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RELEVANT BOUNDS AND ASSOCIATIONS FOR THE ESTIMATION OF STATISTICAL DIVERGENCES

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Abstract *This study introduces a novel inequality on the divergence of Jain and Saraswat, expressed as an estimate of Pearson's divergence. Furthermore, this inequality has been employed to determine bounds and significant relations of the widely recognized statistical divergences.*

1 Introduction and Background

Statistical divergences serve as a metric for the purpose of discrimination among distributions of likelihood. A natural measure of distance from an authentic probability distribution X to an spurious probability distribution Y can be expressed by any arbitrary divergence measure $Ar(X, Y)$. These measures are helpful in probability theory to address different kinds of concerns. Analyzing the diversity of information in the data is the primary focus. Data vanishes when a distribution is rounded up by Y yet the real one is X. Statistical divergences with their symmetric aspect also manifest their metric character if they abide by the triangle inequality.
M. Adil Khan, etc. all [1] utlized the Jensen- Mercer's inequality to introduce the estimates of divergences with regard to Csiszar's divergence. F. Nielsen identified and marked the functional divergences among univariate Cauchy distributions [2] and between Zeta distributions [3], and notably Hyvarinen variation between distributions of exponential family [4]. Divergence networks were developed by T. Nishiyama [5] to facilitate the graphical computation of divergence functions. A. Umar, etc. all [6] explored the divergence under Neutrosophic framework and its implementation in Decision making, also Fuzzy divergence of Intuitionistic kind [7] with applications in Pattern recognition. F. Xiao [8] presented the Evidential divergence for the very same purpose as Y. Song, etc. all [9] and H. Wang, etc. all [10] proposed the Belief function divergences with usefulness in data fusion. G.L. Gilardoni characterized the Pinsker's and Vajda's inequalities for Csiszar's divergence in the article [11]. Further, a function $\psi(z)$ is desginated as convex over an interval (α, β) in case for each $z_1, z_2 \in (\alpha, \beta)$ and $0 \leq \vartheta \leq 1$, we have $\psi[\vartheta z_1 + (1-\vartheta) z_2] \leq \vartheta\psi(z_1) + (1-\vartheta)\psi(z_2)$, and said to be entirely convex on the off chance that balance does not hold as it were in case $\vartheta \neq 0$ or $\vartheta \neq 1$. Specifically, function $\psi(z)$ will be convex in derivative sense, in the domain $(0, \infty) \in (\alpha, \beta)$ if $\psi''(z) \geq 0$ in the given domain, and strictly convex if $\psi''(z) > 0$. Further, the given function is said to be normalized if $\psi(1) = 0$. In addition, X. Chen talked about $s-$ convex and $s-$ Orlicz convex functions in his article [12]. Listed below provide a few general features on convex functions:
1. The composite function $\psi(z) = hog(z) = h[g(z)]$ is convex if g and h are convex functions and h rises consistently. In a broader sense, the combined function $\psi(z) = h[g_1(z), g_2(z), ..., g_k(z)]$ is convex, if the function h is convex as values rise in every argument and every one of the functions $g_i\ \forall i = 1, 2, ..., k$ are convex.
2. The integrated function $\psi(z) = hog(z) = h[g(z)]$ is convex if the function h is convex and monotonically decreasing and g is concave. In an extended form, the merged function $\psi(z) = h[g_1(z), g_2(z), ..., g_k(z)]$ is convex, if the function h is convex with diminishing in

