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Theoretical and Numerical Analysis of Burger's Equations Under Non-Singular Kernel

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Abstract The classical Burger's equations serve as a fundamental model capturing the interaction between diffusion and convection processes. It has been extensively utilized in modeling physical phenomena such as shock wave formation, acoustic wave propagation, and traffic flow. In the present work, we consider the fractional form of Burger's equation involving a non-singular kernel. To obtain approximate analytical solutions, the Shehu decomposition method (SDM) is used. This method synergistically integrates the Adomian decomposition method (ADM), Adomian polynomials, and the Shehu transformation. To assess the reliability and efficiency of the proposed approach, the results are compared with exact solutions and other established numerical schemes. Graphical representations of approximate solutions for different fractional order values, along with exact solutions and corresponding absolute errors, are provided in two- and three-dimensional forms. The numerical results indicate that the proposed method yields accurate and computationally efficient solutions, demonstrating its potential as a robust technique for solving nonlinear fractional partial differential equations.

1 Introduction

The theory of fractional differential equations (FDEs) has greatly advanced mathematical modeling by enabling more precise and realistic representations compared to classical integer-order differential equations. Consequently, FDEs have garnered extensive attention from researchers around the world. Among the various fractional operators, the Caputo and R–L derivatives have been the most frequently used in complex fractional models ([1], [2], [3], [4], [5]). To further enhance the understanding of model dynamics, several fractional operators have been introduced over the years. These include the Riemann–Liouville (RL) and Caputo derivatives ([6], [7], [8], [9], [10]), as well as the Caputo–Fabrizio (CF) ([11], [12], [13], [14]) and Atangana–Baleanu ([15]) operators. In 2016, Atangana and Baleanu [16] proposed a novel non-linear fractional derivative, now known as the Atangana–Baleanu derivative operator. One of the key advantages of the Atangana–Baleanu–Caputo (ABC) fractional derivative operator with a non-singular kernel is its ability to transform a stretched exponential waiting time distribution into a power-law form, and to convert a Gaussian density distribution into a non-Gaussian one.

The Burgers equation was originally introduced by Harry Bateman and later studied by Burgers [17] in 1948, where it was applied to model turbulent fluid motion. This equation plays a fundamental role in various mathematical applications, including nonlinear acoustics, fluid dynamics, and traffic flow. Numerical modeling, a key aspect of mathematical modeling, explores different methods for determining initial values or mathematical properties of the positioning operator. The use of differential derivatives allows equations to be extended into partial differential equations, often referred to as non-standard differential equations. In many cases, fractional derivatives provide a more accurate representation of certain physical phenomena ([18], [19], [20]). Recently, the fractional Burgers-type equation has been formulated and analyzed due to its ability to describe the cumulative effects of friction in the wake region through variable coefficients. Various analytical and approximation techniques for solving fractional differential equations have also been developed in recent years ([21], [22], [23], [24]).
In this paper, we examine the application of Burgers' equation using the Shehu decomposition

method. The proposed approach integrates the Adomian decomposition method (ADM) with the Shehu transformation to enhance the accuracy and efficiency of the solution process. The Shehu transform offers several advantages over conventional transforms, particularly in its ability to achieve more precise and efficient decomposition of signals and systems, especially in complex or nonlinear cases. Unlike traditional transforms, it provides improved time-frequency resolution, allowing greater accuracy in analyzing both temporal and frequency characteristics. Additionally, this method demonstrates superior numerical stability, minimizing computational errors while accommodating a broader range of applications due to its versatility. The Shehu transform's capability to decompose intricate systems into simpler components also proves valuable in signal denoising, feature extraction, and system analysis, making it a powerful tool in engineering, physics, and communication applications.

The structure of this paper is as follows: Section 2 introduces fundamental definitions of fractional calculus. Section 3 outlines the mathematical methodology of the proposed approach, while Section 4 presents two numerical applications to demonstrate its accuracy. Section 5 discusses the results, and concluding remarks are provided in Section 6.

2 Preliminaries

This section provides key definitions, which are essential for understanding the subsequent results.

