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A Study of Multidimensional Fractional Calculus Operators

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Abstract In this paper, we introduce and study a pair of multidimensional fractional calculus operators whose kernel involves the product of general class of polynomials $S_{V_j}^{U_j}(x_j)$, general sequence of functions $R_{N_j}^{\alpha_j,\beta_j}[x_j](j = 1, \ldots, s)$ and the multivariable H -function.Further we obtain the images of certain useful functions in our operators of study. The fractional calculus operators studied by us are quite general in nature and thus the results obtained here provide useful extensions and unification of some (known or new) results for simpler families of fractional calculus operators.

1 Introduction

The following definitions of general sequence of functions and multivariable integral transforms will be used in the subsequent section.

1.1 GENERAL SEQUENCE OF FUNCTIONS

Agarwal and Chaubey [1] see also [15] have introduced and studied a general sequence of functions which is defined by the following Rodrigue's type formula

$$\mathrm{R}_{\mathrm{N}}^{(\alpha,\beta)}[\mathrm{x};\mathrm{A},\ \mathrm{B},\mathrm{c},\ \mathrm{d};\mathrm{p},\mathrm{q};\gamma,\delta;\mathrm{w}(\mathrm{x})]$$

$$= \frac{\left(\mathrm{Ax}^{\mathrm{p}} + \mathrm{B}^{-\alpha}\left(\mathrm{cxx}^{\mathrm{q}} + \mathrm{d}\right)^{-\beta}\right.}{\ell_{\mathrm{n}}'\mathrm{w}(\mathrm{x})}\mathrm{T}_{\lambda,\ell}^{\mathrm{N}}\left[\left(Ax^{\mathrm{p}} + \mathrm{B}\right)^{\alpha+\gamma\mathrm{N}}\left(\mathrm{cx}^{q} + \mathrm{d}\right)^{\beta+\delta\mathrm{N}}\mathrm{w}(\mathrm{x})\right] \tag{1.1}$$

where the differential operator $\mathrm{T}_{\lambda,\ell}^{\mathrm{N}}$ is defined by

$$\mathrm{T}_{\lambda,\ell}^{\mathrm{N}}=\left[\mathrm{x}^{\ell}\left(\lambda+\mathrm{xD}_{\mathrm{x}}\right)\right]^{\mathrm{N}},\mathrm{D}_{\mathrm{x}}=\tfrac{\mathrm{d}}{\mathrm{dx}} \tag{1.2}$$

$\left\{\ell_{\mathrm{n}}'\right\}_{\mathrm{N}=0}^{\infty}$ is a sequence of constants, and $\mathrm{w}(\mathrm{x})$ is independent of N and differentiable an arbitrary number of times.
If we set $w(x) = \exp\left(-sx^{r}\right)$, in 1.1, the generalized sequence of function thus obtained can be written in the series form given as follows [12].

$$\mathrm{R}_{\mathrm{N}}^{(\alpha,\beta)}\left[\mathrm{x;A,B,c,d;p,q;}\gamma,\delta;\mathrm{e}^{-\mathrm{sx}^{\mathrm{r}}}\right] = \sum_{m,v',u',t',e'} \phi\left(m,v',u',t',e'\right)X^{Q}e^{sx^{r}}(1+\frac{c}{d}x^{q})^{\delta N-v'} \tag{1.3}$$

where

$$\phi\left(m,v',u',t',e'\right)=\frac{B^{\gamma\mathrm{N}}(-1)^{t'+\mathrm{m}}\left(-v'\right)_{\mathrm{u}'}\left(-\mathrm{t}^{t'}\right)_{e'}(\alpha)_t}{m!v'!t'!\mathrm{e}'!u'!} \tag{1.4}$$

$$\mathrm{Q}=\ell\mathrm{N}+\mathrm{qvv}'+\mathrm{ptt}'+\mathrm{rm} \tag{1.5}$$

$$\sum_{m,v',u',t',e'} \text{ stands for } \sum_{m=0}^{\infty}\sum_{v'=0}^{n}\sum_{u'=0}^{v'}\sum_{t'=0}^{n}\sum_{e'=0}^{t'}$$

provided that the infinite series on the right-hand side of 1.3 is absolutely convergent. It may be pointed out here that the general sequence of functions defined by 1.1 is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [2, 3], Gould and Hopper [6], Krall and Frink [10], Singh and Srivastava [14], Srivastava and Singhal [17, 18], Thakare and Medhakar [19], Raizada [11], Fujiwara [5]etc.

