

ON \mathbb{G}_m^d -MODULES INDUCING THE SAME HEIGHT

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MSC 2010 Classifications: Primary 11G50; Secondary, 20G30.

Keywords and phrases: Height functions, split algebraic tori, representations.

This work was partially supported by the PRIN project ADVANCES IN MODULI THEORY AND BIRATIONAL CLASSIFICATION and by the “Gruppo Nazionale per le Strutture Algebriche, Geometriche e loro Applicazioni (GNSAGA-INDAM)”

Abstract In this paper, we continue the study of heights associated to representations on split algebraic tori by addressing the following question: is it possible to explicitly characterize the modules inducing the same height? After a series of preliminary reductions, we identify an interesting subset of the set of isomorphism classes of \mathbb{G}_m^d -modules, within which this question can be fully resolved.

1 Introduction

Let k be a number field and let \mathbb{G}_m^d denote the k -split algebraic torus of dimension d . Let E be a \mathbb{G}_m^d -module and let $\rho_E : \mathbb{G}_m^d \rightarrow \mathbf{GL}(E)$ be the associated homomorphism. The height h_E associated to E is defined by setting $h_E := h_s \circ \rho_E$, where h_s denotes the spectral height on $\mathbf{GL}(E)$. For the reader’s convenience, we revisit the definition and key properties of the spectral height in Section 1; additional properties of these heights are discussed in [7]. The height h_E , originally defined by the author in his Ph.D. thesis (see [8]) serves as a specific and concrete example of height functions on an algebraic torus, coming from a torus embedding into a toric variety. The general concept of heights on toric varieties was introduced later by V. Maillot in [5] and subsequently generalized by J. I. Burgos Gil, P. Philippon, and M. Sombra in [2].

Let $\text{Rep}(\mathbb{G}_m^d)$ denote the monoid of isomorphism classes of representations under tensor product and \mathcal{F}^+ the monoid of non-negative real-valued functions on $\mathbb{G}_m^d(\bar{k})$ under addition. Since the height h_E depends only on the isomorphism class of E , the assignment $[E] \mapsto h_E$ gives rise to a map from $\text{Rep}(\mathbb{G}_m^d)$ to \mathcal{F}^+ . The fact that this map is a homomorphism follows directly from the properties of the spectral height. See section 2 for details.

A natural question arises: given $[E] \in \text{Rep}(\mathbb{G}_m^d)$ can we characterize all the isomorphism classes \mathbb{G}_m^d -modules $[E'] \neq [E]$ such that $h_E = h_{E'}$? An analogous question about canonical heights associated with morphisms of projective space was raised and resolved for certain classes of morphisms by Kawaguchi and Silverman in [4].

By its very nature, the height h_E is not uniquely determined by the isomorphism class of the \mathbb{G}_m^d -module E ; in fact many distinct isomorphism classes of \mathbb{G}_m^d -modules induce the same height. To gain a clearer understanding of which modules yield identical heights, it is convenient to work with a concrete model for $\text{Rep}(\mathbb{G}_m^d)$. A well-known such model is provided by the group semiring $\mathbb{N}[\Gamma_d]$, where Γ_d denotes the characters group of \mathbb{G}_m^d , as recalled in section 2. In section 3 we introduce the subset of reduced and concise elements of $\mathbb{N}[\Gamma_d]$, denoted by \mathcal{RC}_d , and prove that the heights associated to elements in \mathcal{RC}_d , exhaust all heights arising from \mathbb{G}_m^d -modules. Nevertheless, many elements in \mathcal{RC}_d still do induce the same height, and a complete characterization of those that do appears to be quite intricate. However, by imposing suitable restrictions on the *primitive support* (see section 4. for the definition) of the elements in \mathcal{RC}_d , we can establish a comparison theorem, for elements in \mathcal{RC}_d having the same primitive support and inducing the same height, which is the main result of this paper (cf. Theorem 4.6). We conclude the paper by presenting examples that demonstrate the failure of the main result when the restriction on the primitive support is removed.

Notations and conventions

Given a number field k we denote by \mathcal{M}_k the set of places of k . We normalize absolute values as follows: if v is archimedean we require that $|\cdot|_v$ restricted to \mathbb{Q} is the standard archimedean absolute value, while if v is a finite place, say $v|p$, then we require that $|p|_v = p^{-1}$. Let k_v be the completion of k with respect to $|\cdot|_v$, and n_v be the local degree. Finally set $d_v = n_v/d$, where d is the degree of k over \mathbb{Q} . With these normalizations the product formula reads $\prod_{v \in \mathcal{M}_k} |\lambda|_v^{n_v} = 1$.