Definition 2.1. The Caputo fractional derivative of $\Phi \in C_{-1}^{m}$ is defined as ([6], [7])

$$ {}^{C}D_{\eta}^{\zeta}\Phi(\xi,\eta) = \begin{cases} \frac{\partial^{p}\Phi(\xi,\eta)}{\partial x^{p}}, & \zeta = p \in \aleph, \\ \frac{1}{\Gamma(p-1)} \int\limits_{0}^{\eta} (\eta-\phi)^{p-1-\zeta} \frac{\partial^{p}\Phi(\xi,\phi)}{\partial \phi^{p}} d\phi, & p-1<\zeta\leq p. \end{cases} \tag{2.1} $$

Definition 2.2. The Caputo–Fabrizio fractional-order derivative (CF) is defined [12] as follows:

$$ {}^{CF}{}_{0}D_{\eta}^{\zeta}\Phi(\xi,\eta) = \frac{1}{(1-\zeta)} \int_{0}^{\eta} \exp\left(\frac{-\zeta(\eta-\phi)}{\zeta-1}\right) \Phi'(\xi,\phi) d\phi, \tag{2.2} $$

such that $0<\zeta\leq 1$.

Definition 2.3. The usual definition of the Atangana-Baleanu-Caputo derivative (ABC) of order $0<\delta<1$ is defined [16] as follows:

$$ {}_{0}^{ABC}D_{\xi}^{\zeta}\left[\Phi(\xi,\eta)\right] = \frac{\mathcal{A}(\zeta)}{1-\zeta} \int\limits_{0}^{\eta} E_{\zeta}\left(-\frac{\zeta(\eta-\phi)^{\zeta}}{1-\zeta}\right) \Phi'\left((\xi,\phi)\right) d\phi, \quad \xi>0, \tag{2.3} $$

where $\mathcal{A}(\zeta)$ is a normalization constant, E_{ζ} is the Mittag-Leffler function, and $\Gamma(\zeta)$ is the Gamma function.

Definition 2.4. S. Maitama and W. Zaho [25] developed a noval transform of exponential order function $\Phi(\eta)$ over the set of $\mathcal{B}$,

$$ \mathcal{B} = \left\{\Phi(\eta) : \exists\ \kappa_{1}, \Xi_{1},\ \Xi_{2}>0,\ |\Phi(\eta)| < \Xi_{1} e^{\frac{|\eta|}{\Xi_{j}}},\ if\ \eta \in (-1)^{j} \times [0,\infty)\right\} $$

by to integral

$$ S\left[\Phi(\eta)\right] = \mathrm{T}(\nu,\rho) = \int\limits_{0}^{\infty} \Phi(\xi)\ e^{\frac{-\nu\eta}{\rho}} d\eta,\ \nu>0,\ \eta>0\ . \tag{2.4} $$

Remark 2.5. If $\rho = 1$, then ST becomes Laplace's transform, and also for $\nu = 1$, this transform converts into Yang's integral transform [26].

Definition 2.6. The Shehu transform (ST) for fractional ABC derivative is given as

$$ S\left({}_{0}^{ABC}D_{\eta}^{\zeta}\left[\Phi(\zeta,\eta)\right]\right) = \frac{\mathcal{A}(\zeta)}{1-\zeta+\zeta\left(\frac{\rho}{\nu}\right)^{\zeta}} \left(V(\nu,\rho) - \frac{\rho}{\nu}\Phi(0)\right), \tag{2.5} $$

here, $V(\nu,\rho)$ is ST of $\Phi(\xi,\eta)$.