2 MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

The multidimensional fractional integral operators of our study will be defined and represented in the following manner

$$I_x\left[f\left(t_1,\ldots,t_s\right)\right]=I_{x:\ \ a;\ \ b;\ \ z}^{\rho,\sigma:e,f;\lambda,\mu;u,v}\left[f\left(t_1,\ldots,t_s\right);x_1,\ldots,x_s\right] \tag{2.1}$$

$$=\prod_{j=1}^{s}\left(x_j\right)^{-\rho_j-\sigma_j}\int_0^{x_1}\cdots\int_0^{x_s}\prod_{j=1}^{s}\left\{t_j^{\rho_j}\left(x_j-t_j\right)^{\sigma_j-1}\right.$$

$$\left.S_{v_j}^{U_j}\left[a_j\left(\frac{t_j}{x_j}\right)^{e_j}\left(1-\frac{t_j}{x_j}\right)^{f}\right]R_{N_j}^{\alpha_j,\beta_j}\left[b_j\left(\frac{t_j}{x_j}\right)^{\lambda_j}\left(1-\frac{t_j}{x_j}\right)^{\mu_j}\right]\right\}$$

$$\times H\left[z_1\left(\frac{t_1}{x_1}\right)^{u_1}\left(1-\frac{t_1}{x_1}\right)^{v_1},\ldots,z_s\left(\frac{t_s}{x_s}\right)^{u_s}\left(1-\frac{t_s}{x_s}\right)^{v_s}\right]f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s$$

$$J_x\left[f\left(t_1,\ldots,t_s\right)\right]=J_x^{\rho,\sigma:e,f;\lambda,\mu;u,v}\left[f\left(t_1,\ldots,t_s\right);x_1,\ldots,x_s\right] \tag{2.2}$$

$$=\prod_{j=1}^{s}\left(x_j\right)^{+\rho_j}\int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty}\prod_{j=1}^{s}\left\{t_j^{-\rho_j-\sigma_j(t_j-x_j)^{\sigma_j-1}}\right.$$

$$\left.S_{v_j}^{U_j}\left[a_j\left(\frac{x_j}{t_j}\right)^{e_j}\left(1-\frac{x_j}{t_j}\right)^{f_j}\right]R_{N_j}^{\alpha_j,\beta_j}\left[b_j\left(\frac{x_j}{t_j}\right)^{\lambda_j}\left(1-\frac{x_j}{t_j}\right)^{\mu_j}\right]\right\}$$

$$\times H\left[z_1\left(\frac{x_1}{t_1}\right)^{u_1}\left(1-\frac{x_1}{t_1}\right)^{v_1},\ldots,z_s\left(\frac{x_s}{t_s}\right)^{u_s}\left(1-\frac{x_s}{t_s}\right)^{v_s}\right]f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s$$

where the general class of polynomials $S_{V_j}^{U_j}\left[x_j\right](j=1,\ldots,s)$, a general sequence of functions $R_{N_j}^{\alpha_j,\beta_j}\left[x_j\right](j=1,\ldots,s)$ and the multivariable H-function occurring in 2.1 and 2.2.
We shall assume throughout this paper that

$$f\left(t_1,\ldots,t_s\right)=\begin{cases}0\left(\prod_{j=1}^{s}|t_j|^{M_j}\right),\max|t_j|\to 0\\ 0\left(\prod_{j=1}^{s}|t_j|^{-A_j}e^{-w_j|t_j|}\right),\min|t_j|\to\infty(j=1,\ldots,s)\end{cases} \tag{2.3}$$

This class of functions will be symbolically represented as $f\left(t_1,\ldots,t_s\right)\in$ A.
We shall also assume that

$$\int_{\Omega_1}\cdots\int_{\Omega_s}\left|\mathrm{f}\left(\mathrm{t}_1,\ldots,\mathrm{t}_s\right)\right|\mathrm{dt}_1\ldots\mathrm{dt}_s<\infty$$

for every bounded s-dimensional region Ω_j (j = 1, . . . , s) excluding the origin.

The operators defined by 2.1 and 2.2 exist if
(i) min $(e_j, f_j, \lambda_j, \mu_j, u_j, v_j) \geq 0 (j = 1, \dots, s)$ not all zero simultaneously.
(ii) minRe $\left(\rho_j + \lambda_j \left(\ell_j N_j + q_j V_j'\right) + M_j\right) + u_j \min_{1\leq k\leq m_j} \left[\mathrm{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(i)}}\right)\right] + 1 > 0$

$$\mathrm{minRe}\left[\sigma_j + \mu_j \left(\ell_j N_j + q_j v_j'\right) + v_j \min_{1\leq k\leq m_j} \left[\mathrm{Re}\left(\frac{d_k^{(i)}}{\delta_k^{(j)}}\right)\right]\right] > 0$$

(iii) minRe $\left[\sigma_j + \mu_j \left(\ell_j N_j + q_j v_j'\right) + v_j \min_{1\leq k\leq m_j} \left[\mathrm{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(j)}}\right)\right]\right] > 0$

$$\mathrm{Re}\,(\mathrm{W_j}) > 0$$
$$\text{or } \mathrm{Re}\,(\mathrm{W_j}) = 0 \text{ and}$$
$$\mathrm{Re}\left[\rho_j + \lambda_j \left(\ell_j N_j + q_j v_j'\right) + A_j\right) + u_j \min_{1\leq k\leq m_j} \left[\mathrm{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(j)}}\right)\right] > 0$$
$$\left(v_j' = 0, \dots, N_j; j = 1, \dots, s\right)$$

If we take $V_j = 0 (j = 1, \dots, s)$, $p_j = d_j = 1$, $\ell'_{n_j} = 1$, $s_j = 0$ and replace by β_j by β_j/τ_j and c_j by $-\tau_j$ in 2.1 and 2.2, the general class of polynomials $S_{V_j}^{U_j}[x_j]$ reduces to unity and general sequence of functions $R_{N_j}^{\alpha_j,\beta_j}[x_j]$ to $S_{N_j}^{\alpha_j,\beta_j,\tau_j}[x_j]$ and we obtain the fractional integral operators defined by Garg and Purohit [7].