2 Representations of the split algebraic torus and associated heights

Let us start by recalling the definition of the classical Northcott-Weil height. Let k be a number field and \bar{k} its algebraic closure, which we fix once and for all. The (*absolute logarithmic*) *Northcott-Weil* height on \bar{k}^n , is defined by setting $h_{NW}(0, \dots, 0) = 0$ and

$$h_{NW}(x_1, \dots, x_n) = \sum_{v \in \mathcal{M}_L} d_v \log(\max\{|x_1|_v, \dots, |x_n|_v\})$$

where $(0, \dots, 0) \neq (x_1, \dots, x_n) \in \bar{k}^n$ and $L \supset k$ is any number field containing all the x_i 's. Our normalization of the absolute values implies that h_{NW} does not depend on the choice of the field L , see e.g. [1].

Next, we recall the definition of the spectral height of linear operators on a k -vector space. Let E be a finite dimensional k -vector space. Given $T \in \text{End}(E)$ and $v \in \mathcal{M}_F$, we also denote by T , with slight abuse of notation, the linear transformation induced by T on $E_v := E \otimes_k k_v$. The *v-adic spectral radius* of T is

$$\rho_v(T) = \sup_{\lambda \in \text{sp}(T)} |\lambda|_{k_v(\lambda)},$$

where $\text{sp}(T)$ denotes the set of characteristic roots of T in an algebraic closure of k_v , and $|\cdot|_{k_v(\lambda)}$ denotes the unique extension of $|\cdot|_v$ to $k_v(\lambda)$. The (*logarithmic*) *spectral height* of T is defined as

$$h_s(T) = \sum_{v \in \mathcal{M}_F} d_v \log(\rho_v(T)),$$

for T not nilpotent. If T is nilpotent we set $h_s(T) = 0$. A proof of the following properties of h_s can be found in [6, Section 2]

- (S1) $h_s(T) \geq 0$, for all $T \in \text{End}(E)$
- (S2) h_s is invariant under field extensions.
- (S3) h_s is invariant under conjugation.
- (S4) If E' is another k -vector space and $S \in \mathbf{GL}(E')$, then $h_s(T \otimes S) = h_s(T) + h_s(S)$
- (S5) If $\text{sp}(T) = \{\lambda_1, \dots, \lambda_r\}$ then

$$h_s(T) = h_{NW}(\lambda_1, \dots, \lambda_r), \tag{2.1}$$

We continue by fixing our notations about characters and representations: a \mathbb{G}_m^d -*module* is a pair (E, ρ_E) , where E is finite dimensional \bar{k} -vector space and $\rho_E : \mathbb{G}_m^d \rightarrow \mathbf{GL}(E)$ is a homomorphism of algebraic groups. We often drop ρ_E and use only E to indicate a G -module. We denote by Γ_d the multiplicative group of characters of \mathbb{G}_m^d , which is isomorphic to \mathbb{Z}^d . Furthermore we let $\varepsilon_1, \dots, \varepsilon_d$ be the standard basis for Γ_d , while ε_0 denotes the trivial character. Given a \mathbb{G}_m^d -module E and a character $\chi \in \Gamma_d$, set

$$E_\chi := \{\mathbf{e} \in E \mid \rho_E(g)\mathbf{e} = \chi(g)\mathbf{e} \ \forall g \in \mathbb{G}_m^d\}, \text{ and } \text{supp}(E) := \{\chi \in \Gamma_d \mid E_\chi \neq 0\}.$$

Clearly $E_\chi \neq 0$ only for finitely many χ 's and we have

$$E = \bigoplus_{\chi \in \text{supp}(E)} E_\chi. \tag{2.2}$$

Set $\mathbf{f}_E := \sum_{\chi \in \Gamma_d} f_E^\chi \chi$, where $f_E^\chi = \dim_{\bar{k}} E_\chi$, and note that if E and E' are isomorphic then $\mathbf{f}_E = \mathbf{f}_{E'}$. Thus, the assignment $E \mapsto \mathbf{f}_E$ give rise to a map from $\text{Rep}(\mathbb{G}_m^d)$ to the group semiring $\mathbb{N}[\Gamma_d]$. Conversely, given $\mathbf{f} = \sum_{\chi \in \Gamma_d} f^\chi \chi \in \mathbb{N}[\Gamma_d]$, we set

$$\text{supp}(\mathbf{f}) = \{\chi \in \Gamma_d \mid f^\chi \neq 0\} \subset \Gamma_d, \text{ and } E_{\mathbf{f}} := \bigoplus_{\chi \in \text{supp}(\mathbf{f})} \bar{k}^{f^\chi} \quad (2.3)$$

