Palestine Journal of Mathematics

Vol 15(Special Issue I)(2026) , 127–132

WHITTAKER TRANSFORM OF GENERALIZED M-SERIES

[1,2]Puja Chavan, [3]Rachana Desai and [1,4]Pallavi Mahadik

AMS 2010 Classifications: 44A20; 33C15; 33C20.

Keywords and phrases: M-Series, Integral transform, Whittaker transform.

Abstract This paper introduces a new integral transform of the generalized M-series using the Whittaker function as the kernel. The results shown here provide a unified approach to derive Whittaker transforms for various well-known special functions. Our results generalize several existing transforms, encompassing them as special cases. The Whittaker transform derived here provides a ready-made and powerful tool for mathematicians, researchers, and engineers working with complex analytical problems in the field of quantum physics, fractional calculus, mathematical physics, economics, biological modeling and computational mathematics.

1 Introduction

In 1903, Gosta Mittag-Leffler introduced the direct generalization of the exponential function, now known as the Mittag-Leffler function [6], which is given by

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+1)} \tag{1.1}$$

where $z, \alpha \in \mathbb{C}, Re(\alpha) \geq 0$. The Mittag-Leffler function is direct generalization of exponential function.

In 1905, Wiman[14] gave the generalization of $E_{\alpha}(z)$ as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)} \tag{1.2}$$

where $z, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0$ which is known as Wiman function.

Further Prabhakar[7] extended (1.2) to the new generalization given by

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^{n}}{n!} \tag{1.3}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $(\gamma)_n$ is the Pochhammer symbol [9] given by
$(\gamma)_n = \gamma(\gamma+1)...(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}, n \geq 1, (\gamma)_0 = 1, \gamma \neq 0.$

Salim[10] introduced the new generalization of Mittag-Leffler function as

$$E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^{n}}{(\delta)_n} \tag{1.4}$$

where $z, \alpha, \beta, \gamma , \delta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$.

For $|z| < 1$, the Confluent hypergeometric function [9] is defined as

$$M(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{1.5}$$

where b is any non negative integer.

For $|z| < 1$, the Gauss hypergeometric function [9] is defined as

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{1.6}$$

where c is any non negative integer.

The generalized hypergeometric function[3] is defined for $|z| < 1$ by

$$ {}_pF_q \begin{bmatrix} \gamma_1, \gamma_2, ...\gamma_p & \\ & ; z \\ \delta_1, \delta_2, ...\delta_q & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\gamma_i)_n}{\prod_{j=1}^{q} (\delta_j)_n} \frac{z^n}{n!} \tag{1.7}$$

such that denominator parameters must be nonzero and non negative integer.

Further in 1933, Wright[15] gave the generalization of hypergeometric function as

$$ {}_p\Psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n)...\Gamma(\alpha_p + \beta_p n)}{\Gamma(\rho_1 + \mu_1 n)...\Gamma(\rho_q + \mu_q n)} \frac{z^n}{n!} \tag{1.8}$$

where β_i and μ_j are real positive numbers such that

$$1 + \sum_{j=1}^{q} \mu_j - \sum_{i=1}^{p} \beta_i > 0$$

When β_i and μ_j are equal to 1, equation (1.8) differs from the generalized hypergeometric function ${}_pF_q(z)$ only by constant multiplier.

Sharma and Jain[11] extended the generalized hypergeometric function to M-series

$$ {}_p^{\alpha}M_q^{\beta}(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\gamma_i)_n}{\prod_{j=1}^{q} (\delta_j)_n} \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{1.9}$$

where $z, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0$.
For $Re(\gamma) > 0$, the Whittaker function [13] defined as

$$W_{\lambda,\mu}(z) = \frac{e^{\frac{-z}{2}} \, z^{\lambda}}{\Gamma(\frac{1}{2} - \lambda + \mu)} \int_0^{\infty} t^{-\lambda - \frac{1}{2} + \mu} (1 + \frac{t}{z})^{\lambda - \frac{1}{2} + \mu} \, e^{-t} \, dt \tag{1.10}$$

where $Re(\lambda - \frac{1}{2} - \mu) \leq 0$ and $(\lambda - \frac{1}{2} - \mu)$ is not an integer.