each argument and all of the functions g_i $\forall i = 1, 2, ..., k$ *are concave.*
3. Given two mappings, $\psi : \Re^n \to \Re$ *and* $\mu : \Re^{n+1} \to \Re$*, defined as* $\mu(z,t) = t\psi\left(\frac{z}{t}\right)$*, then the function* μ *which is the perspective of* ψ*, will be convex if* ψ *is convex.*
Note: The following instances can be used to support the aforementioned three properties:
1. The function $e^{g(z)}$ *is always convex if the augumented function* $g(z)$ *is convex, like: the functions* $e^{z\log z}$, $e^{(z-1)\log z}$, $e^{-\log z}$ *are convex, and many more.*
2. The function $\frac{1}{g(z)}$ *always shows the convex nature, if* $g(z)$ *is concave and positive, such as* $\log z$, $\sqrt{z}$*, and plenty more, are concave in their domain, but the reader can easily verify this by looking up the reciprocals of these functions, which are convex.*
3. Since, the function $\psi(z) = -logz$ *is convex by definition, therefore the unified function* $\mu(z,t) = t\log t - t\log x$ *is also convex in* (z,t).
Not only the conventional algebraic convex functions, but also the transcendental in nature can be seen as well, such as: $\sinh z, \cosh z, \tanh z$, $\sin \pi z$, $z^k \sinh z$, $z^k \cosh z$, $z^k \tanh z$ *for* $z \in [0,1]$ *and* $k \geq 1$*. These kinds of convex functions are further unable of being implemented due to their restricted domain.*
The collection that includes all discrete distributions of probability, is $\digamma_m = \{X = (x_1, x_2, x_3, ..., x_m) : x_n > 0, \sum_{n=1}^{m} x_n = 1\}$, $m \geq 2$*. Here, the probability mass variables* x_n *and* y_n *connect to the discrete distributions* X *and* Y*, accordingly. By taking the function* $\psi : (0,\infty) \to (-\infty,\infty)$ *as convex, in 1961 [13], Reniy introduced a divergence measure* $RL_\psi(X,Y) = \log\left[\sum_{n=1}^{m} \psi^{-1}\left\{x_n\psi\left(\frac{x_n}{y_n}\right)\right\}\right]$*, following that in 1967 Csiszar [14] and Bregman [15] presented* $C_\psi(X,Y) = \sum_{n=1}^{m} y_n\psi\left(\frac{x_n}{y_n}\right)$ *and* $B_\psi(X,Y) = \sum_{n=1}^{m} \psi(x_n) - \psi(y_n) - (x_n - y_n)\psi'(y_n)$*, likewise. Further, Burbea and Rao (1982, [16]) came with* $BR_\psi(X,Y) = \sum_{n=1}^{m} \frac{\psi(x_n)+\psi(y_n)}{2} - \psi\left(\frac{x_n+y_n}{2}\right)$*, the functional divergences were subsequently established by Miquel Salicru in 1994 [17], Jain and Saraswat in 2013 [18], and Lin-Wong in 2018 [19]:*
$MS_\psi(X,Y) = \sum_{n=1}^{m}\left[\sqrt{\psi(y_n)} - \sqrt{\psi(x_n)}\right]^2$, $JS_\psi(X,Y) = \sum_{n=1}^{m} y_n\psi\left(\frac{x_n+y_n}{2y_n}\right)$*, and* $LW_\psi(X,Y) = \sum_{n=1}^{m}\left(\frac{x_n+y_n}{2}\right)\psi\left(\frac{2x_n}{x_n+y_n}\right)$ *consequently, among numerous other things. Beyond that, A few significant parametric divergences are visible in the study [20], notably these ones:* α*- divergence=* $\frac{\sum_{n=1}^{m}\left[x_n^\alpha y_n^{1-\alpha} - \alpha x_n + (\alpha-1)y_n\right]}{\alpha(\alpha-1)}, \alpha \in \Re - \{0,1\}$, β*- divergence=* $\frac{\sum_{n=1}^{m}\left[x_n^{\beta+1} + \beta y_n^{\beta+1} - (\beta+1)x_n y_n^\beta\right]}{\beta(\beta+1)}, \beta \in \Re - \{0,-1\}$ *and* γ*- divergence=* $\frac{\left[\log\left(\sum_{n=1}^{m} x_n^{\gamma+1}\right) + \gamma\log\left(\sum_{n=1}^{m} y_n^{\gamma+1}\right) - (\gamma+1)\log\left(\sum_{n=1}^{m} x_n y_n^\gamma\right)\right]}{\gamma(\gamma+1)}, \gamma \in \Re - \{0,-1\}$*. Furthermore, the directional version of the Triangular distance was described by F. Topse [21], which is* $\sum_{n=1}^{m} \frac{(x_n-y_n)^2}{[x_n+y_n(2^{k+1}-1)]}$*, for* $k = 0, 1, 2....$ *These all are functional or generalized statistical divergences for analyzing two discrete distributions* X *and* Y*, at a moment,* $(X,Y) \in \digamma_m \times \digamma_m$.
With regard to X *and* Y*, we are able to ensure that the disparities* $BR_\psi(X,Y)$ *and* $MS_\psi(X,Y)$ *are symmetric. The remaining divergences, nevertheless, may also be rendered symmetric through the use of a compatible generator* $\frac{\psi(z)+\psi\left(\frac{1}{z}\right)}{2}$.
The above convex function ψ *is actually a generator which generates several divergence or discriminating measures after being used in one of the above mentioned functional measures, or in other words the functional divergences are the producers of the different divergences for different convex functions. For example, if we take the Jain- Saraswat's functional divergence*

$$JS_\psi(X,Y) = \sum_{n=1}^{m} y_n\psi\left(\frac{x_n+y_n}{2y_n}\right), \tag{1.1}$$

as a producer and ψ *as a convex generator, then we can have the followings:*
For $\psi(z) = \frac{2(z-1)^2}{z}, z > 0$ *we have* $JS_\psi(X,Y) = \sum_{n=1}^{m} \frac{(x_n-y_n)^2}{x_n+y_n} = TD_\psi(X,Y)$*= Triangular divergence measure [22]. For* $\psi(z) = \frac{1}{3}\frac{\left(4z^2-2z+1\right)}{z}, z > 0$ *we have* $JS_\psi(X,Y) = \frac{2}{3}\sum_{n=1}^{m} \frac{x_n^2+x_ny_n+y_n^2}{x_n+y_n} = CM_\psi(X,Y)$*= Centroidal mean divergence measure [23]. Similarly, for the function* $\sqrt{2z^2-2z+1}, z > 0$ *we have* $\sum_{n=1}^{m}\sqrt{\frac{x_n^2+y_n^2}{2}} = RMS_\psi(X,Y)$*= Root mean square divergence measure [23], and for* $z\log\frac{z}{\sqrt{2z-1}}, z > \frac{1}{2}$ *we have* $\sum_{n=1}^{m}\left(\frac{x_n+y_n}{2}\right)\log\frac{x_n+y_n}{2\sqrt{x_ny_n}} =$