where the action of \mathbb{G}_m^d on the direct summand \bar{k}^{f^χ} is given by scalar multiplication by χ . The two assignments $E \mapsto \mathbf{f}_E$ and $\mathbf{f} \mapsto E_{\mathbf{f}}$ are inverse of each other and give rise to a bijection between $\text{Rep}(\mathbb{G}_m^d)$ and $\mathbb{N}[\Gamma_d]$. Moreover, both $\text{Rep}(\mathbb{G}_m^d)$ and $\mathbb{N}[\Gamma_d]$ can be given a monoid structures. The monoid structures that we will consider are the following: on $\text{Rep}(\mathbb{G}_m^d)$ we consider the operation induced by the tensor product of \mathbb{G}_m^d -module and we will denote it by \otimes , by an abuse of notation. On $\mathbb{N}[\Gamma_d]$ we consider the convolution¹ product, which will be denote by $*$. It follows that the assignment $E \mapsto \mathbf{f}_E$ defines an isomorphism from $(\text{Rep}(\mathbb{G}_m^d), \otimes)$ to $(\mathbb{N}[\Gamma_d], *)$. In what follows, we will often omit explicit reference to the operation and simply write $\mathbb{N}[\Gamma_d]$, which will always be considered as a monoid under convolution.

Definition 2.1. Let (E, ρ_E) be a \mathbb{G}_m^d -module. The (logarithmic) *height* associated to (E, ρ_E) is the function $h_E : \mathbb{G}_m^d(\bar{k}) \rightarrow \mathbb{R}$, defined by setting $h_E = h_s \circ \rho_E$.

Naturally h_E inherits many properties of the spectral height, the most relevant for us are the following

- (h1) $h_E(g) \geq 0$ for all $g \in \mathbb{G}_m^d(\bar{k})$.
- (h2) If $E \cong E'$, then $h_E = h_{E'}$
- (h3) $h_{E \otimes E'} = h_E + h_{E'}$.
- (h4) If $g \in \mathbb{G}_m^d(L)$, then h_E can be computed by the following formula

$$h_E(g) = \sum_{v \in \mathcal{M}_L} d_v \max_{\chi \in \text{supp}(E)} \log(|\chi(g)|_v) \quad (2.4)$$

From the above properties it follows that an elementary version of the Weil's height machine (cf. [3, Theorem B.3.2.]) holds for \mathbb{G}_m^d -modules. Precisely let \mathcal{F}^+ denote the set of non-negative, real-valued functions on $\mathbb{G}_m^d(\bar{k})$. Then \mathcal{F}^+ is a monoid under point-wise addition of functions, and the assignment $\mathbf{f} \mapsto h_{\mathbf{f}}$ give rise to a homomorphism of monoids $ht : \mathbb{N}[\Gamma_d] \rightarrow \mathcal{F}^+$. Consider the equivalence relation \sim_h on $\mathbb{N}[\Gamma_d]$ defined by setting $\mathbf{f}_1 \sim_h \mathbf{f}_2 \iff h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$. It follows from (h3) that \sim_h is actually a congruence for $\mathbb{N}[\Gamma_d]$. Hence, if we denote by \mathcal{HT}_d the quotient of $\mathbb{N}[\Gamma_d]$ under the equivalence relation \sim_h , we get that \mathcal{HT}_d is a monoid under the operation induced by convolution and, as a monoid, is isomorphic to the image of $\mathbb{N}[\Gamma_d]$ via ht . In other words, understanding all possible heights associated with \mathbb{G}_m^d -modules is equivalent to determining the structure of the quotient \mathcal{HT}_d . This quotient encapsulates the classification of heights by grouping together isomorphism classes of \mathbb{G}_m^d -modules that induce the same height, thereby serving as a fundamental object in the study of these height functions.

3 Concise and reduced elements in $\mathbb{N}[\Gamma_d]$

We now start the study of the structure of equivalence classes under \sim_h and of the quotient \mathcal{HT}_d . Given $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ we denote by $[\mathbf{f}]_h$ the equivalence class of \mathbf{f} , i.e. $[\mathbf{f}]_h = \{\mathbf{g} \in \mathbb{N}[\Gamma_d] : h_{\mathbf{f}} = h_{\mathbf{g}}\}$. We have an action of Γ_d on $\mathbb{N}[\Gamma_d]$ given by

$$\mathbf{f}^\eta = \left(\sum_{\chi \in \Gamma_d} f^\chi \chi \right)^\eta = \sum_{\chi \in \Gamma_d} f^\chi (\eta \chi).$$

¹Recall that the convolution product of $\mathbf{f} = \sum_{\chi \in \Gamma_d} f^\chi \chi$ and $\mathbf{g} = \sum_{\chi \in \Gamma_d} g^\chi \chi$ is given by

$$\mathbf{f} * \mathbf{g} = \sum_{\chi \in \Gamma_d} \left(\sum_{\sigma\tau=\chi} f^\sigma g^\tau \right) \chi$$

