$$

4 Existence and Uniqueness

Equation (1.3) can be written in the form

$$\begin{aligned}
 D_{\vartheta}^{\hbar} S - \wp_S S_{xx} &= -\chi SI^2 + \gamma R, \\
 D_{\vartheta}^{\hbar} E - \wp_E E_{xx} &= \chi SI^2 - \sigma E, \\
 D_{\vartheta}^{\hbar} I - \delta^2 I_{xx} &= \sigma E - \mu I, \\
 D_{\vartheta}^{\hbar} R - \wp_R R_{xx} &= \mu I - \gamma R.
 \end{aligned}
 \tag{4.1}$$

Considering $X = C(\Delta, \mathbb{R})$ and X^4 be Banach spaces equipped with uniform norms

$$\|u\|_X = \text{Sup}_{x \in \Delta} |u(x)|, \quad \forall u \in X,$$

and

$$\|U\|_{X^4} = \text{Sup}_{x \in \Delta} |U(x)|, \quad \forall U \in X^4,$$

with $\|U(x)\| = \sum_{i=1}^4 |U(x)|$ is the Manhattan norm[23].

Let $Y = (Y_1, Y_2, Y_3, Y_4)$, $Y^0 = (Y_1^0, Y_2^0, Y_3^0, Y_4^0)$, $\lambda = (\wp_S, \wp_E, \delta^2, \wp_R)$ and L is a linear operator with

$$\begin{aligned}
 L : D(L) \subset X^4 &\rightarrow X^4 \\
 LY = \lambda Y_{xx} &= (\wp_S(Y_1)_{xx}, \wp_E(Y_2)_{xx}, \delta^2(Y_2)_{xx}, \wp_R(Y_3)_{xx}), \quad \forall Y \in D(L),
 \end{aligned}$$

where

$$D(L) = \left\{ Y \in X^4 : Y_{xx} \in X^4, \frac{\partial Y}{\partial x} = 0_{\mathbb{R}^4} \text{ for } x \in \partial\Delta \right\}.$$

Now consider the function f , which is given by $f : [0, T] \times X^4 \rightarrow X^4$, here

$$f(\vartheta, Y(\vartheta)) := f(Y(\vartheta)) = (f_1(Y(\vartheta)), f_2(Y(\vartheta)), f_3(Y(\vartheta)), f_4(Y(\vartheta))),$$

with

$$\begin{cases}
 f_1(Y(\vartheta)) = -\chi Y_1 Y_3^2 + \gamma Y_4, \\
 f_2(Y(\vartheta)) = \chi Y_1 Y_3^2 - \sigma Y_2, \\
 f_3(Y(\vartheta)) = \sigma Y_2 - \mu Y_3, \\
 f_4(Y(\vartheta)) = \mu Y_3 - \gamma Y_4,
 \end{cases}$$

where, $t \in [0, T]$.

The model can be represented as

$$\begin{cases}
 D_{\vartheta}^{\hbar} Y = LY + f(Y(\vartheta)), \\
 Y(0) = Y^0,
 \end{cases}
 \tag{4.2}$$

where, $Y = (S, E, I, R)$ and $Y^0 = (S^0, E^0, I^0, R^0)$.

Theorem 4.1. Consider $0 < \hbar \leq 1$ for any $Y^0 \in D(L)$, then equation (4.2) that has a unique positive solution $Y \in C([0, T], X^4)$

$$Y(\vartheta) = \int_0^{\infty} \varphi_{\hbar}(\theta) Q(\vartheta^{\hbar} \theta) Y^0 d\theta + F(\vartheta),
 \tag{4.3}$$

here

$$F(\vartheta) = \hbar \int_0^{\vartheta} \int_0^{\infty} \theta(\vartheta - t)^{\hbar-1} \varphi_{\hbar}(\theta) f(t) Q((\vartheta - t)^{\hbar} \theta) d\theta dt,$$

$\varphi_{\hbar}(\theta)$ constitutes a probability density function over the interval $(0, \infty)$.

Proof. Given that L serves as a closed operator of linear type, on a dense set $D(L)$ within X^4 , it consequently generates a Co-semigroup denoted as $\{Q(\vartheta), \vartheta \geq 0\}$ of contractions operating on X^4 . We have already established that f exhibits Lipschitz continuity with respect to y , uniformly across $\vartheta \in [0, T]$, provided that $y_i \geq 0$ for i is 1, 2, 3, 4.

To prove that (4.3) represents the unique solution of equation (4.2), it is sufficient to prove that $F(\vartheta) \in D(L)$ for every $\vartheta \in [0, \infty)$ [24].