3 SOME USEFUL IMAGES

(i) $I_x\left\{\prod_{j=1}^{s} t_j^{g_j}(t_j+w_j)^{-h_j}\right\} = \prod_{j=1}^{s}\left\{\frac{x_j^{g_j}(x_j+w_j)^{-b_j}}{\Gamma(h_j)}\right.$

$$\sum_{R_j=0}^{[v_j/U_j]} \sum_{\eta_j,m_j,v_j,u_j',t_j',e_j'} \frac{(-V_j)_{U_jR_j} A_{jv_jR_j}}{R_j!} \frac{a_j^{R_j} b_j Q_j' s_j' \eta_j}{\eta_j!\Gamma\left(v_j' - \delta_j N_j\right)} \phi_j(m_j, v_j, u_j, t_j, e_j) \quad (3.1)$$
$$* H^{0,n+2s\,;K}_{p+2s,q+s;L}\left[z_1, \dots, z_s, \frac{c_1}{d_1} b_1^{q_1}, \dots, \frac{c_s}{d_s} b_s^{q_s}, \frac{-x_1}{x_1+w_1}, \dots, \frac{-x_s}{x_s+w_s}\Big|\begin{matrix} A: & C \\ B: & D\end{matrix}\right]$$

where $\sum_{\eta_j,m_j,v_j',u_j',t_j',e_j'}$ stands for $\sum_{\eta_j=0}^{\infty} \sum_{m_j,v_j',\mathbf{u}_j',t_j',e_j'}$
$Q_j' = \ell_j N_j + q_j v_j' + p_j t_j' + r_j m_j + r_j n_j (j = 1, \dots, s)$ and $\phi_j (j = 1, \dots, s)$ is defined as in equation 1.4 with appropriate changes in the various symbols.

$$\mathrm{K} = m_1, n_1; \dots; m_s, n_s; \underbrace{1, 1; \dots; 1, 1}_{2\,s} \quad (3.2)$$

$$\mathrm{L} = p_1, q_1; \dots; p_s, q_s; \underbrace{1, 1; \dots; 1, 1}_{2\,s} \quad (3.3)$$

$$\begin{aligned} A = &(a_j, \alpha_j^{(1)}, \dots, \alpha_j^{(s)}, \underbrace{0, \dots, 0}_{2s})_{1,n}, (-\rho_1 - g_1 - e_1 R_1 - \lambda_1 Q_1'; u_1, \underbrace{0, \dots, 0}_{s-1}, \lambda_1 q_1, \\ &\underbrace{0, \dots, 0}_{2s-1}), \dots, (-\rho_s - g_s - e_s R_s - \lambda_s Q_s'; \underbrace{0, \dots, 0}_{s-1}, u_s, \underbrace{0, \dots, 0}_{s-1}, \lambda_s q_s, \underbrace{0, \dots, 0}_{s}), \\ &(1 - \sigma_1 - f_1 R_1 - \mu_1 Q_1'; v_1, \underbrace{0, \dots 0}_{s-1}, \mu_1 q_1, \underbrace{0, \dots, 0}_{s-1}, \underbrace{0, \dots, 0}_{s-1}), \dots, \\ &(1 - \sigma_s - f_s R_s - \mu_s Q_s', \underbrace{0, \dots, 0}_{s-1}, v_s, \underbrace{0, \dots, 0}_{s-1}, \mu_s q_s, \underbrace{0, \dots, 0, 1}_{s-1}) \\ &(a_j, \alpha_j^{(1)}, \dots, \alpha_j^{(s)}, \underbrace{0, \dots, 0}_{2s})_{n+1,p} \end{aligned} \quad (3.4)$$

$$B = (b_j, \beta_j^{(1)}, \ldots, \beta_j^{(s)}, \underbrace{0, \ldots, 0}_{2s})_{1,q}, \Big(-\rho_1 - g_1 - \sigma_1 - (e_1 + f_1) R_1 - (\lambda_1 + \mu_1) Q_1^{'};$$
$$(u_1 + v_1) \underbrace{0, \ldots, 0}_{s-1}, (\lambda_1 + \mu_1) q_1, \underbrace{0, \ldots, 0}_{s-1}, \underbrace{1, 0, \ldots, 0}_{s-1}), \ldots, (-\rho_s - g_s - \sigma_s - (e_s + f_s) R_s \tag{3.5}$$
$$-(\lambda_s + \mu_s) Q_s^{'}; \underbrace{0, \ldots, 0}_{s-1}, (u_s + v_s), \underbrace{0, \ldots, 0}_{s-1}, (\lambda_s + \mu_s) q_s, \underbrace{0, \ldots, 0, 1}_{s-1})$$

$$C = \Big(c_j^{(1)}, \gamma_j^{(1)}\Big)_{1,p_1}; \ldots; \Big(c_j^{(s)}, \gamma_j^{(s)}\Big)_{1,p_s}; \big(1 - v_1^1 + \delta_1 N_1, 1\big), \ldots, \# \tag{3.6}$$
$$\big(1 - v_s^1 + \delta_s N_s, 1\big); (1 - h_1, 1), \ldots, (1 - h_s, 1)$$

$$D = \Big(d_j^{(1)}, \delta_j^{(1)}\Big)_{1,q_1}; \ldots; \Big(d_j^{(s)}, \delta_j^{(s)}\Big)_{1,q_s}; \underbrace{(0,1); \ldots; (0,1)}_{2s} \tag{3.7}$$