3 Proposed Methodology

Consider a nonlinear fractional partial differential equation as

$$ {}^{ABC}D_{\eta}^{\zeta}\Phi\left(\xi,\eta\right)=\Re\Phi\left(\xi,\eta\right)+\aleph\Phi\left(\xi,\eta\right)+P\left(\xi,\eta\right),\ m-1<\zeta\leq m, \tag{3.1} $$

with initial condition

$$ \Phi\left(\xi,\eta\right)=\phi\left(\xi\right), \tag{3.2} $$

where ${}^{ABC}D_{\eta}^{\zeta}=\frac{\partial^{\zeta}}{\partial\eta^{\zeta}}$ represents the fractional AB derivative of order ζ, $\Re$ is a linear function of (ξ,η), $\aleph$ denotes the nonlinear function, and P is the source term.
Applying the Shehu transform (ST) to both sides of equation (3.1), we obtain

$$ S\left[{}^{ABC}D_{\eta}^{\zeta}\Phi\left(\xi,\eta\right)\right]=S\left[\Re\Phi\left(\xi,\eta\right)\right]+S\left[\aleph\Phi\left(\xi,\eta\right)\right]+\ S\left[P\left(\xi,\eta\right)\right]. \tag{3.3} $$

Utilizing the differentiation property of the Shehu transform (ST), we obtain

$$ \begin{aligned} S\left[\Phi\left(\xi,\eta\right)\right]=&\left(\tfrac{\rho}{\upsilon}\right)\left[\Phi\left(\xi,0\right)\right]+\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[P\left(\xi,\eta\right)\right]\\ &+\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[\Re\left[\Phi\left(\xi,\eta\right)\right]+\aleph\left[\Phi\left(\xi,\eta\right)\right]\right]. \end{aligned} \tag{3.4} $$

Now, applying the inverse ST to both sides of the equation, we get (3.4), we get

$$ \begin{aligned} \Phi\left(\xi,\eta\right)=&S^{-1}\left[\left(\tfrac{\rho}{\upsilon}\right)\left[\Phi\left(\xi,0\right)\right]\right]+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[P\left(\xi,\eta\right)\right]\right]\\ &+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[\Re\left[\Phi\left(\xi,\eta\right)\right]+\aleph\left[\Phi\left(\xi,\eta\right)\right]\right]\right]. \end{aligned} \tag{3.5} $$

Let, $\Phi\xi,\eta$ has infinite series solution as

$$ \Phi\left(\xi,\eta\right)=\sum_{\tau=0}^{\infty}\Phi_{\tau}\left(\xi,\eta\right), \tag{3.6} $$

and the nonlinear term $\aleph\Phi(\xi,\eta)$ is expressed as

$$ \aleph\Phi\left(\xi,\eta\right)=\sum_{\tau=0}^{\infty}A_{\tau}\quad, \tag{3.7} $$

where A_{τ} is the Adomian polynomial, given by

$$ A_{\tau}=\frac{1}{\Gamma\left(\tau+1\right)}\left[\frac{d^{\tau}}{d\Pi^{\tau}}\left\{\aleph\left(\sum_{\iota=0}^{\infty}\Pi^{\iota}\xi_{\iota},\sum_{\iota=0}^{\infty}\Pi^{\iota}\eta_{\iota}\right)\right\}\right]_{\Pi=0}. \tag{3.8} $$

Using equations (3.6) and (3.7) in equation (3.5), we get

$$ \begin{aligned} \sum_{\ell=0}^{\infty}\Phi\left(\xi,\eta\right)=&\Phi\left(\xi,0\right)+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[P\left(\xi,\eta\right)\right]\right]\\ &+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[\Re\left[\Phi\left(\sum_{\tau=0}^{\infty}\xi_{\tau},\sum_{\tau=0}^{\infty}\eta_{\tau}\right)\right]+\sum_{\tau=0}^{\infty}A_{\ell}\right]\right]. \end{aligned} \tag{3.9} $$

From equation (3.9), we get

$$ \begin{aligned} &\Phi_{0}\left(\xi,\eta\right)=\Phi\left(\xi,0\right)+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{A(\zeta)}S\left[P\left(\xi,\eta\right)\right]\right],\\ &\Phi_{1}\left(\xi,\eta\right)=S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{A(\zeta)}S\left[\Re\left[\Phi\left(\xi_{0},\eta_{0}\right)\right]+A_{0}\right]\right],\\ &\Phi_{\tau+1}\left(\xi,\eta\right)=S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{A(\zeta)}S\left[\Re\left[\Phi\left(\xi_{\tau},\eta_{\tau}\right)\right]+A_{\tau}\right]\right],\qquad\tau\geq1. \end{aligned} \tag{3.10} $$