$AGM_\psi(X,Y)$= Taneja divergence or Arithmetic- Geometric mean divergence measure [24], also for the function $\frac{(1-\sqrt{2z-1})^2}{2}, z > \frac{1}{2}$ we have $\frac{1}{2}\sum_{n=1}^{m}\left(\sqrt{x_n}-\sqrt{y_n}\right)^2 = HD_\psi(X,Y)$= Hellinger divergence measure [25], for $2|z-1|, z > 0$ we have the famous Kolmogorov distance or Variational divergence $\sum_{n=1}^{m}|x_n - y_n| = VD_\psi(X,Y)$ [26], and for $\psi(z) = 4(z-1)^2, z > 0$ we have

$$JS_\psi(X,Y) = \sum_{n=1}^{m}\frac{(x_n-y_n)^2}{y_n} = \text{Pearson's or Chi- square divergence [27]} = CS_\psi(X,Y). \tag{1.2}$$

Some more divergence measures of the same class will also be discussed in the section 3.

2 A novel inequality about the Pearson's divergence

Utilizing the subsequent theorem and the resulting value, handed down by Dragomir in the work [28], we are ready to construct an absolute type inequality (2.2) on $JS_\psi(X,Y)$ as an expression of the Pearson's divergence.

Theorem 2.1. *Given an absolutely continuous function $f : [a,b] \to \Re$, and constants such as $L, U \in \Re$, in a manner that $-\infty < L \le f'(z) \le U < \infty,\ \forall\, z \in [a,b]$. Afterwards, there is*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(z)\,dz\right| \le \frac{1}{8}(b-a)(U-L). \tag{2.1}$$

The constant $\frac{1}{8}$ is very much sharp. Here $\Re$ stands for the set of all real numbers and L, U are designated as the lower and upper bounds of the function f', respectively.

To figure out the proportional efficacy of two divergence measures concurrently, where one measure Pearson's is stable while the other is an integral part of Jain-Saraswat's class, the subsequent inequality (2.2) needs to be calculated.

Theorem 2.2. *Given a convex and normalized function $\psi : [\mu,\nu] \subset (0,\infty) \to \Re$, i.e, $\psi'' \ge 0$ every single z value in the indicated domain and $\psi(z) = 0$ at $z = 1$, in turn. Along with ψ' is locally absolutely continuous in $[\mu,\nu]$. Then there exist the constants $L, U \in \Re$ with $L < U$, that is to say $L \le \psi''(z) \le U,\ \forall\, z \in [\mu,\nu]$. If $X, Y \in \digamma_m$ is in a manner that $0 < \mu \le 1 \le \nu < \infty, \mu \ne \nu$, Following that, there is*

$$|JS_\psi(X,Y) - P_{\psi'}(X,Y)| \le \frac{B_D}{32}CS_\psi(X,Y), \tag{2.2}$$

where $JS_\psi(X,Y), CS_\psi(X,Y)$, U and L have their usual meaning respectively, and $P_{\psi'}(X,Y) = \sum_{n=1}^{m}\frac{(x_n-y_n)}{4}\psi'\left(\frac{x_n+y_n}{2y_n}\right)$, $B_D = U - L$= Difference of upper and lower bounds. Also, the constant $\frac{1}{32}$ is best possible.

Proof. Replace f by ψ', b by $z \in [\mu,\nu]$ and a by 1 in inequality 2.1, we have

$$\left|\frac{\psi'(1)+\psi'(z)}{2} - \frac{1}{z-1}\int_1^z \psi'(z)\,dz\right| \le \frac{1}{8}(z-1)B_D, i.e.,$$

$$\left|\psi(z) - \frac{1}{2}(z-1)\left[\psi'(z)+\psi'(1)\right]\right| \le \frac{1}{8}(z-1)^2 B_D.$$

After replacing z by $\frac{x_n+y_n}{2y_n}, n = 1,2,3...,m$, we get

$$\left|\psi\left(\frac{x_n+y_n}{2y_n}\right) - \frac{(x_n-y_n)}{4y_n}\left[\psi'\left(\frac{x_n+y_n}{2y_n}\right)+\psi'(1)\right]\right| \le \frac{B_D}{32}\frac{(x_n-y_n)^2}{y_n^2}.$$