In other words, we must prove the existence of the double integral

$$M(\vartheta) = \hbar \int_0^\vartheta \int_0^\infty \theta(\vartheta - t)^{\hbar-1} \varphi_\hbar(\theta) Q\left((\vartheta - t)^\hbar \theta\right) f(t) d\theta dt,$$

or

$$M(\vartheta) = \hbar \int_0^\vartheta \int_0^\infty \theta(\vartheta - t)^{\hbar-1} \varphi_\hbar(\theta) Q\left((\vartheta - t)^\hbar \theta\right) [f(t) - f(\vartheta)] d\theta dt + \int_0^\infty \varphi_\hbar(\theta) Q\left(\vartheta^\hbar \theta\right) f(\vartheta) d\theta - f(\vartheta).$$

In light of f fulfilling the uniform Hölder condition with $\hbar \in (0, 1]$, it consequently follows that

$$\|M(\vartheta)\| \leq K \frac{\vartheta^\hbar}{\hbar} + (K + 1) \|f(\vartheta)\|.$$

Thus $F(\vartheta) \in D(L)$ for each $\vartheta \in [0, \infty)$ and $M(\vartheta) = AF(\vartheta)$.

Hence, proved. □

Theorem 4.2. *The system solution (1.3) stays within finite boundaries for every $\vartheta \geq 0$, signifying that it does not exhibit unbounded growth over time.*

Proof. To provide a detailed demonstration of the boundedness of system (1.3), we sum the three equations together, resulting in:

$$D_\vartheta^\hbar S(x, \vartheta) + D_\vartheta^\hbar E(x, \vartheta) + D_\vartheta^\hbar I(x, \vartheta) + D_\vartheta^\hbar R(x, \vartheta) = \varphi_S(S(x, \vartheta))_{xx} + \varphi_E(E(x, \vartheta))_{xx} + \delta^2(I(x, \vartheta))_{xx} + \varphi_R(R(x, \vartheta))_{xx}.$$

Integrating the above equality in Δ , we have

$$\int_\Delta (D_\vartheta^\hbar S(x, \vartheta) + D_\vartheta^\hbar E(x, \vartheta) + D_\vartheta^\hbar I(x, \vartheta) + D_\vartheta^\hbar R(x, \vartheta)) d\vartheta = \int_\Delta (\varphi_S(S(x, \vartheta))_{xx} + \varphi_E(E(x, \vartheta))_{xx} + \delta^2(I(x, \vartheta))_{xx} + \varphi_R(R(x, \vartheta))_{xx}) d\vartheta.$$

Utilizing Green’s formula and employing the homogeneous Neumann boundary conditions, we obtain

$$\int_\Delta (D_\vartheta^\hbar S(x, \vartheta) + D_\vartheta^\hbar E(x, \vartheta) + D_\vartheta^\hbar I(x, \vartheta) + D_\vartheta^\hbar R(x, \vartheta)) d\vartheta = 0.$$

Let us consider, $\int_\Delta (S + E + I + R) d\vartheta = Q$, from the ([25, Lemma 3, page 4]) and using the property from the linearity of the differential operator, we get

$$D_\vartheta^\hbar Q(x, \vartheta) = 0,$$

on solving, we get

$$Q(x, \vartheta) = Q_0 < \infty.$$

Hence, it is clear that the obtained solution is bounded. □

5 Study the SEIRS Model with Caputo Fractional Derivative

This section is devoted to solving the SEIRS epidemic model with spatial diffusion and nonlinear incidence rates through the application of the HPSTM.

The SEIRS epidemic mathematical model with spatial diffusion for the Caputo fractional derivative is given by

$${}^C D_{\vartheta}^h S = \varphi_S S_{xx} - \chi SI^2 + \gamma R, \tag{5.1}$$

$${}^C D_{\vartheta}^h E = \varphi_E E_{xx} + \chi SI^2 - \sigma E, \tag{5.2}$$

$${}^C D_{\vartheta}^h I = \delta^2 I_{xx} + \sigma E - \mu I, \tag{5.3}$$

$${}^C D_{\vartheta}^h R = \varphi_R R_{xx} + \mu I - \gamma R. \tag{5.4}$$

Applying the Sumudu transform on (5.1), we get

$$\mathbb{S} [{}^C D_{\vartheta}^h S(x, \vartheta)] = \mathbb{S} [\varphi_S S_{xx} - \chi SI^2 + \gamma R],$$

or

$$s^{-h} \{ \mathbb{S} [S(x, \vartheta)] - S(x, 0) \} = \mathbb{S} [\varphi_S S_{xx} - \chi SI^2 + \gamma R],$$

or

$$\mathbb{S} [S(x, \vartheta)] = S(x, 0) + s^h \mathbb{S} [\varphi_S S_{xx} - \chi SI^2 + \gamma R].$$