The Whittaker transform [8] is defined as,

$$W(f(x); \gamma, \lambda, \mu) = \int_0^{\infty} e^{\frac{-t}{2}} \, t^{\gamma - 1} \, W_{\lambda,\mu}(t) \, f(t) dt \tag{1.11}$$

where $W_{\lambda,\mu}$ is a Whittaker function.
The Whittaker integral relation [4] is given by,

$$\int_0^{\infty} e^{\frac{-t}{2}} \, t^{\gamma - 1} \, W_{\lambda,\mu}(t) \, dt = \frac{\Gamma(\frac{1}{2} + \mu + \gamma) \, \Gamma(\frac{1}{2} - \mu + \gamma)}{\Gamma(1 - \lambda + \gamma)} \tag{1.12}$$

where $Re(\gamma \pm \mu) > \frac{-1}{2},\ Re(\gamma - \lambda > -1),\ W_{\lambda,\mu}(t)$ is the Whittaker function & λ, μ are parameters.

Several integral transformations of special functions are studied by different authors and also Whittaker transform of Mittag-Leffler function and few of it's extensions are seen in [4], [5], [10], [12], [8], [11]. The purpose of this investigation is to introduce the Whittaker transform of the generalized M-Series.

2 Whittaker Transform of Generalized M-Series

Theorem 2.1. *For $\alpha, \beta, \gamma_i, \delta_j, \phi, \tau, \sigma, \omega, \lambda, \mu \in \mathbb{C}, Re(\gamma_i), Re(\delta_j), Re(\tau), Re(\sigma) > 0, \forall i = 1, ..., p$ and $\forall j = 1, ..., q, Re(\tau + \sigma n \pm \mu) > \frac{-1}{2}$ and $Re(\tau + \sigma n - \lambda) > -1$, the Whittaker transform of generalized M-series is given by,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t)\ {}_p^{\alpha}M_q^{\beta}(\omega t^\sigma)dt = K\ \phi^{-\tau}\ {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(\gamma_i, 1), (\frac{1}{2} \pm \mu + \tau, \sigma), (1, 1) \\ \\ (\delta_j, 1), (1 - \lambda + \tau, \sigma), (\beta, \alpha)\end{matrix}\ ;\ \frac{\omega}{\phi^\sigma}\right] \tag{2.1}$$

where $K = \frac{\prod_{j=1}^q \Gamma(\delta_j)}{\prod_{i=1}^p \Gamma(\gamma_i)}$.

Proof. put $\phi t = v$ in $L.H.S.$ we get,

$$\begin{aligned}
L.H.S &= \int_0^\infty e^{\frac{-v}{2}} (\frac{v}{\phi})^{\tau-1} W_{\lambda,\mu}(v)\ {}_p^{\alpha}M_q^{\beta}(\omega(\frac{v}{\phi})^\sigma)\frac{dv}{\phi} \\
&= \phi^{-\tau} \int_0^\infty e^{\frac{-v}{2}} v^{\tau-1} W_{\lambda,\mu}(v) \sum_{n=0}^\infty \frac{\prod_{i=1}^p (\gamma_i)_n}{\prod_{j=1}^q (\delta_j)_n} \frac{1}{\Gamma(\alpha n + \beta)} (\frac{\omega v^\sigma}{\phi^\sigma})^n dv \\
&= \phi^{-\tau} \sum_{n=0}^\infty \frac{\prod_{i=1}^p [\frac{\Gamma(\gamma_i+n)}{\Gamma(\gamma_i)}]}{\prod_{j=1}^q [\frac{\Gamma(\delta_j+n)}{\Gamma(\delta_j)}]} \frac{(\frac{\omega}{\phi^\sigma})^n}{\Gamma(\alpha n + \beta)} \int_0^\infty e^{\frac{-v}{2}} v^{\tau+\sigma n-1} W_{\lambda,\mu}(v)dv
\end{aligned}$$