Now multiply the expression by y_n and then sum over all from $n = 1$ to $n = m$ by taking into consideration that $\sum_{n=1}^{m} x_n = \sum_{n=1}^{m} y_n = 1$, we obtain the required result

$$\left|\sum_{n=1}^{m} y_n \psi\left(\frac{x_n+y_n}{2y_n}\right) - \frac{1}{2}\sum_{n=1}^{m} \frac{(x_n-y_n)}{2}\psi'\left(\frac{x_n+y_n}{2y_n}\right)\right| \le \frac{B_D}{32}\sum_{n=1}^{m}\frac{(x_n-y_n)^2}{y_n}, i.e.,$$

$$|JS_\psi(X,Y) - P_{\psi'}(X,Y)| \le \frac{B_D}{32} CS_\psi(X,Y).$$

□

3 Some bounds and relations

The inequality (2.2) concerning the functional divergence of Jain-Saraswat $JS_\psi(X,Y)$ was deduced in the preceding section. We are now going to employ this inequality to establish the limits of a few prominent divergence measures and some intriguing relationships between the measures. The divergence measures in concern fall to the $JS_\psi(X,Y)$ family. ***(Relation among Relative Arithmetic- Geometric divergence measure, Sibson's disparity, and Pearson's divergence measure):***
Let the function be $\psi(z) = z\log z, z \in \Re^+$. It is plain to see $\psi(1) = 0, \psi'(z) = 1 + \log z, \psi''(z) = \frac{1}{z} > 0 \ \forall\ z > 0$, accordingly, the domain-wide normalized function ψ is strictly convex.In lieu of the data prior to this, we have even more

$$JS_\psi(X,Y) = \sum_{n=1}^{m}\left(\frac{x_n+y_n}{2}\right)\log\left(\frac{x_n+y_n}{2y_n}\right) = RAG_\psi(Y,X), \tag{3.1}$$

where, $RAG_\psi(Y,X)$ is Relative Arithmetic- Geometric divergence measure [24], and

$$\begin{aligned}P_{\psi'}(X,Y) &= \sum_{n=1}^{m}\left(\frac{x_n-y_n}{4}\right)\left[1+\log\frac{x_n+y_n}{2y_n}\right] = \sum_{n=1}^{m}\left(\frac{x_n-y_n}{4}\right)\log\frac{x_n+y_n}{2y_n}\\ &= \sum_{n=1}^{m}\left(\frac{y_n-x_n}{4}\right)\log\frac{2y_n}{x_n+y_n} = \sum_{n=1}^{m}\left(\frac{y_n}{2}-\frac{x_n+y_n}{4}\right)\log\frac{2y_n}{x_n+y_n}\\ &= \sum_{n=1}^{m}\left[\frac{y_n}{2}\log\frac{2y_n}{x_n+y_n} + \left(\frac{x_n+y_n}{4}\right)\log\frac{x_n+y_n}{2y_n}\right] = \frac{1}{2}RJS_\psi(Y,X) + \frac{1}{2}RAG_\psi(Y,X),\end{aligned} \tag{3.2}$$

where $\sum_{n=1}^{m} y_n \log\frac{2y_n}{x_n+y_n} = RJS_\psi(Y,X)$= Sibson's disparity or Relative Jensen- Shannon divergence measure [29].
Furthermore, $\psi'''(z) = -\frac{1}{z^2} < 0$ for all $z > 0$, i.e., the function ψ'' is solely falling in the vicinity $[\mu,\nu] \subset (0,\infty)$, so

$$L = \psi''(\nu) = \frac{1}{\nu} \text{ and } U = \psi''(\mu) = \frac{1}{\mu}. \tag{3.3}$$

If we evaluate the inequality (2.2) employing the quantities that have been extracted coming from the equations (3.1), (3.2), and (3.3), we accomplish the anticipated result:

$$|RAG_\psi(Y,X) - RJS_\psi(Y,X)| \le \frac{1}{16}\left(\frac{\nu-\mu}{\mu\nu}\right) CS_\psi(X,Y).$$

Since, these all three measures are non- symmetric in nature with respect to the probability distributions X and Y, so we can put the obtained result in the following form as well:

$$|RAG_\psi(X,Y) - RJS_\psi(X,Y)| \le \frac{1}{16}\left(\frac{\nu-\mu}{\mu\nu}\right) CS_\psi(Y,X). \tag{3.4}$$

(A connection with Relative J- divergence measure, Triangular divergence measure, and Pearson's divergence measure):

Let the function be $\psi(z) = (z-1)\log z, z \in \Re^{+}$. *It is apparent* $\psi(1) = 0, \psi'(z) = \frac{z-1}{z} + \log z, \psi''(z) = \frac{1+z}{z^2} > 0 \ \forall\ z > 0$, *hence the function* ψ *is normalized and strictly convex in the domain. In spite of the data above, we additionally have*

$$JS_{\psi}(X,Y) = \frac{1}{2}\sum_{n=1}^{m}(x_n - y_n)\log\left(\frac{x_n+y_n}{2y_n}\right) = \frac{1}{2}RJ_{\psi}(X,Y), \tag{3.5}$$

where $RJ_{\psi}(X,Y)$ *is the Relative J- divergence measure [30], and*

$$P_{\psi'}(X,Y) = \frac{1}{4}\left[\sum_{n=1}^{m}\frac{(x_n-y_n)^2}{x_n+y_n} + \sum_{n=1}^{m}(x_n-y_n)\log\left(\frac{x_n+y_n}{2y_n}\right)\right]$$
$$= \frac{1}{4}\left[TD_{\psi}(X,Y) + RJ_{\psi}(X,Y)\right]. \tag{3.6}$$