Given $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ we denote by $\mathcal{O}_{\Gamma_d}(\mathbf{f})$ its orbit under Γ_d . It follows from (2.4) and the product formula that ht is constant on the orbit under Γ_d . Thus, if we denote by $[\mathbf{f}]_h$ the equivalence class of \mathbf{f} under \sim_h , we then have that $\mathcal{O}_{\Gamma_d}(\mathbf{f}) \subseteq [\mathbf{f}]_h$. By the same token if $\mathbf{f}' \in [\mathbf{f}]_h$, then $\mathcal{O}_{\Gamma_d}(\mathbf{f}') \subseteq [\mathbf{f}]_h$. Therefore the equivalence class $[\mathbf{f}]_h$ decomposes as the disjoint union of orbits under Γ_d . There is another reduction that can be done, which follows from noting if that \mathbf{f}_1 and \mathbf{f}_2 have the same support then, by (2.4), $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$. This prompts the following definition: $\mathbf{f} = \sum_{\chi \in \Gamma_d} f^\chi \chi$ is called *concise* if $f^\chi \in \{0, 1\}$ for all $\chi \in \Gamma_d$. Given $\mathbf{f} = \sum_{\chi \in \Gamma_d} f^\chi \chi$ we can associate a concise representative: $\mathbf{f}_c := \sum_{\chi \in \text{supp}(\mathbf{f})} \chi$. It follows that if \mathbf{f}_1 and \mathbf{f}_2 have the same concise representative then they define the same height.

It will be useful to have a distinguished representative for the orbits under Γ_d . Unfortunately in order to do it we have to use the standard basis of Γ_d . A $\chi \in \Gamma_d$ is called *positive* if it is a product of non negative powers of the ε_i 's. By extension we say that \mathbf{f} is *positive* if every χ in $\text{supp}(\mathbf{f})$ is a positive. Given a character $\chi = \prod_{i=1}^d \varepsilon_i^{n_i}$, set $\omega_i(\chi) := n_i$. Next for a finite set of characters X , we set

$$\omega_i(X) := \min_{\chi \in X} \omega_i(\chi) \quad \text{and} \quad \text{gcd}(X) := \prod_{i=1}^d \varepsilon_i^{\omega_i(X)}.$$

Given a positive $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ we say that is *reduced* if $\text{gcd}(\text{supp}(\mathbf{f})) = \varepsilon_0$. A moment of reflection show that each orbit under Γ_d contains exactly one reduced element. Given $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ we denote by \mathbf{f}_{rc} the unique reduced element in the orbit of \mathbf{f}_c under Γ_d . The set of reduced concise elements of $\mathbb{N}[\Gamma_d]$ is denoted by \mathcal{RC}_d . Given $\mathbf{f}, \mathbf{g} \in \mathcal{RC}_d$ their convolution product $\mathbf{f} * \mathbf{g}$ is reduced² but not necessarily concise so \mathcal{RC}_d is not a sub-monoid of $(\mathbb{N}[\Gamma_d], *)$. Nevertheless we can define a monoid structure on \mathcal{RC}_d by setting:

$$\begin{aligned} \star : \mathcal{RC}_d \times \mathcal{RC}_d &\rightarrow \mathcal{RC}_d \\ (\mathbf{f}_1, \mathbf{f}_2) &\mapsto \mathbf{f}_1 \star \mathbf{f}_2 := (\mathbf{f}_1 * \mathbf{f}_2)_c \end{aligned}$$

Let $\pi_h : \mathbb{N}[\Gamma_d] \rightarrow \mathcal{HT}_d$ denote the canonical projection sending \mathbf{f} to $[\mathbf{f}]_h$. The restriction of π_h to \mathcal{RC}_d is also a monoid homomorphism from (\mathcal{RC}_d, \star) to $(\mathcal{HT}_d, *)$, which will be denoted by $\tilde{\pi}_h$. The above discussion can be summarized in the following proposition:

Proposition 3.1. *Each map in the following commutative diagram is a surjective monoid homomorphism*

$$\begin{array}{ccc} (\mathbb{N}[\Gamma_d], *) & \xrightarrow{\Psi} & (\mathcal{RC}_d, \star) \\ & \searrow \pi_h & \downarrow \tilde{\pi}_h \\ & & (\mathcal{HT}_d, *) \end{array}$$

In particular the projection map $\pi_h : \mathbb{N}[\Gamma_d] \rightarrow \mathcal{HT}_d$ factors through Ψ .

It follows from the proposition that every possible height associated with a \mathbb{G}_m^d -module can be realized as the height corresponding to some element of \mathcal{RC}_d .