Applying the inverse Sumudu transform,

$$S(x, \vartheta) = F(x) + \mathbb{S}^{-1} \{ s^h \mathbb{S} [\varphi_S S_{xx} - \chi SI^2 + \gamma R] \}.$$

Where,

$$F(x) = \mathbb{S}^{-1} [S(x, 0)].$$

Presently, by employing the Homotopy perturbation method and expanding S , E , I , and R as

$$S = \sum_{n=0}^{\infty} p^n S_n, \quad E = \sum_{n=0}^{\infty} p^n E_n, \quad I = \sum_{n=0}^{\infty} p^n I_n \quad \text{and} \quad R = \sum_{n=0}^{\infty} p^n R_n.$$

We get

$$\sum_{n=0}^{\infty} p^n S_n(x, \vartheta) = F(x) + p \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\varphi_S \left(\sum_{n=0}^{\infty} p^n S_n(x, \vartheta) \right)_{xx} - \chi \left(\sum_{n=0}^{\infty} p^n S_n(x, \vartheta) \right) * \left(\sum_{n=0}^{\infty} p^n I_n(x, \vartheta) \right) * \left(\sum_{n=0}^{\infty} p^n I_n(x, \vartheta) \right) + \gamma \sum_{n=0}^{\infty} p^n R_n(x, \vartheta) \right] \right\}.$$

By examining the coefficients of p raised to various powers, we acquire

$$\begin{aligned} p^0 : S_0(x, \vartheta) &= F(x), \\ p^1 : S_1(x, \vartheta) &= \mathbb{S}^{-1} \{ s^h \mathbb{S} [\varphi_S (S_0(x, \vartheta))_{xx} - \chi (S_0 * I_0 * I_0) + \gamma R_0] \}, \\ p^2 : S_2(x, \vartheta) &= \mathbb{S}^{-1} \{ s^h \mathbb{S} [\varphi_S (S_1(x, \vartheta))_{xx} - \chi S_1 * I_0 * I_0 - 2\chi S_0 * I_0 * I_1 + \gamma R_1] \}, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Hence, we can find the solution by

$$S(x, \vartheta) = \sum_{n=0}^{\infty} S_n(x, \vartheta),$$

or

$$S = S_0 + S_1 + S_2 + \dots$$

Now, through the application of the Sumudu transform on (5.2), we get

$$\mathbb{S} [{}^C D_{\vartheta}^h E(x, \vartheta)] = \mathbb{S} [\wp_E E_{xx} + \chi SI^2 - \sigma E],$$

or

$$s^{-h} \{\mathbb{S} [E(x, \vartheta)] - E(x, 0)\} = \mathbb{S} [\wp_E E_{xx} + \chi SI^2 - \sigma E],$$

or

$$\mathbb{S} [E(x, \vartheta)] = E(x, 0) + s^h \mathbb{S} [\wp_E E_{xx} + \chi SI^2 - \sigma E].$$

Applying the inverse Sumudu transform,

$$E(x, \vartheta) = \phi(x) + \mathbb{S}^{-1} \{s^h \mathbb{S} [\wp_E E_{xx} + \chi SI^2 - \sigma E]\}.$$

Where,

$$\phi(x) = \mathbb{S}^{-1} [E(x, 0)].$$

Hence, with the help of HPM, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n E_n(x, \vartheta) = \phi(x) + p \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\wp_E \left(\sum_{n=0}^{\infty} p^n E_n(x, \vartheta) \right)_{xx} + \chi \left(\sum_{n=0}^{\infty} p^n S_n(x, \vartheta) \right) * \right. \right. \\ \left. \left. \left(\sum_{n=0}^{\infty} p^n I_n(x, \vartheta) \right) * \left(\sum_{n=0}^{\infty} p^n I_n(x, \vartheta) \right) - \sigma \sum_{n=0}^{\infty} p^n E_n(x, \vartheta) \right] \right\}. \end{aligned}$$