Likewise, $\psi'''(z) = -\frac{2+z}{z^3} < 0$ *for all* $z > 0$, *i.e., the function* ψ'' *is necessarily dropping in dimension* $[\mu,\nu] \subset (0,\infty)$, *so*

$$L = \psi''(\nu) = \frac{1+\nu}{\nu^2} \text{ and } U = \psi''(\mu) = \frac{1+\mu}{\mu^2}. \tag{3.7}$$

Enter the information generated by the equations (3.5), (3.6), and (3.7) via the inequality (2.2), the anticipated result is made:

$$\left|RJ_{\psi}(X,Y) - TD_{\psi}(X,Y)\right| \le \frac{1}{8}\frac{(\nu-\mu)(\mu+\nu+\mu\nu)}{\mu^2\nu^2}CS_{\psi}(X,Y). \tag{3.8}$$

(Association of Sibson's disparity, Harmonic mean divergence measure, and Pearson's divergence measure):
Let the function be $\psi(z) = -\log z, z \in \Re^{+}$. *It is obvious* $\psi(1) = 0, \psi'(z) = -\frac{1}{z}, \psi''(z) = \frac{1}{z^2} > 0 \ \forall\ z > 0$, *this means the domain-wide normalized function* ψ *is strictly convex. Additionally, for the data preceding, there's*

$$JS_{\psi}(X,Y) = -\sum_{n=1}^{m} y_n \log\frac{x_n+y_n}{2y_n} = \sum_{n=1}^{m} y_n\log\frac{2y_n}{x_n+y_n} = RJS_{\psi}(Y,X), \tag{3.9}$$

and

$$P_{\psi'}(X,Y) = -\frac{1}{4}\sum_{n=1}^{m}(x_n-y_n)\frac{2y_n}{x_n+y_n} = \frac{1}{2}\sum_{n=1}^{m}\frac{(y_n-x_n)y_n}{x_n+y_n} = \frac{1}{2}\sum_{n=1}^{m}\frac{y_n^2 - 2x_ny_n + x_ny_n}{x_n+y_n}$$
$$= \frac{1}{2}\sum_{n=1}^{m}y_n - \frac{1}{2}\sum_{n=1}^{m}\frac{2x_ny_n}{x_n+y_n} = \frac{1}{2} - \frac{1}{2}HM_{\psi}(X,Y) = \frac{1}{2}\left[1 - HM_{\psi}(X,Y)\right], \tag{3.10}$$

where $\sum_{n=1}^{m}\frac{2x_ny_n}{x_n+y_n} = HM_{\psi}(X,Y)$ *is the Harmonic mean divergence measure [23].*
Even, $\psi'''(z) = -\frac{2}{z^3} < 0$ *for all* $z > 0$, *i.e., the function* ψ'' *is consistently reducing in the space* $[\mu,\nu] \subset (0,\infty)$, *so*

$$L = \psi''(\nu) = \frac{1}{\nu^2} \text{ and } U = \psi''(\mu) = \frac{1}{\mu^2}. \tag{3.11}$$

Once the inputs using the three equations (3.9), (3.10), and (3.11) are plugged into the inequality (2.2), the result that is wanted is obtained:

$$\left|RJS_{\psi}(Y,X) - \frac{1}{2}\left[1 - HM_{\psi}(X,Y)\right]\right| \le \frac{1}{32}\left(\frac{1}{\mu^2} - \frac{1}{\nu^2}\right)CS_{\psi}(X,Y).$$

We can additionally state our findings in an alternative fashion as the measures $RJS_{\psi}(Y,X), CS_{\psi}(X,Y)$ *are Asymmetric, and* $HM_{\psi}(X,Y)$ *is symmetric with regard to the probability distributions* X *and* Y, *accordingly:*

$$\left|RJS_{\psi}(X,Y) - \frac{1}{2}\left[1 - HM_{\psi}(X,Y)\right]\right| \le \frac{1}{32}\left(\frac{1}{\mu^2} - \frac{1}{\nu^2}\right)CS_{\psi}(Y,X). \tag{3.12}$$