4 Application of Burger's equation

Application 1. Consider the one-dimensional fractional Burger's equation

$$ {}^{ABC}D_{\eta}^{\zeta}\varpi(\xi,\eta)+\alpha\varpi(\xi,\eta)\frac{\partial\varpi(\xi,\eta)}{\partial\xi}-\gamma\frac{\partial^{2}\varpi(\xi,\eta)}{\partial\xi^{2}}=0,\ 0<\zeta\leq 1, \tag{4.1} $$

with the initial condition

$$ \varpi(\xi,0)=\frac{2\left[\beta-\beta\gamma\tanh(\beta\xi)\right]}{\alpha}, \tag{4.2} $$

where α, β, and γ are arbitrary constants and $\alpha\neq 0$.
Applying the ST to both sides of equation (4.1) and on simplification, we get

$$ S\left[\varpi(\xi,\eta)\right]=\left(\frac{\rho}{\upsilon}\right)\varpi(\xi,0)+\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}S\left[-\alpha\varpi\frac{\partial\varpi}{\partial\xi}+\gamma\frac{\partial^{2}\varpi}{\partial\xi^{2}}\right]. \tag{4.3} $$

Taking inverse Shehu transform (ST) to both sides of (4.3), we get

$$ \varpi(\xi,\eta)=\varpi(\xi,0)+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{A(\zeta)}S\left[-\alpha\varpi\frac{\partial\varpi}{\partial\xi}+\gamma\frac{\partial^{2}\varpi}{\partial\xi^{2}}\right]\right]. \tag{4.4} $$

Assume that solution of unknown function $\varpi(\xi,\eta)$ in an infinite series as

$$ \varpi(\xi,\eta)=\sum_{\ell=0}^{\infty}\varpi_{\ell}(\xi,\eta), $$

and the nonlinear term by means of the Adomian polynomials is described as $\varpi\frac{\partial\varpi}{\partial\xi}=\sum_{\ell=0}^{\infty}A_{\ell}$.
Hence, the equation (4.4) is rewritten as

$$ \sum_{\ell=0}^{\infty}\varpi(\xi,\eta)=\varpi(\xi,0)+S^{-1}\left[\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{A(\zeta)}S\left[-\alpha\sum_{\ell=0}^{\infty}A_{\ell}+\gamma\sum_{\ell=0}^{\infty}\frac{\partial^{2}\varpi_{\ell}}{\partial\xi^{2}}\right]\right]. \tag{4.5} $$

Thus, according to our proposed scheme, by equating both sides of equation (4.5), we get

$$ \varpi_{0}(\xi,\eta)=\frac{2\left[\beta-\beta\gamma\tanh(\beta\xi)\right]}{\alpha}, $$

$$ \varpi_{1}(\xi,\eta)=\frac{4\beta^{3}\gamma\sec h^{2}(\beta\xi)}{\alpha}\left(\frac{1-\zeta+\zeta\left(\frac{\eta^{\zeta}}{\Gamma(\zeta+1)}\right)}{\mathcal{A}(\zeta)}\right), $$

