**Palestine Journal of Mathematics**

**Vol 15(Special Issue I)(2026)** , 124–126 

# MELLIN TRANSFORM OF GENERALIZED *k*-HYPERGEOMETRIC FUNCTION

Pallavi Mahadik[1,2], Rachana Desai[3] and Puja Chavan[4,5]

MSC 2010 Classifications: Primary 44A15; 33C20.

Keywords and phrases: mellin Transform, generalized k-Hypergeometric function, $k$- Hypergeometric function of three parameters.

**Abstract** This paper aims to obtain the mellin transform of the generalized $k$-hypergeometric function. The result obtained here is verified by referring to existing results of the mellin transform of some special functions and used to deduce the mellin transform of $k$-Hypergeometric function of three parameters ${}_2F_{1,k}(\alpha,\beta;\gamma;z)$.
The outcome gained here provide a consistent framework for determining the mellin transforms of a variety of special functions, ranging from intermediate functions like Wright, M-series, and K-functions to classical functions like exponential, hypergeometric, and Mittag-Leffler functions. This so greatly enhances the study of special functions and provides a useful resource for mathematicians and researchers in related fields.

## 1 Introduction

The Hypergeometric function enables us to address various fascinating problems and thus plays a crucial role in mathematical analysis and its applications. Many of the functions encountered in the analysis are specific cases of hypergeometric functions.

In 2008, Diaz and Pariguan [1] defined the new form of hypergeometric function namely the generalized $k$-hypergeometric function as follows.

for $z \in \mathbb{C}, \alpha = (\alpha_1, \alpha_2, ..., \alpha_p), \beta = (\beta_1, \beta_2, ...\beta_q)$ and $k > 0$,

$$ {}_pF_{q,k}(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}...(\alpha_p)_{n,k}}{(\beta_1)_{n,k}...(\beta_q)_{n,k}} \frac{z^n}{n!}. \tag{1.1}$$

where $\alpha_i' s, \beta_j' s \in \mathbb{C}$, and $\beta_j \neq 0, -k, -2k, ....$

The series (1.1) is convergent for all finite $z$ if $p \leq q$; it diverges for $p > q+1$. When $p = q+1$ the series converges for $|z| < 1$, diverges for $|z| > 1$ and is absolutely convergent on $|z| = 1$ provided the parameters satisfy specific conditions.

The Pochhammer $k$-symbol [1] used in 1.1 is defined for $k > 0$ as,
$(\alpha)_{n,k} = \alpha(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k)$, for $n \geq 1, \alpha \neq 0$ and $(\alpha)_{0,k} = 0$.

For $k > 0$ and $\Re(\alpha) > 0$, the $k$-gamma function is defined as,

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.2}$$

The relation between Pochhammer $k$-symbol and $k$-gamma function is given as follows,

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)}. \tag{1.3}$$

Additionally, the following is the relationship between the $k$-gamma function and the regular gamma function

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right). \tag{1.4}$$

It is certain that,

$$\Gamma_k \to \Gamma \text{ as } k \to 1 \tag{1.5}$$

Aditionally, for $\Re(p) > 0, \Re(q) > 0$,

$$\beta_k(p,q) = \frac{\Gamma_k(p)\Gamma_k(q)}{\Gamma_k(p+q)}. \tag{1.6}$$

**Definition-1**: **Mellin Transform**
Mellin transform of a function $f(z)$ is defined in [2] as

$$M[f(z);s] \equiv F(s) = \int_0^{\infty} f(z) z^{s-1} dz, \; s \in \mathbb{C}. \tag{1.7}$$

provided the function is well-behaved under the integral.

**Definition-2**: **Mellin Inversion Formula**
If $F(s)$ is holomorphic in a strip $S(a_1, a_2)$ then for $a_1 < a < a_2$ the inverse mellin transform of $F(z)$ defined in [2] as.

$$M^{-1}\{F(s); z\} = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} F(s) z^{-s} ds. \tag{1.8}$$

The following section provides the mellin transform of the *k*-Hypergeometric function.