By examining the coefficients of p raised to various powers, we acquire

$$\begin{aligned} p^0 : E_0(x, \vartheta) &= \phi(x), \\ p^1 : E_1(x, \vartheta) &= \mathbb{S}^{-1} \{s^h \mathbb{S} [\wp_E (E_0(x, \vartheta))_{xx} + \chi (S_0 * I_0 * I_0) - \sigma E_0]\}, \\ p^2 : E_2(x, \vartheta) &= \mathbb{S}^{-1} \{s^h \mathbb{S} [\wp_E (E_1(x, \vartheta))_{xx} + \chi S_1 * I_0 * I_0 + 2\chi S_0 * I_0 * I_1 - \sigma E_1]\}, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Hence, we can find the solution by

$$E(x, \vartheta) = \sum_{n=0}^{\infty} E_n(x, \vartheta),$$

or

$$E = E_0 + E_1 + E_2 + \dots$$

Now, through the application of the Sumudu transform to equation (5.3), we yield

$$\mathbb{S} [{}^C D_{\vartheta}^h I(x, \vartheta)] = \mathbb{S} [\delta^2 I_{xx} + \sigma E - \mu I],$$

or

$$\mathbb{S} [I(x, \vartheta)] = I(x, 0) + s^h \mathbb{S} [\delta^2 I_{xx} + \sigma E - \mu I].$$

Applying the inverse Sumudu transform,

$$S(x, \vartheta) = G(x) + \mathbb{S}^{-1} \{s^h \mathbb{S} [\delta^2 I_{xx} + \sigma E - \mu I]\}.$$

Where,

$$G(x) = \mathbb{S}^{-1} [I(x, 0)].$$

Hence, with the help of HPM, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n I_n(x, \vartheta) = G(x) + p \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\delta^2 \left(\sum_{n=0}^{\infty} p^n I_n(x, \vartheta) \right)_{xx} \right. \right. \\ \left. \left. + \sigma \left(\sum_{n=0}^{\infty} p^n E_n(x, \vartheta) \right) - \mu \sum_{n=0}^{\infty} p^n I_n(x, \vartheta) \right] \right\} \end{aligned}$$

Now, on collecting coefficients of various powers of p , we get

$$\begin{aligned}
 p^0 : I_0(x, \vartheta) &= G(x), \\
 p^1 : I_1(x, \vartheta) &= \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\delta^2(I_0(x, \vartheta))_{xx} + \sigma E_0 - \mu I_0 \right] \right\}, \\
 p^2 : I_2(x, \vartheta) &= \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\delta^2(I_1(x, \vartheta))_{xx} + \sigma E_1 - \mu I_1 \right] \right\}, \\
 &\vdots \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

Hence, the solution can be obtained by

$$I(x, \vartheta) = \sum_{n=0}^{\infty} I_n(x, \vartheta),$$

or

$$I = I_0 + I_1 + I_2 + \dots$$

With the help of the Sumudu transform, we get from equation (5.4):

$$\mathbb{S} \left[{}^C D_{\vartheta}^h R(x, \vartheta) \right] = \mathbb{S} \left[\varphi_R R_{xx} + \mu I - \gamma R \right],$$

or

$$\mathbb{S} \left[R(x, \vartheta) \right] = R(x, 0) + s^h \mathbb{S} \left[\varphi_R R_{xx} + \mu I - \gamma R \right].$$

Applying the inverse Sumudu transform,

$$R(x, \vartheta) = H(x) + \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\varphi_R R_{xx} + \mu I - \gamma R \right] \right\}.$$

Where,

$$H(x) = \mathbb{S}^{-1} \left[R(x, 0) \right].$$

Hence, with the help of HPM, we get

$$\sum_{n=0}^{\infty} p^n R_n(x, \vartheta) = H(x) + p \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\varphi_R \left(\sum_{n=0}^{\infty} p^n R_n(x, \vartheta) \right)_{xx} + \mu \sum_{n=0}^{\infty} p^n I_n(x, \vartheta) - \gamma \sum_{n=0}^{\infty} p^n R_n(x, \vartheta) \right] \right\}.$$

On collecting the terms of various p , we get

$$\begin{aligned}
 p^0 : R_0(x, \vartheta) &= H(x), \\
 p^1 : R_1(x, \vartheta) &= \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\varphi_R (R_0(x, \vartheta))_{xx} + \mu I_0 - \gamma R_0 \right] \right\}, \\
 p^2 : R_2(x, \vartheta) &= \mathbb{S}^{-1} \left\{ s^h \mathbb{S} \left[\varphi_R (R_1(x, \vartheta))_{xx} + \mu I_1 - \gamma R_1 \right] \right\}, \\
 &\vdots \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

Hence, the solution can be obtained by the following equation

$$R(x, \vartheta) = \sum_{n=0}^{\infty} R_n(x, \vartheta),$$

or

$$R = R_0 + R_1 + R_2 + \dots$$

By the above obtained results, we can easily find the behavior of susceptible, infected, and recovered people.