(Bounds of the Taneja divergence associated with the Pearson's divergence measure): *Let the function be* $\psi(z) = z\log\left(\frac{z}{\sqrt{2z-1}}\right), z > \frac{1}{2}$*. It can be noted that* $\psi(1) = 0, \psi'(z) = \log\left(\frac{z}{\sqrt{2z-1}}\right) + \frac{z-1}{2z-1}, \psi''(z) = \frac{2z^2-2z+1}{z(1-2z)^2} > 0 \ \forall \ z > \frac{1}{2}$*, thus the function* ψ *is normalized and strictly convex in the domain. Besides the data above, we possess*

$$JS_\psi(X,Y) = \sum_{n=1}^{m}\left(\frac{x_n+y_n}{2}\right)\log\frac{x_n+y_n}{2\sqrt{x_n y_n}} = AGM_\psi(X,Y), \tag{3.13}$$

and

$$P_{\psi'}(X,Y) = \frac{1}{4}\sum_{n=1}^{m}(x_n-y_n)\left[\sqrt{\frac{y_n}{x_n}}\log\frac{x_n+y_n}{2y_n} + \frac{x_n-y_n}{2x_n}\right] = A_\psi(X,Y) + \frac{1}{8}CS_\psi(Y,X), \tag{3.14}$$

where $\frac{1}{4}\sum_{n=1}^{m}(x_n-y_n)\sqrt{\frac{y_n}{x_n}}\log\frac{x_n+y_n}{2y_n} = A_\psi(X,Y)$.
Moreover, $\psi'''(z) = -\frac{4z^3-6z^2+6z-1}{z^2(2z-1)^3} < 0$ *for all* $z > \frac{1}{2}$*, i.e., the function* ψ'' *is definitely lowering in terms of area* $[\mu,\nu] \subset \left(\frac{1}{2},\infty\right)$*, so*

$$L = \psi''(\nu) = \frac{2\nu^2-2\nu+1}{\nu(1-2\nu)^2} \text{ and } U = \psi''(\mu) = \frac{2\mu^2-2\mu+1}{\mu(1-2\mu)^2}. \tag{3.15}$$

When the values from the equations (3.13), (3.14), and (3.15) are applied to the inequality (2.2), the desired outcome is achieved:

$$\left|AGM_\psi(X,Y) - \frac{1}{8}CS_\psi(Y,X) - A_\psi(X,Y)\right| \le \frac{1}{32}\left(\frac{2\mu^2-2\mu+1}{\mu(1-2\mu)^2} - \frac{2\nu^2-2\nu+1}{\nu(1-2\nu)^2}\right)CS_\psi(X,Y). \tag{3.16}$$

(A relationship involving Information radius, Relative J- divergence measure, and Pearson's divergence measure):
Let the function be $\psi(z) = \left(z-\frac{1}{2}\right)\log(2z-1) - z\log z, z > \frac{1}{2}$*. It is evident that* $\psi(1) = 0, \psi'(z) = \log(2z-1) - \log z = \log\left(\frac{2z-1}{z}\right) = \log\left(2-\frac{1}{z}\right), \psi''(z) = \frac{1}{z(2z-1)} > 0 \ \forall \ z > \frac{1}{2}$*, in turn, the domain-wide normalized function* ψ *is strictly convex. Along with the data above, we have*

$$JS_\psi(X,Y) = \frac{1}{2}\left[\sum_{n=1}^{m}x_n\log\left(\frac{2x_n}{x_n+y_n}\right) + \sum_{n=1}^{m}y_n\log\left(\frac{2y_n}{x_n+y_n}\right)\right] = JSH_\psi(X,Y), \tag{3.17}$$

where $JSH_\psi(X,Y)$ *is the Information radius or Jensen- Shannon divergence measure [16, 29]. And*

$$\begin{aligned} P_{\psi'}(X,Y) &= \frac{1}{4}\sum_{n=1}^{m}(x_n-y_n)\log\left(2-\frac{2y_n}{x_n+y_n}\right) = \frac{1}{4}\sum_{n=1}^{m}(x_n-y_n)\log\left(\frac{2x_n}{x_n+y_n}\right) \\ &= \frac{1}{4}\sum_{n=1}^{m}(y_n-x_n)\log\left(\frac{x_n+y_n}{2x_n}\right) = \frac{1}{4}RJ_\psi(Y,X). \end{aligned} \tag{3.18}$$

In addition, $\psi'''(z) = \frac{1-4z}{z^2(2z-1)^2} < 0 \ \forall \ z > \frac{1}{2}$*, i.e., the function* ψ'' *is rigorously diminishing in extent* $[\mu,\nu] \subset \left(\frac{1}{2},\infty\right)$*, so*

$$L = \psi''(\nu) = \frac{1}{\nu(2\nu-1)} \text{ and } U = \psi''(\mu) = \frac{1}{\mu(2\mu-1)}. \tag{3.19}$$

Put the obtained values from the equations (3.17), (3.18), and (3.19) into the inequality (2.2), we have the required result

$$|4JSH_\psi(X,Y) - RJ_\psi(Y,X)| \le \frac{1}{8}\left[\frac{\nu(2\nu-1)-\mu(2\mu-1)}{\mu\nu(2\mu-1)(2\nu-1)}\right]CS_\psi(X,Y). \tag{3.20}$$

(Bounds of the Hellinger divergence measure relating to the Pearson's divergence measure):