# 2 mellin Transform of *k*-Hypergeometric function

**Theorem 2.1.** - *The mellin transform of the k-Hypergeometric function under the conditions defined in (1.1) , for* $|arg(z)| \leq \pi - \delta, \delta > 0$ *and for* $0 < \Re(s) < min\{\Re(\frac{\alpha_1}{k}), \Re(\frac{\alpha_2}{k}), ..., \Re(\frac{\alpha_p}{k})\}$ *is*

$$M\{{}_pF_{q,k}(\alpha, \beta; -z)\} = \frac{\prod_{j=1}^{q} \Gamma_k(\beta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}}. \tag{2.1}$$

*Proof.* For $z \in \mathbb{C}, \alpha = (\alpha_1, \alpha_2, ..., \alpha_p),\ \beta = (\beta_1, \beta_2, ...\beta_q)$ and $\Re(\alpha_i), \Re(\beta_j) > 0$ for all $i$ and $j$ ,

Consider the function $F(s) = \dfrac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}}$.

By the definition of Inverse mellin transform (1.8) for $\Re(s) > 0$,

$$M^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_B F(s)\, z^{-s} ds.$$

Here B is the contour of integration in the s-plane starting from $a - i\infty$ and runs to $a + i\infty$, where $0 < a < min\{\Re(\frac{\alpha_1}{k}), \Re(\frac{\alpha_2}{k}), ..., \Re(\frac{\alpha_p}{k})\}$ to put the poles of $\Gamma_k(ks)$ to the left and of $\Gamma_k(\alpha_1 - ks), \Gamma_k(\alpha_2 - ks), ..., \Gamma_k(\alpha_p - ks)$ to the right of the path.

$$M^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_B \frac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}}\, z^{-s} ds.$$

Consider the contour $C_R$ which consists of a large anticlockwise oriented semicircle of radius R with center at the origin $[C_R = Re^{i\theta},\ \frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi]$, situated to the left of B and bounded away from poles and the curve B. Let $C$ be the closed contour consisting of $C_R$ and a portion of B terminating above and below $C_R$, covering all singularities $s = 0, -1, -2, ....$

By the Cauchy-Residue theorem, as $R \to \infty$

$$\begin{aligned}
M^{-1}\{F(s)\} &= \frac{1}{2\pi j} \int_C \frac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}}\, z^{-s} ds \\
&= \sum_{n=0}^{\infty} \underset{s \to -n}{Res} \frac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}} z^{-s} \\
&= \sum_{n=0}^{\infty} \underset{s \to -n}{lim} (s+n) \frac{k^{s-1}\Gamma(s) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}} z^{-s} \\
&= \sum_{n=0}^{\infty} \frac{1}{\Gamma(1-s)\cos s\pi} \frac{\prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks)} z^{-s} \\
&= \frac{\prod_{i=1}^{p} \Gamma_k(\alpha_i)}{\prod_{j=1}^{q} \Gamma_k(\beta_j)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_{n,k}}{\prod_{j=1}^{q} (\beta_j)_{n,k}} \frac{(-z)^n}{n!} \\
&= \frac{\prod_{i=1}^{p} \Gamma_k(\alpha_i)}{\prod_{j=1}^{q} \Gamma_k(\beta_j)}\, {}_pF_{q,k}(\alpha; \beta; -z).
\end{aligned}$$

$$\therefore M\{{}_pF_{q,k}(\alpha; \beta; -z)\} = \frac{\prod_{j=1}^{q} \Gamma_k(\beta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}}. \tag{2.2}$$

This result gives the mellin transform of *k*-Hypergeometric function.

For $\omega \in \mathbb{C}$, the result (2.2) becomes

$$\therefore M\{{}_pF_{q,k}(\alpha; \beta; -\omega z)\} = \frac{\prod_{j=1}^{q} \Gamma_k(\beta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\Gamma_k(ks) \prod_{i=1}^{p} \Gamma_k(\alpha_i - ks)}{\prod_{j=1}^{q} \Gamma_k(\beta_j - ks) \quad k^{s-1}.\omega^s}. \tag{2.3}$$

□

## 3 Mellin Transform of Special Functions

Theorem 2.1 establishes a unified framework that encapsulates and generalizes several existing results found in the current literature. The following corollaries show the wider application and generality of our theorem 2.1 and validate it.

**Corollary 3.1.** *For $k = 1$ and $p = 0, q = 0$, by Theorem 2.1, the mellin transform of $e^{-z}$ is*

$$\therefore M\{e^{-z}\} = M\{{}_0F_{0,1}(-;-;-z)\} = \Gamma_k(ks) = \Gamma(s). \tag{3.1}$$

In line with the result given by Fikioris [3].