To examine the numerical outcomes, we present visualizations of several results for different values of \hbar . We begin by showing the results obtained from the SEIRS model using the Caputo fractional derivative with fractional order $\hbar = 0.7$."

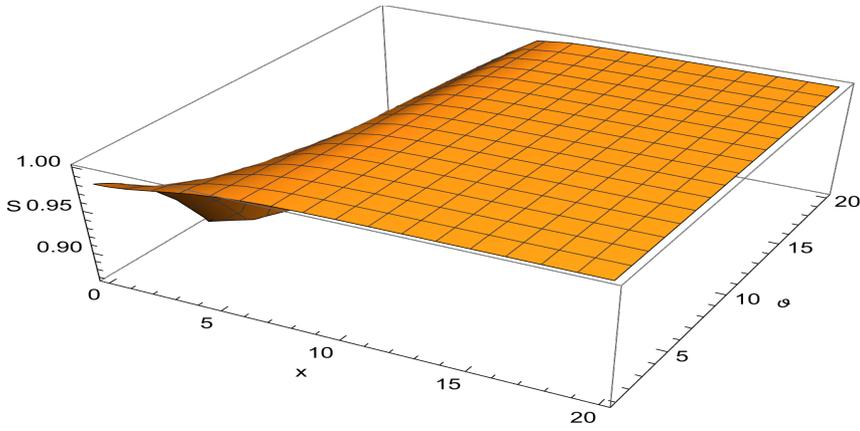


Figure 1. Susceptible population dynamics at order $\hbar = 0.7$

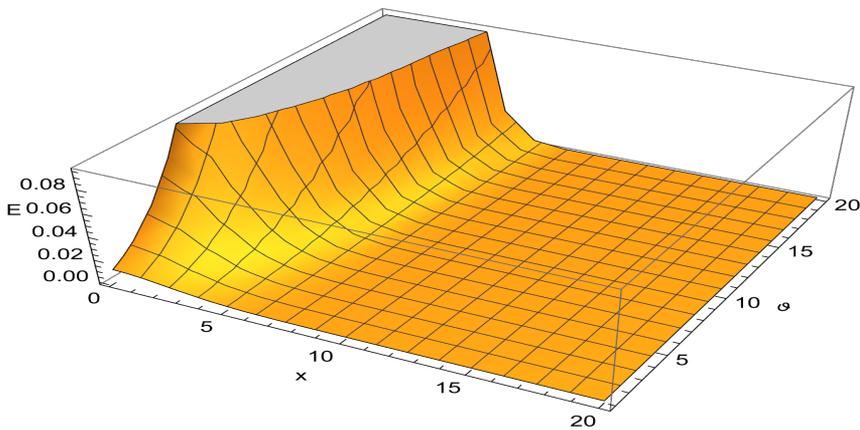


Figure 2. Exposed population dynamics at order $\hbar = 0.7$

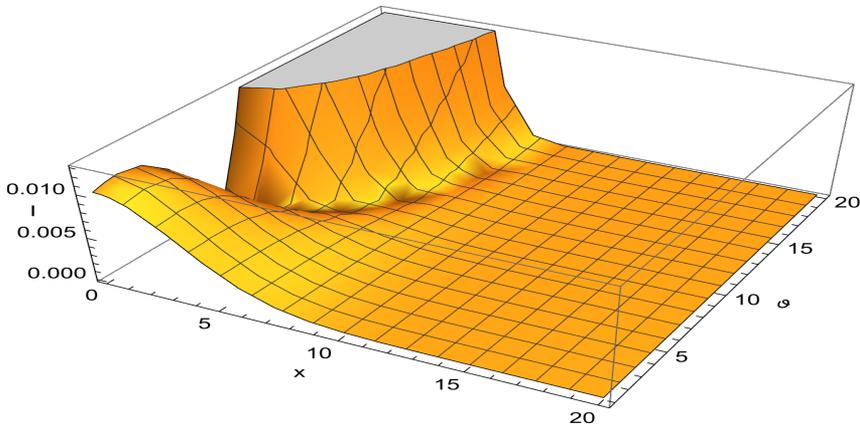


Figure 3. Infected population dynamics at order $\hbar = 0.7$

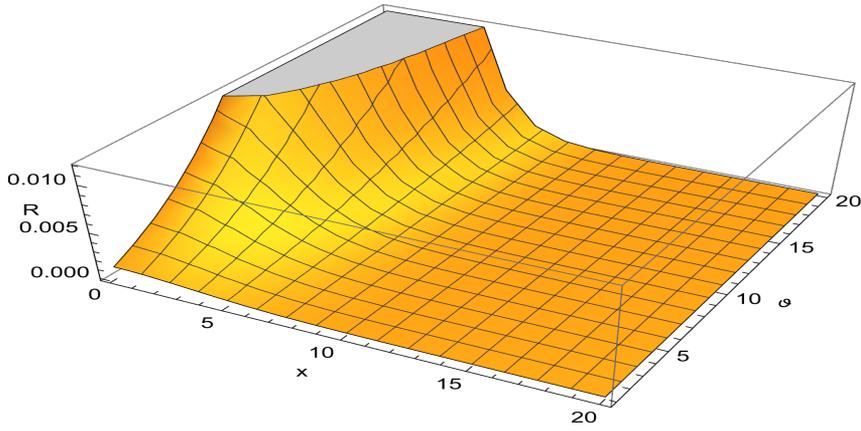


Figure 4. Recovered population dynamics at order $\hbar = 0.7$

By considering, $x = 5.2$, we can readily examine the behavior of all the variables across various values of \hbar .