Using the whittaker integral relation mentioned in (1.12), we get

$$\begin{aligned}
&= \phi^{-\tau} \sum_{n=0}^\infty \frac{\prod_{i=1}^p [\frac{\Gamma(\gamma_i+n)}{\Gamma(\gamma_i)}]}{\prod_{j=1}^q [\frac{\Gamma(\delta_j+n)}{\Gamma(\delta_j)}]} \frac{(\frac{\omega}{\phi^\sigma})^n}{\Gamma(\alpha n + \beta)} \frac{\Gamma(\frac{1}{2} + \mu + \tau + \sigma n)\Gamma(\frac{1}{2} - \mu + \tau + \sigma n)}{\Gamma(1 - \lambda + \tau + \sigma n)} \\
&= K\ \phi^{-\tau}\ {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(\gamma_i, 1), (\frac{1}{2} \pm \mu + \tau, \sigma), (1, 1) \\ \\ (\delta_j, 1), (1 - \lambda + \tau, \sigma), (\beta, \alpha)\end{matrix}\ ;\ \frac{\omega}{\phi^\sigma}\right]
\end{aligned}$$

here $K = \frac{\prod_{j=1}^q \Gamma(\delta_j)}{\prod_{i=1}^p \Gamma(\gamma_i)}$.
which proves the theorem.

□

Theorem 2.1 is the generalization of Whittaker transform of the exponential function, Mittag-Leffler function[6] and its generalizations given by Wiman [14], Prabhakar[7] and Salim[10]. Equation (2.1)also gives the Whittaker transform of family of hypergeometric functions such as Confluent hypergeometric function, Gauss hypergeometric function and generalized hypergeometric function.

Corollary 2.2. *On putting $\alpha = 1, \beta = 1$, no numerator parameter and denominator parameters in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t) \exp(\omega t^\sigma) dt = \phi^{-\tau} \,{}_2\Psi_1 \left[\begin{matrix} (\frac{1}{2} \pm \mu + \tau, \sigma) \\ \\ (1-\lambda+\tau, \sigma) \end{matrix} ; \frac{\omega}{\phi^\sigma} \right] \tag{2.2}$$

where $k = 1, Re(\tau + \sigma n \pm \mu) > \frac{-1}{2}$ *and* $Re(\tau + \sigma n - \lambda) > -1$
which represents Whittaker transform of exponential function, seen in [5].

Corollary 2.3. *Put $\beta = 1$, no numerator and denominator parameters in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t) E_\alpha(\omega t^\sigma) dt = \phi^{-\tau} \,{}_3\Psi_2 \left[\begin{matrix} (1,1), (\frac{1}{2} \pm \mu + \tau, \sigma) \\ \\ (1,\alpha), (1-\lambda+\tau, \sigma) \end{matrix} ; \frac{\omega}{\phi^\sigma} \right] \tag{2.3}$$

where $k = 1, Re(\alpha) > 0, Re(\tau + \sigma n \pm \mu) > \frac{-1}{2}$ *and* $Re(\tau + \sigma n - \lambda) > -1$
which is Whittaker transform of Mittag-Leffler function.

Corollary 2.4. *With no numerator and denominator parameters in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t) E_{\alpha,\beta}(\omega t^\sigma) dt = \phi^{-\tau} \,{}_3\Psi_2 \left[\begin{matrix} (1,1), (\frac{1}{2} \pm \mu + \tau, \sigma) \\ \\ (\beta,\alpha), (1-\lambda+\tau, \sigma) \end{matrix} ; \frac{\omega}{\phi^\sigma} \right] \tag{2.4}$$

where $k = 1, Re(\alpha) > 0, Re(\beta) > 0, Re(\tau + \sigma n \pm \mu) > \frac{-1}{2}$ *and* $Re(\tau + \sigma n - \lambda) > -1$
which represents Whittaker transform of Wiman's function, seen in [5].

Corollary 2.5. *Consider, one numerator parameter γ and one denominator parameter $\delta = 1$ in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t) E^\gamma_{\alpha,\beta}(\omega t^\sigma) dt = \frac{\phi^{-\tau}}{\Gamma(\gamma)} \,{}_3\Psi_2 \left[\begin{matrix} (\gamma,1), (\frac{1}{2} \pm \mu + \tau, \sigma) \\ \\ (\beta,\alpha), (1-\lambda+\tau, \sigma) \end{matrix} ; \frac{\omega}{\phi^\sigma} \right] \tag{2.5}$$

where $k = \frac{1}{\Gamma(\gamma)}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\tau+\sigma n \pm \mu) > \frac{-1}{2}$ *and* $Re(\tau+\sigma n-\lambda) > -1$
which represents Whittaker transform of Prabhakar's function given in (See [4], Page no. 102) and [5].