**Corollary 3.2.** *For $k = 1, p = 2$ and $q = 1$ the Theorem 2.1 provides the mellin Transform of the Gauss' Hypergeometric function.*

$$\begin{aligned}\therefore M\{{}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta\end{matrix}\middle| -\omega z\right]\} = M\{{}_2F_{1,1}\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta\end{matrix}\middle| -\omega z\right]\} &= \frac{\Gamma(\beta)\Gamma(s)\Gamma(\alpha_1-s)\Gamma(\alpha_2-s)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta-s)\omega^s}\\ &= \frac{\beta(s,\alpha_1-s)\beta(s,\alpha_2-s)}{\beta(s,\beta-s)\omega^s}.\\ &\textit{for } 0<\Re(s)<min\{\Re(\alpha_1),\Re(\alpha_2)\}\end{aligned} \tag{3.2}$$

The result is identical to the one listed in the Table of Integral Transforms given by Bateman et al.[4, p. 336].

**Corollary 3.3.** *For $p = 2$ and $q = 1$ the k-Hyperbolic function gets reduced to the Gauss' k-Hypergeometric function of three parameters [3]. It's mellin transform by the Theorem 2.1 is*

$$\begin{aligned}\therefore M\{{}_2F_{1,k}(\gamma_1,\gamma_2;\gamma_3;-\omega z)\} &= \frac{\Gamma_k(\gamma_3)}{\Gamma_k(\gamma_1)\Gamma_k(\gamma_2)}\,\frac{\Gamma_k(ks)\Gamma_k(\gamma_1-ks)\Gamma)k(\gamma_2-ks)}{\Gamma_k(\gamma_3-ks)\ k^{s-1}\omega^s}\\ &= \frac{\beta_k(ks,\gamma_1-ks)\beta_k(ks,\gamma_2-ks)}{\beta_k(ks,\gamma_3-ks)\ k^{s-1}\omega^s}\\ &\textit{for } 0<\Re(s)<min\{\Re(\frac{\gamma_1}{k}),\Re(\frac{\gamma_2}{k})\}\end{aligned} \tag{3.3}$$

The result justifies the mellin transform of Gauss' $k$-Hypergeometric function of three parameters given by Kiymaz [5].

## 4 Conclusion

The paper presents accurate result regarding the mellin transform of the generalized $k$-hypergeometric function. The corollaries included in this article demonstrate the validity and broad applicability of our study. This result may play a significant role in solving the $k$-hypergeometric differential equation. Moreover, since this function generalizes many special functions, the findings will be valuable for solving fractional differential equations using the mellin transform.

## References

[1] Diaz, R. & Pariguan, E. On hypergeometric functions and Pochhammer k -symbol. *ArXiv Preprint Math/0405596*. (2004)

[2] Debnath, L. & Bhatta, D. Integral transforms and their applications. (Chapman,2016)

[3] Fikioris, G. Integral evaluation using the mellin transform and generalized hypergeometric functions: Tutorial and applications to antenna problems. *IEEE Transactions On Antennas And Propagation*. **54**, 3895-3907 (2006)

[4] Bateman, H., Erdélyi, A., Magnus, W. & Oberhettinger, F. Tables of integral transforms. (McGraw-Hill New York,1954)

[5] Kiymaz, I., Cetinkaya, A. & Agarwal, P. A study on the k-generalizations of some known functions and fractional operators. (UNIV PRISHTINES,2017)

#### Author information

Pallavi Mahadik[1,2], [1]Somaiya School of Basic and Applied Sciences, Somaiya Vidyavihar University,400077, India
[2]SIES Graduate School of Technology, Nerul-400706, India,.
E-mail: pallavi.m@somaiya.edu

Rachana Desai[3], [3] K.J. Somaiya School of Engineering, Somaiya Vidyavihar University, 400077, India.
E-mail: rachanadesai@somaiya.edu

Puja Chavan[4,5], [4]Somaiya School of Basic and Applied Sciences, Somaiya Vidyavihar University,400077, India
[5]A. P. Shah Institute of Technology, Thane-400615, India,.
E-mail: puja.chavan@somaiya.edu