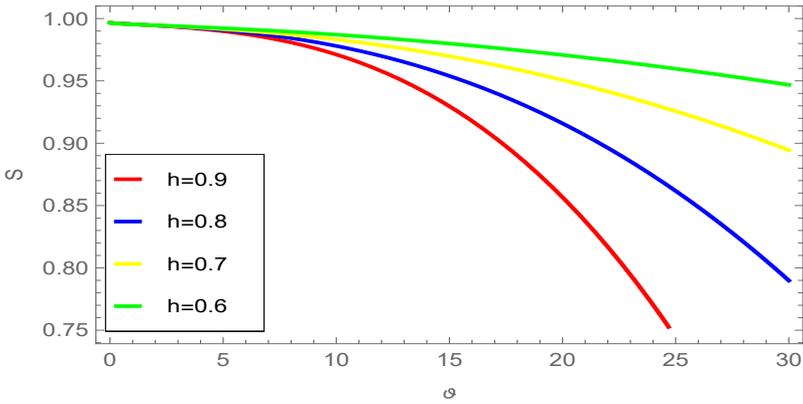


Figure 5. Susceptible population dynamics at different values of \hbar

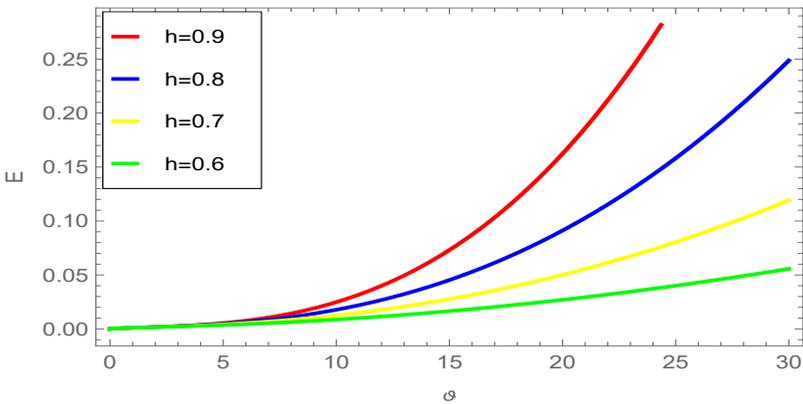


Figure 6. Exposed population dynamics at different values of \hbar

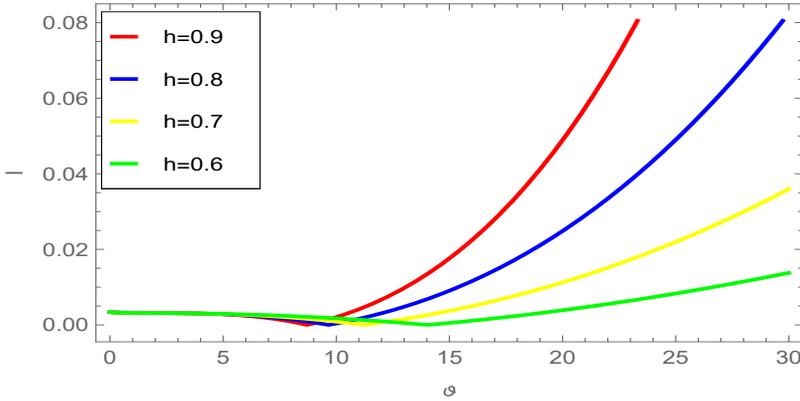


Figure 7. Infected population dynamics at different values of h

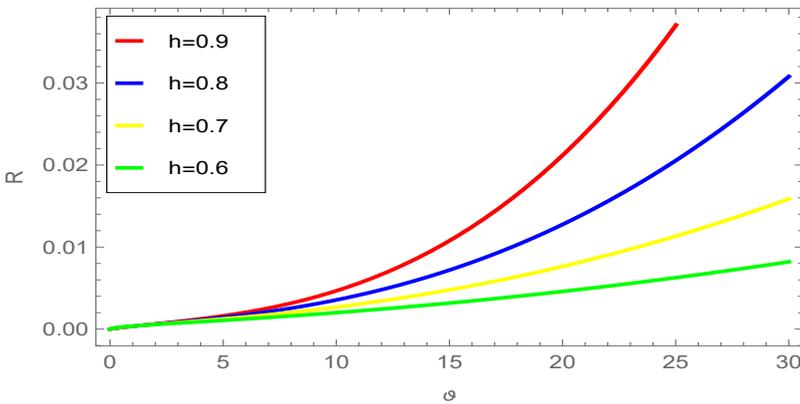


Figure 8. Recovered population dynamics at different values of h

6 Conclusion

In this work, we proposed a fractional SEIRS epidemic model that incorporates spatial diffusion and nonlinear incidence rates. By applying the Homotopy perturbation Sumudu transform method, approximate analytical solutions were obtained. The use of Caputo fractional operators allowed us to capture the influence of memory and nonlocality on system dynamics. Theoretical analysis ensured the existence, uniqueness, and boundedness of solutions, while numerical simulations and graphical illustrations highlighted the distinct dynamical behaviors of the fractional orders. Notably, the Caputo operator showed faster growth in the susceptible population and exhibited stronger effects on the infected and recovered compartments. These findings confirm the accuracy and efficiency of the proposed framework, demonstrating its suitability for analyzing fractional epidemic models with diffusion.

References

[1] Gomez-Aguilar, J. F.; Abdon Atangana, eds. Applications of fractional calculus to modeling in dynamics and chaos. *CRC Press*, 2022.

[2] Srivastava, H.M.; Saad, K.M. Some new models of the time-fractional gas dynamics equation. *Adv. Math. Models Appl*, 2018, 3(1), 5–17.