Corollary 2.6. *On taking one numerator parameter γ and one denominator parameter δ in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t) E^{\gamma,\delta}_{\alpha,\beta}(\omega t^\sigma) dt = \frac{\phi^{-\tau}\Gamma(\delta)}{\Gamma(\gamma)} \,{}_4\Psi_3 \left[\begin{matrix} (\gamma,1), (1,1), (\frac{1}{2} \pm \mu + \tau, \sigma) \\ \\ (\beta,\alpha), (\delta,1), (1-\lambda+\tau, \sigma) \end{matrix} ; \frac{\omega}{\phi^\sigma} \right] \tag{2.6}$$

where $k = \frac{\Gamma(\delta)}{\Gamma(\gamma)}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\tau+\sigma n \pm \mu) > \frac{-1}{2}$ *and* $Re(\tau+\sigma n - \lambda) > -1$
which is Whittaker transform of 4-parametric Mittag-Leffler function by Salim.

Corollary 2.7. *On substituting $\alpha = 1, \beta = 1$, one numerator parameters $\gamma_1 = a$ and One denominator parameter $\delta_1 = b$ in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t)\, M(a;b;\omega t^\sigma) dt = \phi^{-\tau} \frac{\Gamma(b)}{\Gamma(a)} \,{}_3\Psi_2 \left[\begin{matrix} (a,1), (\frac{1}{2} \pm \mu + \tau, \sigma) \\ \\ (b,1), (1-\lambda+\tau, \sigma) \end{matrix} ; \frac{\omega}{\phi^\sigma} \right] \tag{2.7}$$

where $Re(a), Re(b), Re(\tau), Re(\sigma) > 0, Re(\tau + \sigma n \pm \mu) > \frac{-1}{2}, Re(\tau + \sigma n - \lambda) > -1.$
which is Whittaker transform of Confluent hypergeometric function [1].

Corollary 2.8. *On taking $\alpha = 1, \beta = 1,$ two numerator parameters $\gamma_1 = a$, $\gamma_2 = b$ and One denominator parameter $\delta_1 = c$ in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t)\, {}_2F_1(\omega t^\sigma) dt = \phi^{-\tau} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, {}_4\Psi_2 \left[\begin{matrix} (a,1),(b,1),(\frac{1}{2}\pm\mu+\tau,\sigma) \\ \\ (c,1),(1-\lambda+\tau,\sigma) \end{matrix} ;\ \frac{\omega}{\phi^\sigma}\right] \tag{2.8}$$

where $Re(a)$, $Re(b)$, $Re(c)$, $Re(\tau)$, $Re(\sigma) > 0$, $Re(\tau+\sigma n\pm\mu) > \frac{-1}{2}$, $Re(\tau+\sigma n-\lambda) > -1$. which represents Whittaker transform of Gauss hypergeometric function [1].

Corollary 2.9. *Consider, $\alpha = 1$ and $\beta = 1$ in (2.1) we get,*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\tau-1} W_{\lambda,\mu}(\phi t)\, {}_pF_q(\omega t^\sigma) dt = K\phi^{-\tau}\ {}_{p+2}\Psi_{q+1} \left[\begin{matrix} (\gamma_i,1),(\frac{1}{2}\pm\mu+\tau,\sigma) \\ \\ (\delta_j,1),(1-\lambda+\tau,\sigma) \end{matrix} ;\ \frac{\omega}{\phi^\sigma}\right] \tag{2.9}$$

*where $Re(\gamma_i)$, $Re(\delta_j)$, $Re(\tau)$, $Re(\sigma) > 0$, $\forall i = 1,...,p$ and $\forall j = 1,...,q$, $K = \frac{\prod_{j=1}^{q}\Gamma(\delta_j)}{\prod_{i=1}^{p}\Gamma(\gamma_i)}$, $Re(\tau+\sigma n\pm\mu) > \frac{-1}{2}$, $Re(\tau+\sigma n-\lambda) > -1$.
which represents Whittaker transform of generalized hypergeometric function [1].*