The result holds true provided that

$$\min(e_j, f_j, \lambda_j, \mu_j, u_j, v_j, q_j) \geq 0 (\forall j = 1, \ldots, \mathrm{s}) \text{ not all zero simultaneously.}$$

$$\mathrm{Re}\,(\rho_j + \lambda_j \ell_j N_j + g_j) + u_j \min_{1 \leq k \leq m_j} \left[\mathrm{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(j)}}\right)\right] + 1 > 0$$
$$\mathrm{Re}\,(\sigma_j + \mu_j \ell_j N_j) + v_j \min_{1 \leq k \leq m_j} \left[\mathrm{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(j)}}\right)\right] > 0 \tag{3.8}$$
$$(\mathrm{j} = 1, \ldots, \mathrm{s})$$

(ii) $$J_x\left[\prod_{j=1}^{s} t_j g_j (t_j + w_j)^{-h_j}\right] = \prod_{j=1}^{s} \left\{\frac{x_j^{g_j}(x_j + w_j)^{-h_j}[v_j/U_j]}{\Gamma(h_j)} \sum_{R_j=0}\right.$$
$$\left.\sum_{\eta_j, m_j, v_j, u_j^{'} \,j t_j^{'}, e_j^{'}} \frac{(-V_j)_{U_j R_j} A_{j v_j, R_j}}{R_j!} \frac{a_j^{R_j} b_j Q_j s_j \eta_j}{\eta_j! \Gamma\big(v_j^{'} - \delta_j N_j\big)} \phi_j\Big(m_j, v_j^{'}, u_j^{'}, t_j^{'}, e^{'}\Big)\right\} \tag{3.9}$$
$$H_{p+2s,q+s;L}^{0,n+2s;K}\left[\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{s}, \frac{\mathrm{c}_1}{\mathrm{d}_1} \mathrm{b}_1^{\mathrm{q}_1}, \ldots, \frac{\mathrm{c}_\mathrm{s}}{\mathrm{d}_\mathrm{s}} \mathrm{b}_\mathrm{s}^{\mathrm{q}_\mathrm{s}} - \frac{W_1}{X_1 + W_1}, \ldots, -\frac{W_s}{X_s + W_s} \middle| \begin{matrix} I; & C \\ J; & D \end{matrix}\right]$$

where

$$I = (a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(s)}, \underbrace{0, \ldots, 0}_{2s})_{1,n}, \Big(1 - h_1 - \rho_1 + g_1 - e_1 R_1 - \lambda_1 Q_1^{'};$$
$$u_1, \underbrace{0, \ldots, 0}_{s-1}, \lambda_1 q_1, \underbrace{0, \ldots, 0}_{2s-1}), \ldots, \Big(1 - h_s - \rho_s + g_s - e_s R_s - \lambda_s Q_s^{'};$$
$$\underbrace{0, \ldots, 0}_{s-1}, u_s, \underbrace{0, \ldots, 0}_{s-1}, \lambda_s q_s, \underbrace{0, \ldots, 0}_{s}), \Big(1 - \sigma_1 - f_1 R_1 - \mu_1 Q_1^{'}; \tag{3.10}$$
$$v_1, \underbrace{0, \ldots, 0}_{s-1}, \mu_1 q_1, \underbrace{0, \ldots, 0}_{s-1}, 1, \underbrace{0, \ldots, 0}_{s-1}), \ldots, \Big(1 - \sigma_s - f_s R_s - \mu_s Q_s^{'};$$
$$\underbrace{0, \ldots, 0}_{s-1}, v_s, \underbrace{0, \ldots, 0}_{s-1}, \mu_s q_s, \underbrace{0, \ldots, 0}_{s}, 1), (a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(s)},, \underbrace{0, \ldots, 0}_{2s})_{\mathrm{n+1,p}}$$

$$J = (b_j; \beta_j^{(1)}, \ldots, \beta_j^{(s)}, \underbrace{0, \ldots, 0}_{2s})_{1,q}, (1 - \sigma_1 - \rho_1 - h_1 + g_1 - (e_1 + f_1) R_1$$
$$-(\lambda_1 + \mu_1) Q_1^{'}; (u_1 + v_1), \underbrace{0, \ldots, 0}_{s-1}, (\lambda_1 + \mu_1) q_1, \underbrace{0, \ldots, 0}_{s-1},, \underbrace{0, \ldots, 0}_{s-1}), \ldots, \tag{3.11}$$
$$(1 - \sigma_s - \rho_s - h_s + g_s - (e_s + f_s) R_s - (\lambda_s + \mu_s) Q_s^{'};$$
$$\underbrace{0, \ldots, 0}_{s-1}(u_s + v_s), \underbrace{0, \ldots, 0}_{s-1}(\lambda_s + \mu_s) q_s, \underbrace{0, \ldots, 0}_{s-1}, 1)$$

K, L, C, D, ϕ_j and $Q_j^{'}$(j = 1, . . . , s) are as given with result 3.1 and the following conditions are satisfied.