4 Modules with lean primitive support

Thus we have reduced the problem of determining when two isomorphism classes of \mathbb{G}_m^d -modules define the same height to the problem of determining when two elements of \mathcal{RC}_d induce the same height. Let us start with a couple of examples of pairs of elements in \mathcal{RC}_d inducing the same height:

Example 4.1. Let $\chi_1, \dots, \chi_r \in \Gamma_d$, and suppose that $\text{gcd}(\chi_1, \dots, \chi_r) = \varepsilon_0$. Consider the following elements of \mathcal{RC}_d

$$\begin{aligned} \mathbf{f}_1 &= \chi_1^{M_1} + \dots + \chi_r^{M_r} + \varepsilon_0 \\ \mathbf{f}_2 &= \chi_1^{M_1} + \dots + \chi_r^{M_r} + \chi_1^{d_1} + \dots + \chi_r^{d_r} + \varepsilon_0 \end{aligned}$$

²To check that the convolution product of two reduced elements $\mathbf{f} = \sum_{\chi \in \Gamma_d} f^\chi \chi$ and $\mathbf{g} = \sum_{\chi \in \Gamma_d} g^\chi \chi$ is reduced, write $\text{supp}(\mathbf{f}) = \{\sigma_1, \dots, \sigma_r\}$ and $\text{supp}(\mathbf{g}) = \{\tau_1, \dots, \tau_\ell\}$. Then for every $k=1, \dots, d$ there exist $r_k \in \{1, \dots, r\}$ (respectively $\ell_k \in \{1, \dots, \ell\}$) such that $\omega_k(\sigma_{r_k})=0$ (respectively $\omega_k(\tau_{\ell_k})=0$). Then $\omega_k(\sigma_{r_k} \tau_{\ell_k})=0$ and since $\text{supp}(\mathbf{f} * \mathbf{g}) = \{\sigma_i \tau_j, i=1, \dots, r, j=1, \dots, \ell\}$ we have that $\mathbf{f} * \mathbf{g}$ is reduced.

with $0 < d_i < M_i$, for $i = 1, \dots, r$. Since ε_0 is in the support of both \mathbf{f}_1 and \mathbf{f}_2 , we have that the only contribution to $h_{\mathbf{f}_1}(g)$ and $h_{\mathbf{f}_2}(g)$, comes from those absolute values v for which $\max_{1 \leq i \leq r} \{|\chi_i(g)|_v\} > 1$. But for those v we have:

$$\max_{1 \leq i \leq r} \{|\chi_i(g)|_v^{M_i}\} = \max_{1 \leq i \leq r} \{\max\{|\chi_i(g)|_v^{M_i}, |\chi_i(g)|_v^{d_i}\}\}$$

and so $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$

Example 4.2. Let $\chi_1, \dots, \chi_r \in \Gamma_d$,

$$\begin{aligned} \mathbf{f}_1 &= \chi_1^{M_1} + \dots + \chi_r^{M_r} + \chi_1^{m_1} + \dots + \chi_r^{m_r} \\ \mathbf{f}_2 &= \chi_1^{M_1} + \dots + \chi_r^{M_r} + \chi_1^{m_1} + \dots + \chi_r^{m_r} + \chi_1^{d_1} + \dots + \chi_r^{d_r} \end{aligned}$$

with $m_i < d_i < M_i$, for $i = 1, \dots, r$. Then

$$\max\{|\chi_i^{M_i}(g)|_v, |\chi_i^{d_i}(g)|_v, |\chi_i^{m_i}(g)|_v\} = \begin{cases} |\chi_i(g)|_v^{M_i}, & \text{if } |\chi_i(g)|_v \geq 1, \\ |\chi_i(g)|_v^{m_i}, & \text{if } |\chi_i(g)|_v \leq 1 \end{cases}$$

for each $i = 1, \dots, r$. It follows, as in the above example, that $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$.

Recall that a character χ is *primitive* if it is either the trivial character or is not a positive power of another character, (i.e. $\chi = \eta^n$ with $\eta \in \Gamma_d$ and $n \geq 1$ implies $n = 1$ and $\chi = \eta$). The above examples prompt the following definition:

Definition 4.3. Let $\mathbf{f} \in \mathbb{N}[\Gamma_d]$, we define $\text{prim}(\mathbf{f})$, the *primitive support* of \mathbf{f} , as follows:

$$\text{prim}(\mathbf{f}) = \{\eta \in \Gamma_d \mid \eta \text{ is primitive and there exists } \ell > 0 \text{ such that } \eta^\ell \in \text{supp}(\mathbf{f})\}$$

Moreover, for each $\chi \in \text{prim}(\mathbf{f})$, $\chi \neq \varepsilon_0$, we let:

$$m_{\mathbf{f}}(\chi) = \min\{n > 0 \mid \eta^n \in \text{supp}(\mathbf{f})\} \quad \text{and} \quad M_{\mathbf{f}}(\chi) = \max\{n > 0 \mid \eta^n \in \text{supp}(\mathbf{f})\}$$

as a matter of notation if $\varepsilon_0 \in \text{prim}(\mathbf{f})$ we set $m_{\mathbf{f}}(\varepsilon_0) = M_{\mathbf{f}}(\varepsilon_0) = 1$

Lemma 4.4. Let $\mathbf{f}_1, \mathbf{f}_2$ belong to \mathcal{RC}_d , and suppose that $\text{prim}(\mathbf{f}_1) = \text{prim}(\mathbf{f}_2) = X$.