3 Conclusion remarks

We have derived the Whittaker transform of generalized M-Series with Whittaker function as a kernel. Although classical transforms such as Fourier, Laplace, and Mellin are powerful and widely used, the Whittaker transform offers unique advantages when dealing with special functions, particularly those arising in quantum mechanics, fractional calculus, and generalized heat equations. The Whittaker transform of the generalized M-series defined here can be instrumental in solving initial and boundary value problems for linear differential equations, mainly arising in fractional calculus and mathematical physics. The results derived here provide a unified framework. With these results, the authors open up the possibilities to connect various classical transforms and special functions. This transform can also aid in modeling complex systems with memory effects and non-local behavior. The Whittaker transform of Mittag-Leffler functions can play a crucial role in fractional-order systems, viscoelastic modeling, and anomalous diffusion. The Whittaker transform of a family of hypergeometric functions can be widely used in quantum mechanics, wave propagation, and heat conduction models. Additionally it may find relevance in statistical distributions, approximation theory, and modeling of random processes. The results obtained from this research can be further extended to find Whittaker transformations of several well-known special functions and polynomials like Bessel, Laguerre, Lagrange's polynomials etc. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

References

[1] P. Chavan and R. Desai, *Whittaker Transform of Generalized Hypergeometric Function*, presented at 90^{th} annual conference-an international meet of Indian Mathematical Society, India, (2024).

[2] L. Debnath, *Integral Transforms and Their Applications*,CRC Press,New York, (1995).

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, (1953).

[4] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer, Berlin (2014).

[5] M. A. Khan and S. Ahmed,*On some properties of the generalised Mittag-Leffler function*, SpringerPlus, (2013), 1-9.

[6] G. M. Mittag-Leffler,*Sur la nouvelle fonction $E_a(x)$*, CR Acad. Sci. Paris, 137(1903), 554-558.

[7] T. R. Prabhakar, *A singular integral equation with a generalised mittag leffler function in the kernel*, Yokohama Mathematical Journal, 19 (1971), 7-15.

[8] J. C. Prajapati, *Generalization of Mittag-Leffler function and it's applications*, Department of Mathematics, SVNIT, Surat, India (2008).

[9] E.D. Rainville, *Special Functions*, Macmillan Company, New York, (1960), Reprinted by Chelsea Publishing Company, Bronx, New York, (1971).

[10] T. O. Salim, *Some properties relating to the generalized Mittag-Leffler function*, Advances in Applied Mathematics and Analysis, 4(1) (2009), 21-30.

[11] M. Sharma and R. Jain, *A note on a generalised M-Series as a special function of fractional calculus*, Fractional Calculus and Applied Analysis, 12(4) (2009), 449-452.

[12] A. K. Shukla and J. C. Prajapati, *On a generalization of Mittag-Leffler function and its properties*, Journal of Mathematical Analysis and Applications, 336(2) (2007), 797-811.

[13] E. Whittaker and G. Watson,*A course of modern analysis*, Cambridge University press, Cambridge, (1962).

[14] A. Wiman, *Über den Fundamentalsatz in der Teorie der Funktionen* $E_a(x)$, Acta Mathematica, 29 (1) (1905), 191-201.

[15] E. M. Wright, *On the coefficient of power series having exponential singularities*, Journal of London Mathematical Society, 5(1933), 71-79.

Author information

[1,2]Puja Chavan, [1]Department of Mathematics, S. K. Somaiya School of Basic and Applied Sciences, Somaiya Vidyavihar University, 400077, India.
[2]Department of Humanities and Applied Sciences, A. P. Shah Institute of Technology, Thane-400615, India.
E-mail: puja.chavan@somaiya.edu

[3]Rachana Desai, [3]Department of Science and Humanities, K. J. Somaiya School of Engineering, Somaiya Vidyavihar University, 400077, India.
E-mail: rachana.132@gmail.com

[1,4]Pallavi Mahadik, [1]Department of Mathematics, S. K. Somaiya School of Basic and Applied Sciences, Somaiya Vidyavihar University, 400077, India.
[4]SIES Graduate School of Technology, Nerul-400706, India.
E-mail: pallavi.m@somaiya.edu