$$\min\left(e_j, f_j, \lambda_j, \mu_j, u_j, v_j q_j\right) \geq 0 (j = 1, \ldots, s) \text{ not all zero simultaneously.}$$

$$\begin{aligned} \text{Re}\left(\rho_j - g_j + \lambda_j \ell_j N_j + h_j\right) + u_{j_{1 \leq k \leq m_j}} \left[\text{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(j)}}\right)\right] > 0 \\ \text{Re}\left(\sigma_j + \mu_j \ell_j N_j\right) + v_{j_1} \min_{1 \leq k \leq m_j} \left[\text{Re}\left(\frac{d_k^{(j)}}{\delta_k^{(j)}}\right)\right] > 0 \end{aligned} \tag{3.12}$$

$(j = 1, 2, ..., s)$

Proof To prove 3.1, we first express the operator I_x with the help of 2.1 and get

$$\begin{aligned} & I_x \left[\prod_{j=1}^{s} t_j^{g_j} (t_j + w_j)^{-h_j}\right] = \prod_{j=1}^{s} \left(x_j^{-\rho_j - \sigma_j}\right) \int_0^{x_1} \cdots \int_0^{x_s} \prod_{j=1}^{s} \left\{t_j^{\rho_j + g_j}\right. \\ & (x_j - t_j)^{\sigma_j - 1} (t_j + w_j)^{-h_j} S_{v_j}^{U_j} \left[a_j \left(\frac{t_j}{x_j}\right)^{e_j} \left(1 - \frac{t_j}{x_j}\right)^{f_j}\right] R_{N_j}^{\alpha_j, \beta_j} \left.\left[b_j \left(\frac{t_j}{x_j}\right)^{\lambda_j} \left(1 - \frac{t_j}{x_j}\right)^{\mu_j}\right]\right\} \\ & H\left[z_1 \left(\frac{t_1}{x_1}\right)^{u_1} \left(1 - \frac{t_1}{x_1}\right)^{v_1}, \ldots, z_s \left(\frac{t_s}{x_s}\right)^{u_s} \left(1 - \frac{t_s}{x_s}\right)^{v_s}\right] dt_1 \ldots dt_s \end{aligned} \tag{3.13}$$

Next, expressing the general class of polynomials $S_{V_j}^{U^j}[x_j]$ (j = 1, . . . , s) and the general sequence of functions $R_{N_j}^{\alpha_j;\beta_j}[x_j]$ (j= 1, . . . ,s) in their respective series forms, interchanging the order of series and t_j-integrals (j = 1, . . . , s), the right hand side of 3.13 assumes the following form

$$\begin{aligned} & \prod_{j=1}^{s} \left\{x^{-p_j - \sigma_j} \sum_{R_j=0}^{[v_j/U_j]} \sum_{\eta_j, m_j, v_j^{'}, u_j^{'}, {}_j^{'}, e_j^{'}} \frac{(-V_j)_{U_j R_j} A_{j V_j, R_j}}{R_j!} \frac{a_j^{R_j} b_j Q_j s_{j\ j}}{\eta_j!}\right. \\ & \left.\phi_j\left(m_j, v_j^{'}, u_j^{'}, t_j^{'}, e_j^{'}\right) x_j^{-(e_j + f_j) R_j - (\lambda_j + \mu_j) Q_j^{'}}\right\} \\ & \int_0^{x_1} \cdots \int_0^{x_s} \prod_{j=1}^{s} \left\{t_j \rho_j + g_j + e_j R_j + \lambda_j Q_j^{'} (x_j - t_j)^{\sigma_j + f_j R_j + \mu_j Q_j^{'} - 1}\right. \\ & \left.(t_j + w_j)^{-h_j} \left[1 + \frac{c_j}{d_j} b_j^{q_j} \left(\frac{t_j}{x_j}\right)^{\lambda_j q_j} \left(1 - \frac{t_j}{x_j}\right)^{\mu_j q_j}\right]^{\delta_j N_j - v_j^{'}}\right\} \\ & H\left[z_1 \left(\frac{t_1}{x_1}\right)^{u_1} \left(1 - \frac{t_1}{x_1}\right)^{v_1}, \ldots, z_s \left(\frac{t_s}{x_s}\right)^{u_s} \left(1 - \frac{t_s}{x_s}\right)^{v_s}\right] dt_1 \ldots dt_s \end{aligned} \tag{3.14}$$

Now, we express the terms $\left[1 + \frac{c_j}{d_j} b_j^{q_j} \left(\frac{t_j}{x_j}\right)^{\lambda_j q_j} \left(1 - \frac{t_j}{x_j}\right)^{\mu_j q_j}\right]^{\delta_j N_j - v_j^{'}}$ (j = 1, . . . , s) and the multivariable H function in terms of Mellin-Barnes-type contour integrals and interchange the order of ξ_j and t_j-integral (which is permissible under the conditions stated with 3.1, so that equation 3.14 assumes the following form after a little simplification.