Let the function be $\psi(z) = \frac{\left(1-\sqrt{2z-1}\right)^2}{2}, z > \frac{1}{2}$. *We can see that* $\psi(1) = 0, \psi'(z) = 1 - \frac{1}{\sqrt{2z-1}}, \psi''(z) = \frac{1}{(2z-1)^{\frac{3}{2}}} > 0\ \forall\ z > \frac{1}{2}$, *as a result, the function* ψ *is normalized and strictly convex in the domain. In furtherance of the data above, we have*

$$JS_{\psi}(X,Y) = \frac{1}{2}\sum_{n=1}^{m}\left(\sqrt{x_n} - \sqrt{y_n}\right)^2 = HD_{\psi}(X,Y), \tag{3.21}$$

and

$$P_{\psi'}(X,Y) = \frac{1}{4}\sum_{n=1}^{m}(x_n - y_n)\left[1 - \frac{\sqrt{y_n}}{\sqrt{x_n}}\right] = \frac{1}{4}\sum_{n=1}^{m}(y_n - x_n)\frac{\sqrt{y_n}}{\sqrt{x_n}} = D_{\psi}(X,Y). \tag{3.22}$$

Also, $\psi'''(z) = \frac{-3}{(2z-1)^{\frac{5}{2}}} < 0\ \forall\ z > \frac{1}{2}$, *i.e., the function* ψ'' *is absolutely declining in the area* $[\mu,\nu] \subset \left(\frac{1}{2},\infty\right)$, *so*

$$L = \psi''(\nu) = \frac{1}{(2\nu-1)^{\frac{3}{2}}} \text{ and } U = \psi''(\mu) = \frac{1}{(2\mu-1)^{\frac{3}{2}}}. \tag{3.23}$$

The intended result is obtained when we solve the inequality (2.2) with the values that were derived from the equations (3.21), (3.22), and (3.23):

$$\left|HD_{\psi}(X,Y) - D_{\psi}(X,Y)\right| \leq \frac{1}{32}\left[\frac{(2\nu-1)^{\frac{3}{2}} - (2\mu-1)^{\frac{3}{2}}}{(2\mu-1)^{\frac{3}{2}}(2\nu-1)^{\frac{3}{2}}}\right]CS_{\psi}(X,Y). \tag{3.24}$$

(Bounds of the Jain- Srivastava divergence based on the Pearson's divergence measure):

Let the function be $\psi(z) = \frac{4(z-1)^2}{\sqrt{2z-1}}, z > \frac{1}{2}$. *It can be observed that* $\psi(1) = 0, \psi'(z) = \frac{4\left(3z^2-4z+1\right)}{(2z-1)^{\frac{3}{2}}}, \psi''(z) = \frac{4\left(3z^2-2z+1\right)}{(2z-1)^{\frac{5}{2}}} > 0\ \forall\ z > \frac{1}{2}$, *consequently, the domain-wise normalized function* ψ *is strictly convex. In addition to the data above, we have*

$$JS_{\psi}(X,Y) = \sum_{n=1}^{m}\frac{(x_n - y_n)^2}{\sqrt{x_n y_n}} = JS_{\psi}^{R}(X,Y), \tag{3.25}$$

where $JS_{\psi}^{R}(X,Y)$ *is the Jain- Srivastava variational measure [31]. And*

$$P_{\psi'}(X,Y) = \sum_{n=1}^{m}(x_n - y_n)\left[\frac{3\left(\frac{x_n+y_n}{2y_n}\right)^2 - 4\left(\frac{x_n+y_n}{2y_n}\right) + 1}{\left(2\frac{x_n+y_n}{2y_n} - 1\right)^{\frac{3}{2}}}\right] = \sum_{n=1}^{m}(x_n - y_n)\left[\frac{3x_n^2 - 2x_n y_n - y_n^2}{4x_n^{\frac{3}{2}}y_n^{\frac{1}{2}}}\right]$$

$$= \sum_{n=1}^{m}\frac{(x_n - y_n)^2(3x_n + y_n)}{4x_n^{\frac{3}{2}}y_n^{\frac{1}{2}}} = \frac{3}{4}JS_{\psi}^{R}(X,Y) + \frac{1}{4}E_{\psi}(X,Y), \tag{3.26}$$

where $\sum_{n=1}^{m}\frac{\sqrt{y_n}(x_n-y_n)^2}{x_n^{\frac{3}{2}}} = E_{\psi}(X,Y)$.