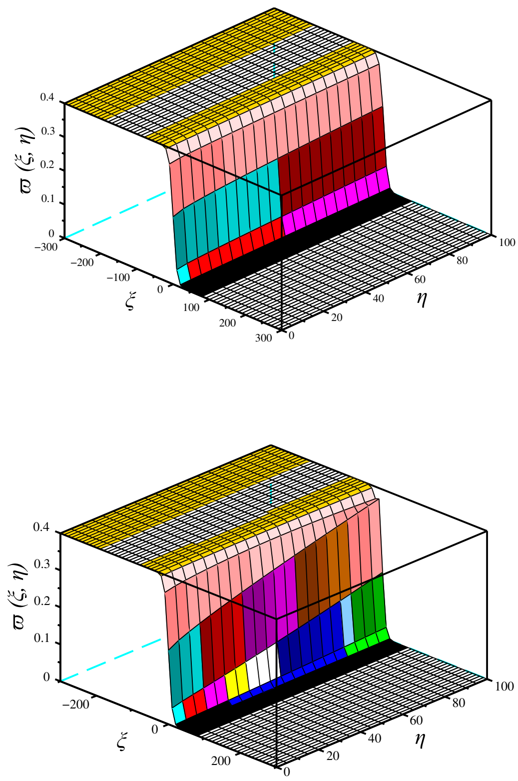
$$ \varpi_{2}(\xi,\eta)=-\frac{16\beta^{5}\gamma\sec h^{2}(\beta\xi)\tanh(\beta\xi)}{(\mathcal{A}(\zeta))^{2}\alpha}\left(1+2\zeta\left(1+\frac{\eta^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^{2}\left(1+\frac{\eta^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\eta^{\zeta}}{\Gamma(\zeta+1)}\right)\right), $$

similarly, we obtain next terms in the same manner. Hence, the approximate solution of (4.1) is given as

$$ \varpi(\xi,\eta)=\frac{2[\beta-\beta\gamma\tanh(\beta\xi)]}{\alpha}+\frac{4\beta^{3}\gamma\sec h^{2}(\beta\xi)}{\alpha}\left(\frac{1-\zeta+\zeta\left(\frac{\eta^{\zeta}}{\Gamma(\zeta+1)}\right)}{\mathcal{A}(\zeta)}\right) -\frac{16\beta^{5}\gamma\sec h^{2}(\beta\xi)\tanh(\beta\xi)}{(\mathcal{A}(\zeta))^{2}\alpha}\left(1+2\zeta\left(1+\frac{\eta^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^{2}\left(1+\frac{\eta^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\eta^{\zeta}}{\Gamma(\zeta+1)}\right)\right)+.. \tag{4.6} $$

In particular, for $\zeta=1$, equation (4.1) converges rapidly to the exact solution and is given as

$$ \varpi(\xi,\eta)=\frac{2\left[\beta-\beta\gamma\tanh\left[\beta(\xi-2\beta\eta)\right]\right]}{\alpha}. \tag{4.7} $$



(a)

(b)

Figure 1. $3D$ nature of $\varpi(\xi,\eta)$ for (a) $\zeta = 0.5$, (b) $\zeta = 0.8$ at $\alpha = 1$, $\beta = 0.1$ and $\gamma = 1$ for application **1**.

Application 2. Consider the (3+1)-dimensional fractional Burger's equation

$$^{ABC}D_{\eta}^{\zeta}\varpi(\xi,\vartheta,\omega,\eta) = \frac{\partial^2\varpi}{\partial\xi^2} + \frac{\partial^2\varpi}{\partial\vartheta^2} + \frac{\partial^2\varpi}{\partial\omega^2} + \varpi\frac{\partial\varpi}{\partial\xi},\ 0 < \zeta \leq 1, \tag{4.8}$$

with the initial condition

$$\varpi(\xi,\vartheta,\omega,0) = \xi + \vartheta + \omega. \tag{4.9}$$

Applying the ST to both sides of equation (4.8) and on simplification, we get

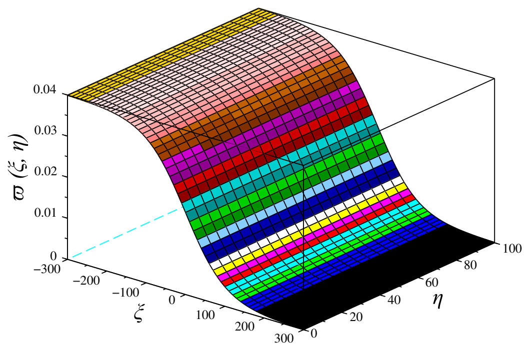
$$[\varpi(\xi,\vartheta,\omega,\eta)] = \left(\frac{\rho}{\upsilon}\right)\varpi(\xi,\vartheta,\omega,0) + \frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)} S\left[\frac{\partial^2\varpi}{\partial\xi^2} + \frac{\partial^2\varpi}{\partial\vartheta^2} + \frac{\partial^2\varpi}{\partial\omega^2} + \varpi\frac{\partial\varpi}{\partial\xi}\right]. \tag{4.10}$$