$$\begin{aligned} & \prod_{j=1}^{s} \left\{x_j^{-\rho_j - \sigma_j} \sum_{R_j^{[v_j/U_j]}} \sum_{\eta_j, m_j, v_j^{'}, u_j^{'}, t_j^{'}, e_j^{'}}\right. \\ & \frac{(-V_j)_{U_j R_j} A_{j v_j, R_j}}{R_j!} \frac{R_{j\ j} b_j^{Q_j^{'}} S_j^{\eta_j}}{\eta_j!} \frac{\phi_j(m_j, v_j^{\prime}, u_j^{\prime}, t_j^{\prime}, e_j^{\prime})}{\Gamma(v_j^{\prime} - \delta_j N_j)} x_j^{-(e_j + f_j) R_j - (\lambda_j + \mu_j) Q_j^{\prime}} \\ & \frac{1}{(2\pi\omega)^{2s}} \int_{L_1} \cdots \int_{L_s} \int_{L_{s+1}} \cdots \int_{L_{2s}} \Psi(\zeta_1, ..., \zeta_s) \prod_{j=1}^{s} \{\phi_j(\zeta_j) z_j^{\zeta_j} x_j^{-u_j j - N \zeta_j}\} \\ & \prod_{j=1}^{s} \{\Gamma(v_j^{\prime} - \delta_j N_j + \zeta_{s+j}) \Gamma(-\zeta_{s+j}) (\frac{c_j}{d_j} b_j^{q_j} x_j^{-(\lambda_j + \mu_j) q_j}) \zeta_s + j\} \\ & \int_0^{x_1} \cdots \int_0^{x_s} \prod_{j=1}^{s} \{t_j^{\rho_j + g_j + e_j R_j + \lambda_j Q_j^{\prime} + \lambda_j q_j \zeta_{s+j} + u_j \zeta_j} \\ & (x_j - t_j)^{\sigma_j + f_j R_j + \mu_j Q^{\prime}{}_j + \mu_j q_j \zeta_{s+j} + v_j \zeta_j - 1} \\ & (t_j + w_j)^{-} h_j\} dt_1 \ldots dt_s d\xi_{1 \ldots} \ldots d\xi_s d\xi_{s+1} \ldots d\xi_{2\ s} \end{aligned} \tag{3.15}$$

Next, evaluating the inner t_j-integrals (say Δ) with the help of a known result [9], we get

$$\Delta =\prod_{j=1}^{s} \left\{(w_j)^{-h_j} x_j^{\rho_j+\sigma_j+g_j+(e_j+f_j)R_j+(e_j+f_j)R_j+(\lambda_j+\mu_j)Q_j^{'}+(\lambda_j+\mu_j)q_j\xi_{s+j}+(u_j+v_j)\xi_j} \right.$$
$$\times B\left(\sigma_j+f_jR_j+\mu_jQ_j^{'}+\mu_jq_j\xi_{s+j}+v_j\xi_j,\rho_j+g_j+e_jR_j+\lambda_jQ_j^{'}\right.$$
$$\left.+\lambda_jq_j\xi_{s+j}+u_j\xi_j+1\right)$$
$$ {}_2F_1\left[h_j,\rho_j+g_j+e_jR_j+\lambda_jQ_j^{'}+\lambda_jq_j\xi_{s+j}+u_j\xi_j+1\right.$$
$$\left.\rho_j+\sigma_j+g_j+(e_j+f_j)R_j+(\lambda_j+\mu_j)Q_j^{'}+(\lambda_j+\mu_j)q_j\xi_{s+j}+(u_j+v_j)\xi_j+1;-\frac{X_j}{W_j}\right] \tag{3.16}$$

where B(x, y) denotes the well known beta function.
On using the well-known transformation formula for Gauss hypergeometric function ${}_2F_1$ [4], we get

$$\Delta=\prod_{j=1}^{s} \left\{(w_j)^{-h_j} x_j\rho_j+\sigma_j+g_j+(e_j+f_j)R_j+(\lambda_j+\mu_j)Q_j^{'}(\lambda_j+\mu_j)q_j\xi_{s+j}+(u_j+v_j)\xi_j \right.$$
$$\times B\left(\sigma_j+f_jR_j+\mu_jQ_j^{'}+\mu_jq_jq_{s+j}+v_j\xi_j,\rho_j+g_j+e_jR_j+\lambda_jQ_j^{'}\right.$$
$$\left.+\lambda_jq_j\xi_{s+j}+u_j\xi_j+1\right)$$
$$\left(1+\frac{x_j}{w_j}\right)^{-h_j} {}_2F_1\left[h_j,\sigma_j+f_jR_j+\mu_jQ_j^{'}+\mu_jq_j\xi_{s+j}+V_j\xi_j\right.$$
$$\left.\rho_j+\sigma_j+g_j+(e_j+f_j)R_j+(\lambda_j+\mu_j)Q_j^{'}+(\lambda_j+\mu_j)q_j\xi_{s+j}+(u_j+v_j)\xi_j+1-\frac{X_j}{X_j+W_j}\right] \tag{3.17}$$

Now we substitute the value of Δ from 3.16 in the right-hand side of 3.15 and interpreted the Mellin-Barnes contour integrals thus obtained in terms of the multivariable H -function, we easily arrive at the desired result 3.1 after a little simplification.
The proof of result 3.9 can be developed on similar lines.
In the results 3.1 and 3.9 if we take $w_j=0(j=1,\ldots,s)$ and replace (g_j-h_j) by g_j, we get the following two results after a little simplification