- [3] Althobaiti, S.; Dubey, R.S.; Prasad, J.G. Solution of local fractional generalized fokker-planck equation using local fractional Mohand Adomian decomposition method, *Fractal Fractional*, **2022**, 1 (30), 2240028.
- [4] Jafari, H.; Daftardar-Gejji, V. Solving a system of nonlinear fractional differential equations using Adomian decomposition. *Journal of Computational and Applied Mathematics*, **2006**, 196(2), 644–651.
- [5] Van Gorder, R.A.; Vajravelu, K. On the selection of auxiliary functions, operators, and convergence control parameters in the application of the homotopy analysis method to nonlinear differential equations: a general approach. *Communications in Nonlinear Science and Numerical Simulation*, **2009**, 14(12), 4078–4089.
- [6] Odibat, Z.; Momani, S. A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters*, **2008**, 21(2), 194–199.
- [7] Yıldırım, A. Analytical approach to fractional partial differential equations in fluid mechanics by means of the homotopy perturbation method. *International Journal of Numerical Methods for Heat and Fluid Flow*, **2010**, 20(2), 186–200.
- [8] Mtawal, A.A.; Alkaleeli, S.R. A new modified homotopy perturbation method for fractional partial differential equations with proportional delay. *Journal of Advances In Mathematics*, **2020**, 19, 58–73.
- [9] Wang, F., Fang, Q. and Hu, Y., 2025. Homotopy Analysis Transform Method for Solving Systems of Fractional-Order Partial Differential Equations. *Fractal and Fractional*, **2025**, 9(4), 253.
- [10] He, J.H. Homotopy perturbation technique. *Computer methods in applied mechanics and engineering*, **1999**, 178, 257–262. [https://doi.org/10.1016/S0045-7825\(99\)00018-3](https://doi.org/10.1016/S0045-7825(99)00018-3).
- [11] Friedman, A. Epidemiological models with seasonality. In *Mathematical methods and models in biomedicine* **2012**, 389–410.
- [12] Brauer, F. Some simple epidemic models. *Mathematical Biosciences and Engineering*, **2006**, 3(1), 1.
- [13] Driessche, P. Van Den; Yakubu, A. Demographic population cycles and R0 in discrete-time epidemic models, *Journal of Biological Dynamics*, **2019**, 179-200.
- [14] Alazman, I.; Albalawi, K. S.; Goswami, P.; Malik, K. A Restricted SIR Model with Vaccination Effect for the Epidemic Outbreaks Concerning COVID-19. *CMES-Computer Modeling in Engineering and Sciences*, **2023**, 137 (3), DOI: 10.32604/cmcs.2023.028674.
- [15] Mouaouine, A.; Boukhouima, A.; Hattaf, K.; Yousfi, N. A. fractional order SIR epidemic model with nonlinear incidence rate. *Advances in difference Equations*, **2018**, (1), 1-9.
- [16] El-Shahed, M.; Alsaedi, A. The fractional SIRC model and influenza A. *Mathematical problems in Engineering* **2011**, 9, Art. ID 480378 (2011).