- (a) Suppose $\varepsilon_0 \in X$. If $M_{\mathbf{f}_1}(\chi) = M_{\mathbf{f}_2}(\chi), \forall \chi \in X$, then $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$.
- (b) Suppose $\varepsilon_0 \notin X$. If $M_{\mathbf{f}_1}(\chi) = M_{\mathbf{f}_2}(\chi)$ and $m_{\mathbf{f}_1}(\chi) = m_{\mathbf{f}_2}(\chi), \forall \chi \in X$, then $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$.

Proof. First of all recall that by (2.4) we have:

$$h_{\mathbf{f}}(g) = \sum_{v \in \mathcal{M}_k} d_v \max_{\chi \in \text{supp}(\mathbf{f})} \log(|\chi(g)|_v)$$

(a) Now for a generic $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ with $\varepsilon_0 \in \text{supp}(\mathbf{f})$, we have

$$\max_{\chi \in \text{supp}(\mathbf{f})} \log(|\chi(g)|_v) = \max\{0, \max_{\eta \in \text{prim}(\mathbf{f})} \log(|\eta(g)|_v^{M_\eta})\}, \quad (4.1)$$

for all $v \in \mathcal{M}_k$, and so (a) follows at once.

(b) For a generic $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ with $\varepsilon_0 \notin \text{supp}(\mathbf{f})$ we have

$$\max_{\chi \in \text{supp}(\mathbf{f})} \log(|\chi(g)|_v) = \max\{\max_{\eta \in \text{prim}(\mathbf{f})} \log(|\eta(g)|_v^{m_\eta}), \max_{\eta \in \text{prim}(\mathbf{f})} \log(|\eta(g)|_v^{M_\eta})\}, \quad (4.2)$$

for all $v \in \mathcal{M}_k$, and so (b) follows at once. \square

Unfortunately neither of the above conditions is, in general, necessary, see the appendix for some examples. But there is a class of elements of \mathcal{RC}_d for which these conditions are also necessary.

Definition 4.5. A subset X of Γ_d is called *lean* if

$$\text{rank}\langle X \setminus \{\chi\} \rangle < \text{rank}\langle X \rangle \quad (\text{R})$$

for each $\chi \in X, \chi \neq \varepsilon_0$. Moreover, $\mathbf{f} \in \mathbb{N}[\Gamma_d]$ is called *lean* if its primitive support is lean.

The converse of the above lemma holds if we restrict to lean elements of \mathcal{RC}_d having the same primitive support. Before proving it we recall, for convenience of the reader some basic facts about algebraic subgroups of \mathbb{G}_m^d and their relation with subgroups of Γ_d . For more details and proofs see [1, Chapter 4]. Given a subgroup Γ of Γ_d set

$$H_\Gamma = \{g \in \mathbb{G}_m^d \mid \chi(g) = 1, \forall \chi \in \Gamma\}.$$

Then H_Γ is an algebraic subgroup of \mathbb{G}_m^d of dimension $n = d - \text{rank } \Gamma$. Moreover, all subgroups of \mathbb{G}_m^d arise in this way. Given a subset $X \subset \Gamma_d$, we denote by $\langle X \rangle$ the subgroup generated by X and by H_X the associated algebraic subgroup of \mathbb{G}_m^d . In particular given a character χ we let H_χ denote the kernel of χ .

Theorem 4.6. Let $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{RC}_d$ be two lean elements having the same primitive support X . Then

- (a) Suppose $\varepsilon_0 \in X$. Then $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$ if and only if $M_{\mathbf{f}_1}(\chi) = M_{\mathbf{f}_2}(\chi)$ for all $\chi \in X$.
- (b) Suppose $\varepsilon_0 \notin X$. Then $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$ if and only if $M_{\mathbf{f}_1}(\chi) = M_{\mathbf{f}_2}(\chi)$ and $m_{\mathbf{f}_1}(\chi) = m_{\mathbf{f}_2}(\chi)$ for all $\chi \in X$.

Proof. The if part of both (a) and (b) was proven, in a more general setting, in Lemma 4.4. We start by dealing with the almost degenerate cases, i.e. $\text{rank}\langle X \rangle = 1$. Since the elements of X are primitive, we have that X contains a single non trivial character and so (a) and (b) follow from (4.1) and (4.2) respectively. So from now on we assume $\text{rank}\langle X \rangle > 1$.

Next we make some considerations that will be used in both (a) and (b). Let $\chi \in X, \chi \neq \varepsilon_0$. Set $Y = X \setminus \{\chi\}$, since X is lean we have that $\text{rank}\langle Y \rangle < \text{rank}\langle X \rangle$ and so $\dim H_Y > \dim H_X$. The latter inequality implies that H_χ does not contain H_Y or equivalently that χ restricted to H_Y is not trivial.