(iii) $$I_x\left[\prod_{j=1}^{s}\left(t_j^{g_j}\right)\right]=\prod_{j=1}^{s}\left\{x_j^{g_j}\sum_{R_j=0}^{[V_j/U_j]}\sum_{\eta_j,m_j,v_j,u_j^{'},t_j,e_j^{'}}\frac{(-V_j)_{U_jR_j}A_{jV_j,R_j}}{R_j!}\right.$$
$$\left.\frac{a_j^{R_j}b_j^{Q_j^{'}}s_j^{\eta_j}}{\eta_j!\Gamma\left(v_j^{'}-\delta_jN_j\right)}\phi_j\left(m_j,v_j^{'},u_j^{'},t_j^{'},e_j^{'}\right)\right\}$$

$$\times H_{p+2s,q+s;L^*}^{0,n+2s;K^*}\left[z_1,\ldots,z_s,\frac{c_1}{d_1}b_1^{q_1},\ldots,\frac{c_s}{d_s}b_s^{q_s}\;\middle|\;\begin{matrix}A^*:C^*\\B^*:D^*\end{matrix}\right] \tag{3.18}$$

where

$$K^*=m_1,n_1;\ldots;m_s,n_s;1,\underbrace{1;\ldots;1}_{s},1 \tag{3.19}$$

$$L^*=p_1,q_1;\ldots;p_s,q_s;1,\underbrace{1;\ldots;1,1}_{s} \tag{3.20}$$

$$A^*=(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(s)},\underbrace{0,\ldots,0}_{s})_{1,n},(-\rho_1-g_1-e_1R_1-\lambda_1Q_1^{'};u_1,\underbrace{0,\ldots,0}_{s-1},\lambda_1q_1, \tag{3.21}$$

$$\underbrace{0,\ldots,0}_{s-1}),\ldots,(-\rho_s-g_s-e_sR_s-\lambda_sQ_s^{'};\underbrace{0,\ldots,0}_{s-1},u_s,\underbrace{0,\ldots,0}_{s-1},\lambda_sq_s),$$

$$(1-\sigma_1-f_1R_1-\mu_1Q_1^{'};v_1,\underbrace{0,\ldots,0}_{s-1},\mu_1q_1,\underbrace{0,\ldots,0}_{s-1}),\ldots,$$

$$(1-\sigma_s-f_sR_s-\mu_sQ_s^{'},\underbrace{0,\ldots,0}_{s-1},v_s,\underbrace{0,\ldots,0}_{s-1},\mu_sq_s)$$

$$(a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(s)}, \underbrace{0, \ldots, 0}_{s})_{n+1,p}$$

$$B^* = (b_j, \beta_j^{(1)}, \ldots, \beta_j^{(s)}, \underbrace{0, \ldots, 0}_{s})_{1,q}, (-\rho_1 - g_1 - \sigma_1(e_1 + f_1)R_1 - (\lambda_1 + \mu_1)Q_1^{'};$$

$$(\mathrm{u}_1 + v_1), \underbrace{0, \ldots, 0}_{s-1}, (\lambda_1 + \mu_1)q_1, \underbrace{0, \ldots, 0}_{s-1}), \ldots, (-\rho_s - g_s - \sigma_s - (e_s + f_s)R_s -$$

$$(\lambda_s + \mu_s)Q_s^{'}; \underbrace{0, \ldots, 0}_{s-1}, u_s + v_s, \underbrace{0, \ldots, 0}_{s-1}, (\lambda_s + \mu_s)q_s)(3.22)$$

$$\mathrm{C}^* = \left(\mathrm{C}_\mathrm{j}^{(1)}, \gamma_\mathrm{j}^{(1)}\right)_{1,\mathrm{p}_1}; \ldots; \left(\mathrm{d}_\mathrm{j}^{(\mathrm{s})}, \gamma_\mathrm{j}^{(\mathrm{s})}\right)_{1,\mathrm{p}_s}; (1 - v_1' + \delta_1 N_1, 1), ..., (1 - v_s' + \delta_s N_s, 1) \quad (3.23)$$

$$\mathrm{D}^* = \left(\mathrm{d}_\mathrm{j}^{(1)}, \delta_\mathrm{j}^{(1)}\right)_{1,\mathrm{q}_1}; \ldots; \left(\mathrm{d}_\mathrm{j}^{(\mathrm{s})}, \delta_\mathrm{j}^{(\mathrm{S})}\right)_{1,\mathrm{q}_\mathrm{s}}; \underbrace{(0,1); \ldots; (0,1)}_{\mathrm{s}} \quad (3.24)$$

$$Q_j' = \ell_j N_j + q_j v_j' + p_j t_j' + r_j m_j + r_j n_j (j = 1, \ldots, s) \quad (3.25)$$

$$\sum_{\eta_j, m_j, v_j', u_j', t_j', e_j'} \text{ stands for } \sum_{\eta_j=0}^{\infty} \sum_{m_j, v_j', u_j', t_j', e_j'}$$

ϕ_j is defined by equation 1.4 with appropriate changes in the various symbols. The above result 3.18 holds good under the conditions easily derivable from those mentioned with 3.1.