- [17] Demirci, E.; Unal, A. A fractional order SEIR model with density dependent death rate. *Hacettepe journal of mathematics and statistics*, **2011**, 40(2), 287–295.
- [18] Alshammari, N.A.; Alharthi, N.S.; Mohammed Saeed, A.; Khan, A.; Ganie, A.H. Numerical solutions of a fractional order SEIR epidemic model of measles under Caputo fractional derivative. *PloS one*, **2025**, 20(5).
- [19] Dubey, R.S.; Baleanu, D.; Mishra, M.; Goswami, P. Solution of Modified Bergman’s Minimal Blood Glucose Insulin Model Using Caputo-Fabrizio Fractional Derivative. *Computer Modeling in Engineering & Sciences*, **2021**, 128(3), <https://doi.org/10.32604/cmcs.2021.015224>.
- [20] Demiray, S.T.; Bulut, H.; Belgacem, F.B.M. Sumudu Transform Method for Analytical Solutions of Fractional Type Ordinary Differential Equations. *Mathematical Problems in Engineering*, **2014**, 2015, 131690. <https://doi.org/10.1155/2015/131690>.
- [21] Baleanu, D.; Aydoğdu, S.M.; Mohammadi, H. On modelling of epidemic childhood diseases with the Caputo-Fabrizio derivative by using the Laplace Adomian decomposition method. *Alexandria Engineering Journal*, **2020**, 59(5), 3029-3039, <https://doi.org/10.1016/j.aej.2020.05.007>.
- [22] Hasan, B.; Mehmet, B. H.; Muhammad, B. F. B. The analytical solution of some fractional ordinary differential equations by the Sumudu transform method. *Abstract and Applied Analysis*, **2013**, doi.org/10.1155/2013/203875.
- [23] Clapham, C.; Nicholson, J.; Nicholson, J. R. Te Concise Oxford Dictionary of Mathematics. *Oxford University Press*. **2014**.
- [24] El-Borai, M. M. Some probability densities and fundamental solutions of fractional evolution equations. *Chaos, Solitons & Fractals*, **2002**, 14(3), 433–440, [https://doi.org/10.1016/S0960-0779\(01\)00208-9](https://doi.org/10.1016/S0960-0779(01)00208-9).
- [25] Li, H. L.; Zhang, L.; Hu, C.; Jiang, Y. L.; Teng, Z. Dynamical analysis of a fractional-order predator-prey model incorporating prey refuge. *Journal of Applied Mathematics and Computing*, **2017**, 54, 435-449, <https://doi.org/10.1007/s12190-016-1017-8>.

Author information

A. Gouri^{1,2}, ¹Department of Mathematics, ASAS, AMITY University Rajasthan, Jaipur 302002, Rajasthan.
²Department of Mathematics, Vivekananda Global University, Jaipur 303012, Rajasthan, India.
E-mail: aafringouri30@gmail.com

R. S. Dubey¹, Department of Mathematics, ASAS, AMITY University Rajasthan, Jaipur 302002, Rajasthan, India.
E-mail: ravimath13@gmail.com

S. Sharma¹, Department of Mathematics, ASAS, AMITY University Rajasthan, Jaipur 302002, Rajasthan, India.
E-mail: sharmashivani045@gmail.com

K. Modi¹, Department of Mathematics, ASAS, AMITY University Rajasthan, Jaipur 302002, Rajasthan, India.
E-mail: kmodi@jpr.amity.edu