(a) Let $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{RC}_d$ be as in the hypothesis of the theorem. Let $\bar{\chi} \in X, \bar{\chi} \neq \varepsilon_0$, be such that $M_{\mathbf{f}_1}(\bar{\chi}) \neq M_{\mathbf{f}_2}(\bar{\chi})$ and set $Y = X \setminus \{\bar{\chi}\}$. If $\text{rank}\langle Y \rangle = 0$, then $X = \{\bar{\chi}, \varepsilon_0\}$ and we immediately get a contradiction. So we can assume $\text{rank}\langle Y \rangle \geq 1$. For any $g \in H_Y$ we have

$$h_{\mathbf{f}_i}(g) = \sum_{v \in \mathcal{M}_k} d_v \max\{0, M_{\mathbf{f}_i}(\bar{\chi}) \log |\bar{\chi}(g)|_v\}$$

for $i = 1, 2$. Since, $M_{\mathbf{f}_1}(\bar{\chi}) \neq M_{\mathbf{f}_2}(\bar{\chi})$ and $h_{\mathbf{f}_1}(g) = h_{\mathbf{f}_2}(g)$ we must have $|\bar{\chi}(g)|_v \leq 1$ for all $v \in \mathcal{M}_k$. But then, by Kronecker's theorem, $\bar{\chi}(g)$ has to be a root of unity for all $g \in H_Y$, which is impossible because $\bar{\chi}$ restricted to H_Y is not trivial.

(b) Let $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{RC}_d$ be as in the hypothesis of the theorem. First of all note that since \mathbf{f}_1 and \mathbf{f}_2 are reduced we must have that $\#X > 1$. We first show that if $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$ then $M_{\mathbf{f}_1}(\chi) = M_{\mathbf{f}_2}(\chi)$ for all $\chi \in X$. Suppose $\bar{\chi} \in X$, is such that $M_{\mathbf{f}_1}(\bar{\chi}) \neq M_{\mathbf{f}_2}(\bar{\chi})$ and set $Y = X \setminus \{\bar{\chi}\}$. As before for any $g \in H_Y$, we have

$$\begin{aligned} h_{\mathbf{f}_i}(g) &= \sum_{v \in \mathcal{M}_k} d_v \max\{0, m_{\mathbf{f}_i}(\bar{\chi}) \log |\bar{\chi}(g)|_v, M_{\mathbf{f}_i}(\bar{\chi}) \log |\bar{\chi}(g)|_v\} \\ &= \sum_{v \in \mathcal{M}_k} d_v \max\{0, M_{\mathbf{f}_i}(\bar{\chi}) \log |\bar{\chi}(g)|_v\} \end{aligned} \quad (4.3)$$

for $i = 1, 2$. As in case (a) we get a contradiction. Proving that $m_{\mathbf{f}_1}(\chi) = m_{\mathbf{f}_2}(\chi)$, for all $\chi \in X$ requires more work. So fix $\chi \in X$. By hypothesis we have that $r = \text{rank}\langle X \rangle = \#X \leq d$. So we write $X = \{\chi = \chi_1, \dots, \chi_r\}$ and we consider the surjective homomorphism:

$$\begin{aligned} \Phi_X : \mathbb{G}_m^d &\rightarrow \mathbb{G}_m^r \\ g &\mapsto (\chi_1(g), \dots, \chi_r(g)) \end{aligned}$$

Since Φ_X is surjective there exists $g_0 \in \mathbb{G}_m^d(\bar{k})$ such that $\Phi_X(g_0) = (p, p^{n_2}, \dots, p^{n_r})$, with $p \in \mathbb{N}$ and where the n_i 's are such that $m_{\mathbf{f}_1}(\chi_1) < n_i m_{\mathbf{f}_1}(\chi_i)$ and $m_{\mathbf{f}_2}(\chi_1) < n_i m_{\mathbf{f}_2}(\chi_i)$ for all $i = 2, \dots, r$. Since $M_{\mathbf{f}_1}(\chi) = M_{\mathbf{f}_2}(\chi)$ for all $\chi \in X$, it follows that:

$$\max_{1 \leq i \leq r} n_i M_{\mathbf{f}_1}(\chi_i) = n_h M_{\mathbf{f}_1}(\chi_h) = N = n_h M_{\mathbf{f}_2}(\chi_h) = \max_{1 \leq i \leq r} n_i M_{\mathbf{f}_2}(\chi_i)$$

Then

$$\begin{aligned} h_{\mathbf{f}_1}(g) &= h_{N_W} \left(p^{m_{\mathbf{f}_1}(\chi_1)}, p^{M_{\mathbf{f}_1}(\chi_1)}, \dots, p^{n_r m_{\mathbf{f}_1}(\chi_2)}, p^{n_r M_{\mathbf{f}_1}(\chi_r)} \right) \\ &= h_{N_W} \left(1, p^{M_{\mathbf{f}_1}(\chi_1) - m_{\mathbf{f}_1}(\chi_1)}, \dots, p^{n_r m_{\mathbf{f}_1}(\chi_2) - m_{\mathbf{f}_1}(\chi_1)}, p^{n_r M_{\mathbf{f}_1}(\chi_r) - m_{\mathbf{f}_1}(\chi_1)} \right) \\ &= p^{N - m_{\mathbf{f}_1}(\chi_1)} \end{aligned}$$