Taking inverse Shehu transform (ST) to both sides of (4.10), we get

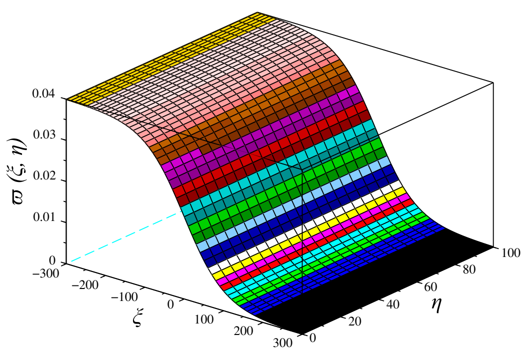
$$\begin{aligned}\varpi(\xi,\vartheta,\omega,\eta) &= \varpi(\xi,\vartheta,\omega,0) \\ &+ S^{-1}\left[\left(\frac{\left(1-\zeta+\zeta\left(\frac{\rho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}\right) S\left[\frac{\partial^2\varpi}{\partial\xi^2} + \frac{\partial^2\varpi}{\partial\vartheta^2} + \frac{\partial^2\varpi}{\partial\omega^2} + \varpi\frac{\partial\varpi}{\partial\xi}\right]\right].\end{aligned} \tag{4.11}$$

Assume that solution of unknown function $\varpi(\xi,\vartheta,\omega,\eta)$ in an infinite series as

$$\varpi(\xi,\vartheta,\omega,\eta) = \sum_{\ell=0}^{\infty}\varpi_{\ell}(\xi,\vartheta,\omega,\eta),$$



(Exact Sol.)



(Approximate Sol.)

Figure 2. $3D$ nature of $\varpi(\xi, \eta)$ at $\alpha = 1$, $\beta = 0.01$ and $\gamma = 1$ for application **1**.

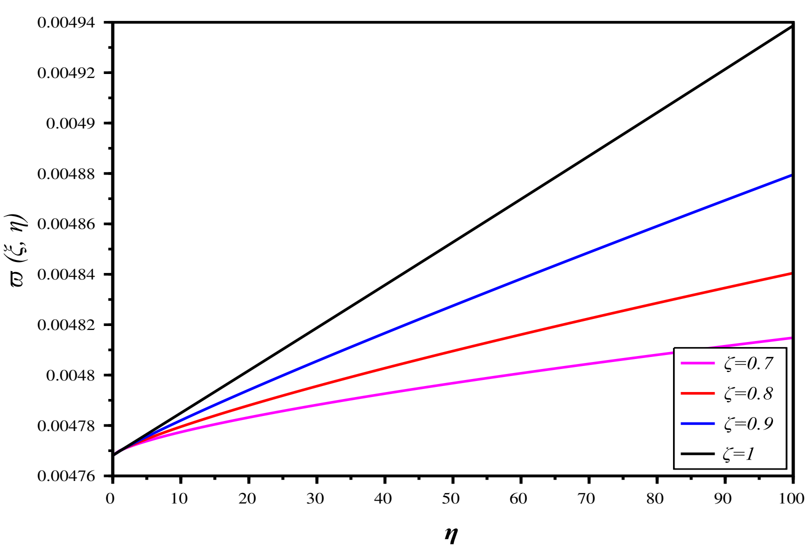


Figure 3. Graphical nature of $\varpi(\xi, \eta)$ with respect to η at $\alpha = 1$, $\beta = 0.01$ and $\gamma = 1$ for application **1**.

and the nonlinear term by means of the Adomian polynomials is described as $\varpi \frac{\partial \varpi}{\partial \xi} = \sum_{\ell=0}^{\infty} A_\ell$.

Table 1: The error between the exact and third order approximation solution of ϖ_{SDM} with FVIM for application 1.