Further, $\psi'''(z) = \frac{-12\left(z^2+1\right)}{(2z-1)^{\frac{7}{2}}} < 0\ \forall\ z > \frac{1}{2}$, *i.e., the function* ψ'' *is strictly decreasing in the region* $[\mu,\nu] \subset \left(\frac{1}{2},\infty\right)$, *so*

$$L = \psi''(\nu) = \frac{4\left(3\nu^2 - 2\nu + 1\right)}{(2\nu-1)^{\frac{5}{2}}} \text{ and } U = \psi''(\mu) = \frac{4\left(3\mu^2 - 2\mu + 1\right)}{(2\mu-1)^{\frac{5}{2}}}. \tag{3.27}$$

When we solve the inequality (2.2) using the values that were obtained from the equations (3.25), (3.26), and (3.27), we achieve the desired outcome:

$$\left|JS_{\psi}^{R}(X,Y) - E_{\psi}(X,Y)\right| \leq \frac{1}{2}\left[\frac{\left(3\mu^2 - 2\mu + 1\right)}{(2\mu-1)^{\frac{5}{2}}} - \frac{\left(3\nu^2 - 2\nu + 1\right)}{(2\nu-1)^{\frac{5}{2}}}\right]CS_{\psi}(X,Y). \tag{3.28}$$

(Bounds of the Directed divergence identified through the Pearson's divergence measure): *Let's say the function is* $\psi(z) = (2z-1)\log(2z-1), z > \frac{1}{2}$. *It is evident that* $\psi(1) = 0, \psi'(z) = 2[1+\log(2z-1)], \psi''(z) = \frac{4}{(2z-1)} > 0 \ \forall\ z > \frac{1}{2}$, *consequently, the domain-wise normalized function* ψ *is strictly convex. Along with the information above, we also have*

$$JS_\psi(X,Y) = \sum_{n=1}^{m} x_n \log \frac{x_n}{y_n} = DD_\psi(X,Y), \tag{3.29}$$

where $DD_\psi(X,Y)$ *is the Directed divergence or Relative entropy or Kullback- Leibler divergence [32]. And*

$$P_{\psi'}(X,Y) = \frac{1}{2}\sum_{n=1}^{m}(x_n - y_n)\left[1+\log\left(2\times\frac{x_n+y_n}{2y_n}-1\right)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{m}(x_n-y_n) + \frac{1}{2}\sum_{n=1}^{m}(x_n-y_n)\log\frac{x_n}{y_n} = \frac{1}{2}\left[DD_\psi(X,Y) + DD_\psi(Y,X)\right]. \tag{3.30}$$

Further, $\psi'''(z) = \frac{-8}{(2z-1)^2} < 0 \ \forall\ z > \frac{1}{2}$, *i.e., the function* ψ'' *is firmly descending in the region* $[\mu,\nu] \subset \left(\frac{1}{2},\infty\right)$, *so*

$$L = \psi''(\nu) = \frac{4}{(2\nu-1)} \text{ and } U = \psi''(\mu) = \frac{4}{(2\mu-1)}. \tag{3.31}$$

The desired result is reached when the inequality (2.2) is solved with the values obtained from the equations (3.29), (3.30), and (3.31):

$$|DD_\psi(X,Y) - DD_\psi(Y,X)| \leq \frac{(\nu-\mu)}{2(2\mu-1)(2\nu-1)} CS_\psi(X,Y). \tag{3.32}$$

(Bounds of the Jain- Chhabra exponential divergence in terms of the Pearson's divergence measure):
Let's say the function is $\psi(z) = 2(z-1)e^{2z-1}, z > 0$. *Obviously,* $\psi(1) = 0, \psi'(z) = 2(2z-1)e^{2z-1}, \psi''(z) = 8ze^{2z-1} > 0 \ \forall\ z > 0$, *so the given function is normalized and strictly convex in its domain. We also have*

$$JS_\psi(X,Y) = \sum_{n=1}^{m}(x_n-y_n)e^{\frac{x_n}{y_n}} = G_\psi^{exp}(X,Y), \tag{3.33}$$

where $G_\psi^{exp}(X,Y)$ *is the Jain- Chhabra divergence [33], which is exponential in nature. And*

$$P_{\psi'}(X,Y) = \frac{1}{2}\sum_{n=1}^{m}\frac{x_n(x_n-y_n)e^{\frac{x_n}{y_n}}}{y_n} = F_\psi(X,Y). \tag{3.34}$$

Also, $\psi'''(z) = 8(1+2z)e^{2z-1} > 0 \ \forall\ z > 0$, *i.e., the function* ψ'' *is strictly increasing in the domain* $[\mu,\nu] \subset (0,\infty)$, *so*

$$L = \psi''(\mu) = 8\mu e^{2\mu-1} \text{ and } U = \psi''(\nu) = 8\nu e^{2\nu-1}. \tag{3.35}$$

Put the data from the equations (3.33), (3.34), and (3.35), into the relation (2.2), we have the following result:

$$\left|G_\psi^{exp}(X,Y) - F_\psi(X,Y)\right| \leq \frac{\left(\nu e^{2\nu-1} - \mu e^{2\mu-1}\right)}{4} CS_\psi(X,Y). \tag{3.36}$$

4 Conclusion remarks

Using an estimate of Pearson's divergence, this study presents a fresh inequality on the differences between Jain and Saraswat. Additionally, the bounds and significant relations of the commonly known statistical divergences have been determined using this inequality. As a result, the findings of this work are diverse, important, and intriguing, making it suitable for further research.

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