(iv) $J_x \left[\prod_{j=1}^{s} (t_j^{g_j})\right] = \prod_{j=1}^{s} \left\{ x_j^{g_j} \sum_{R_j=0}^{[V_j/U_j]} \sum_{n_j m_j v_j' j' y_j' j_j' f_j' e_j'} \frac{(-V_j)_{U_j R_j} A_{j v_j R_j}}{R_j!} \right.$

$$\left. \frac{a_j^{R_j} b_j^{Q_j'} \, {}_j^{\eta_j}}{\eta_j \Gamma\left(v_j' - \delta_j N_j\right)} \phi_j \left(m_j, v_j', u_j', t_j', e_j'\right) \right\}$$

$$H_{p+2s, q+s; L}^{0, n+2s; K} \left[\mathrm{z}_1^*, \ldots, \mathrm{z}_\mathrm{s}, \frac{\mathrm{c}_1}{\mathrm{d}_1} \mathrm{b}_1^{\mathrm{q}_1}, \ldots, \frac{\mathrm{c}_\mathrm{s}}{\mathrm{d}_\mathrm{s}} \mathrm{b}_\mathrm{s}^{\mathrm{q}_\mathrm{s}} \middle| \begin{matrix} A^{**}:C^* \\ B^{**}:\mathrm{D}^* \end{matrix} \right] \quad (3.26)$$

where

$$\begin{aligned} A^{**} = {} & (a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(s)}, \underbrace{0, \ldots, 0}_{s})_{1,n}, (1 - \rho_1 + g_1 - e_1 R_1 - \lambda_1 Q_1^{'}; u_1, \underbrace{0, \ldots, 0}_{s-1}, \lambda_1 q_1, \\ & \underbrace{0, \ldots, 0}_{s-1}), \ldots, (1 - \rho_s + g_s - e_s R_s - \lambda_s \mathrm{Q_s}; \underbrace{0, \ldots, 0}_{s-1}, \mathrm{u_s}, \underbrace{0, \ldots, 0}_{\mathrm{s}-1}, \lambda_\mathrm{s} \mathrm{q_s}), \\ & (1 - \sigma_1 - f_1 R_1 - \mu_1 Q_1^{'}; v_1, \underbrace{0, \ldots, 0}_{s-1}, \mu_1 q_1, \underbrace{0, \ldots, 0}_{s-1}), \ldots, \\ & (1 - \sigma_s - f_s R_s - \mu_s Q_s^{'}, \underbrace{0, \ldots, 0}_{s-1}, v_s, \underbrace{0, \ldots, 0}_{s-1}, \mu_s q_s)(a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(s)}, \underbrace{0, \ldots, 0}_{s})_{\mathrm{n}+1,\mathrm{p}} \end{aligned} \quad (3.27)$$

$$\begin{aligned} B^{**} = {} & (b_j, \beta_j^{(1)}, \ldots, \beta_j^{(\mathrm{s})}, \underbrace{0, \ldots, 0}_{\mathrm{s}})_{1,\mathrm{q}}. \left(1 - \rho_1 + \mathrm{g}_1 - \sigma_1 - (e_1 + \mathrm{f}_1) R_1 - (\lambda_1 + \mu_1) Q_1^{'}; \right. \\ & (u_1 + v_1), 0, \ldots, 0, (\lambda_1 + \mu_1) q_1, \underbrace{0, \ldots, 0}_{s-1}), \ldots, (1 - \rho_s + g_s - \sigma_s - (e_s + f_s) R_s - \\ & (\lambda_s + \mu_s) Q_s'; \underbrace{0, \ldots, 0}_{\mathrm{s}}, u_s + v_s, \underbrace{0, \ldots, 0}_{\mathrm{s}}, (\lambda_s + \mu_s) q_s) \end{aligned} \quad (3.28)$$

and the other symbols have the same meaning as given in result 3.18. The result hold goods under the conditions easily obtainable from those mentioned with result 3.9.

4 Special Cases

By appropriately choosing the parameters, we recover several well-known fractional operators and their results. For example, setting $v_j = 0$ and $S_{v_j}^{U_j}[x_j] = 1$ reduces our operators to those studied by Garg and Purohit [7].In the result 3.26 if we replace g_j by $-(g_j+1)$ $(j= 1, \ldots, s)$ and compare it with the result given by 3.18, after a little simplification,we get interesting relationship.

5 Applications

The generalized operators presented in this paper can be applied to solve problems in:

- Signal and image processing, where fractional derivatives model non-local interactions.
- Mathematical physics, particularly in the study of anomalous diffusion and elasticity.
- Control theory for fractional-order systems.

6 Conclusion

This paper presents and analyzes a pair of multidimensional fractional calculus operators with kernels involving general polynomials, sequences of functions, and the multi-variable H-function. The results obtained are broad and versatile, offering a framework that encompasses and extends several known fractional operators. By appropriate parameter choices, the findings can generate new fractional integrals involving functions beyond traditional ones, such as the some useful images of multidimensional fractional operators,the multi-variable H- function and polynomials. These results pave the way for further exploration in the theory of special functions. Future research may delve into numerical evaluation methods and investigate real-world applications of these operators. The results unify and extend various known fractional operators, providing a comprehensive framework for further research and applications. Future work may explore numerical methods for evaluating these operators and their applications in solving real-world problems.

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