ξ	η	*Exact*	*Approximate* $\zeta = 1$	$\lvert\varpi_{Exact} - \varpi_{SDM}\rvert$ $\zeta = 1$	$\lvert\varpi_{Exact} - \varpi_{FVIM}\rvert$ $\zeta = 1$
	0	0.0399011	0.0399011	0	0
	20	0.0399019	0.0399019	1.6237×10^{-14}	8.36083×10^{-12}
	40	0.0399027	0.0399027	2.59348×10^{-13}	6.67572×10^{-11}
-300	60	0.0399034	0.0399034	1.31092×10^{-12}	2.2487×10^{-10}
	80	0.0399042	0.0399042	4.13682×10^{-12}	5.31996×10^{-10}
	100	0.039905	0.039905	1.00842×10^{-11}	1.03705×10^{-9}
	0	0.0352319	0.0352319	0	0
	20	0.0352654	0.0352654	1.40942×10^{-13}	2.47381×10^{-10}
	40	0.0352987	0.0352987	2.23993×10^{-12}	1.98016×10^{-9}
-100	60	0.0353318	0.0353318	1.12634×10^{-11}	6.68676×10^{-9}
	80	0.0353646	0.0353646	3.5358×10^{-11}	1.58587×10^{-8}
	100	0.0353973	0.0353973	8.57394×10^{-11}	3.09908×10^{-8}
	0	0.00476812	0.00476812	0	0
	20	0.00480182	0.00480182	1.42837×10^{-13}	1.78436×10^{-11}
	40	0.00483572	0.00483572	2.30062×10^{-12}	1.41591×10^{-10}
100	60	0.00486984	0.00486984	1.17243×10^{-11}	4.73909×10^{-10}
	80	0.00490415	0.00490415	3.73003×10^{-11}	1.11383×10^{-9}
	100	0.00493868	0.00493868	9.16666×10^{-11}	2.15664×10^{-9}
	0	0.0000989049	0.0000989049	0	4.20128×10^{-19}
	20	0.0000996974	0.0000996974	1.62847×10^{-14}	8.22805×10^{-22}
	40	0.000100496	0.000100496	2.60942×10^{-13}	6.59551×10^{-11}
300	60	0.000101301	0.000101301	1.32305×10^{-12}	2.2304×10^{-10}
	80	0.000102113	0.000102113	4.18795×10^{-12}	5.29741×10^{-10}
	100	0.000102931	0.000102931	1.02402×10^{-11}	1.03671×10^{-9}

Hence, the equation (4.11) is rewritten as

$$\sum_{\ell=0}^{\infty} \varpi(\xi,\vartheta,\omega,\eta) = \varpi(\xi,\vartheta,\omega,0) + S^{-1}\left[\left(\frac{\left(1-\zeta+\zeta\left(\frac{\varrho}{\upsilon}\right)^{\zeta}\right)}{\mathcal{A}(\zeta)}\right) S\left[\sum_{\ell=0}^{\infty}\frac{\partial^2\varpi_\ell}{\partial\xi^2} + \sum_{\ell=0}^{\infty}\frac{\partial^2\varpi_\ell}{\partial\vartheta^2} + \sum_{\ell=0}^{\infty}\frac{\partial^2\varpi_\ell}{\partial\omega^2} + \sum_{\ell=0}^{\infty} A_\ell\right]\right]. \tag{4.12}$$

Thus, according to our proposed scheme, by equating both sides of equation (4.12), we get

$$\varpi_0(\xi,\vartheta,\omega,\eta) = \xi+\vartheta+\omega,$$

$$\varpi_1(\xi,\vartheta,\omega,\eta) = (\xi+\vartheta+\omega)\left(\frac{1-\zeta+\zeta\left(\frac{\eta^\zeta}{\Gamma(\zeta+1)}\right)}{\mathcal{A}(\zeta)}\right),$$

$$\varpi_2(\xi,\vartheta,\omega,\eta) = (\xi+\vartheta+\omega)\frac{1}{(\mathcal{A}(\zeta))^2}\left(1+2\zeta\left(1+\frac{\eta^\zeta}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1+\frac{\eta^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\eta^\zeta}{\Gamma(\zeta+1)}\right)\right),$$