By hypothesis $h_{\mathbf{f}_1}(g) = h_{\mathbf{f}_2}(g)$ and so we get

$$p^{N - m_{\mathbf{f}_1}(\chi_1)} = p^{N - m_{\mathbf{f}_2}(\chi_1)}$$

which yields $m_{\mathbf{f}_1}(\chi_1) = m_{\mathbf{f}_2}(\chi_1)$, completing the proof. \square

Appendix: Counterexamples

For non lean elements of \mathcal{RC}_d Theorem 4.6 does not hold in general as it can be seen by the next two examples.

Example (A1). Let $d \geq 2$. We will exhibit two elements $\mathbf{f}_1, \mathbf{f}_2$ in \mathcal{RC}_d , such that

- (a1) $\varepsilon_0 \in \text{prim}(\mathbf{f}_1) = \text{prim}(\mathbf{f}_2) = X$
- (a2) X is not lean
- (a3) $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$
- (a4) There exist $\bar{\chi} \in X$ such that $M_{\mathbf{f}_1}(\bar{\chi}) \neq M_{\mathbf{f}_2}(\bar{\chi})$

Let $\chi_1, \chi_2 \in \Gamma_d$, be primitive characters and suppose that $\gcd(\chi_1, \chi_2) = \varepsilon_0$

$$\begin{aligned} \mathbf{f}_1 &= \chi_1^n + \chi_2^n + \chi_1 \chi_2 + \varepsilon_0 \\ \mathbf{f}_2 &= \chi_1^n + \chi_2^n + (\chi_1 \chi_2)^2 + \varepsilon_0 \end{aligned}$$

It follows that $\text{prim}(\mathbf{f}_1) = \{\chi_1, \chi_2, \chi_1 \chi_2, \varepsilon_0\} = \text{prim}(\mathbf{f}_2)$. Apparently $\{\chi_1, \chi_2, \chi_1 \chi_2, \varepsilon_0\}$ is not a lean set. Furthermore, $M_{\mathbf{f}_1}(\chi_1 \chi_2) = 1$ and $M_{\mathbf{f}_2}(\chi_1 \chi_2) = 2$. Now, if $n \geq 5$ we have

$$|\chi_1(g) \chi_2(g)|_v \leq |(\chi_1(g)^2 \chi_2(g)^2)|_v \leq \max\{|\chi_1(g)^n|_v, |\chi_2(g)^n|_v\}$$

for all g such that $|(\chi_1 \chi_2)(g)|_v \geq 1$ and all $v \in \mathcal{M}_k$. Now using (4.1) we get that for $i = 1, 2$ we have

$$\max_{\chi \in \text{supp}(\mathbf{f}_i)} \log(|\chi(g)|_v) = \max\{0, \log(|(\chi_1)(g)|_v^n), \log(|(\chi_2)(g)|_v^n)\}$$

for all $v \in \mathcal{M}_k$, and all $g \in \mathbb{G}_m^d$. It follows, by means of (2.4), that $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$.

Example (A2). Let $d = 4$, a slight modification of the above example will produce two reduced and concise element \mathbf{f}_1 and \mathbf{f}_2 , which have the same primitive support and satisfy (a2)-(a4) of the previous example. Consider

$$\begin{aligned} \mathbf{f}_1 &= \chi_1^n + \chi_2^n + \chi_1 \chi_2 + (\chi_1 \chi_2)^2 \\ \mathbf{f}_2 &= \chi_1^n + \chi_2^n + \chi_1 \chi_2 + (\chi_1 \chi_2)^3 \end{aligned}$$

As in the previous example, $\text{prim}(\mathbf{f}_1) = \{\chi_1, \chi_2, \chi_1 \chi_2\} = \text{prim}(\mathbf{f}_2)$, and so we have that the primitive support is not lean. Moreover, $M_{\mathbf{f}_1}(\chi_1 \chi_2) = 2$ and $M_{\mathbf{f}_2}(\chi_1 \chi_2) = 3$, Suppose $n \geq 7$, we have

$$\max_{\chi \in \text{supp}(\mathbf{f}_i)} |\chi(g)|_v = \begin{cases} \max\{|\chi_1^n(g)|_v, |\chi_2^n(g)|_v\} & \text{if } |(\chi_1 \chi_2)(g)|_v \geq 1 \\ \max\{|\chi_1^n(g)|_v, |\chi_2^n(g)|_v, |(\chi_1 \chi_2)(g)|_v\} & \text{if } |(\chi_1 \chi_2)(g)|_v < 1 \end{cases}$$

Using (2.4), we conclude that $h_{\mathbf{f}_1} = h_{\mathbf{f}_2}$

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