similarly, we obtain next terms in the same manner. Hence, the approximate solution of (4.8) is given as

$$\varpi(\xi,\vartheta,\omega,\eta) = (\xi+\vartheta+\omega) + (\xi+\vartheta+\omega)\left(\frac{1-\zeta+\zeta\left(\frac{\eta^\zeta}{\Gamma(\zeta+1)}\right)}{\mathcal{A}(\zeta)}\right) + (\xi+\vartheta+\omega)\frac{1}{(\mathcal{A}(\zeta))^2}\left(1+2\zeta\left(1+\frac{\eta^\zeta}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1+\frac{\eta^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\eta^\zeta}{\Gamma(\zeta+1)}\right)\right)+\ldots. \tag{4.13}$$

In particular, for $\zeta = 1$, equation (4.8) converges rapidly to the exact solution and is given as

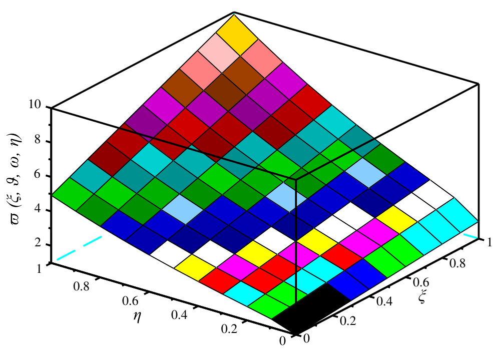
$$\varpi(\xi,\vartheta,\omega,\eta) = \frac{\xi+\vartheta+\omega}{1-\eta}. \tag{4.14}$$

5 Results and discussions

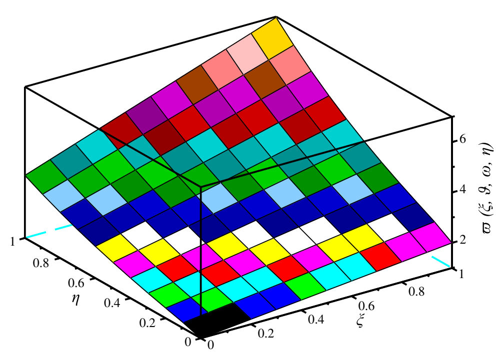
In this section, we present and analyze the results of numerical simulations for various values of the fractional order ζ. Table 1 displays the approximate numerical solutions along with the

Table 2: The error between the exact and third order approximation solution of ϖ_{SDM} for application 2.

ξ	η	$Approximate$ $\zeta = 0.9$	$Approximate$ $\zeta = 1$	$Exact$	$\lvert\varpi_{Exact} - \varpi_{SDM}\rvert$ $\zeta = 1$
	0	0.2	0.2	0.2	0
-0.8	0.2	0.267012	0.249999	0.25	6.4×10^{-7}
	0.4	0.383913	0.333115	0.333333	0.000218453
	0.6	0.639169	0.491602	0.5	0.00839808
	0	0.5	0.5	0.5	0
-0.5	0.2	0.66753	0.624998	0.625	1.6×10^{-6}
	0.4	0.959782	0.832787	0.83333	0.000546133
	0.6	1.59792	1.229	1.25	0.0209952
	0	1	1	1	0
0	0.2	1.33506	1.25	1.25	3.2×10^{-6}
	0.4	1.91956	1.66557	1.66667	0.00109227
	0.6	3.19585	2.45801	2.5	0.0419904
	0	1.5	1.5	1.5	0
0.5	0.2	2.00259	1.875	1.875	4.8×10^{-6}
	0.4	2.87935	2.49836	2.5	0.0016358
	0.6	4.79377	3.68701	3.75	0.0629856
	0	2	2	2	0
1	0.2	2.67012	2.49999	2.5	6.4×10^{-6}
	0.4	3.83913	3.33115	3.33333	0.00218453
	0.6	6.39169	4.91602	5	0.0839808



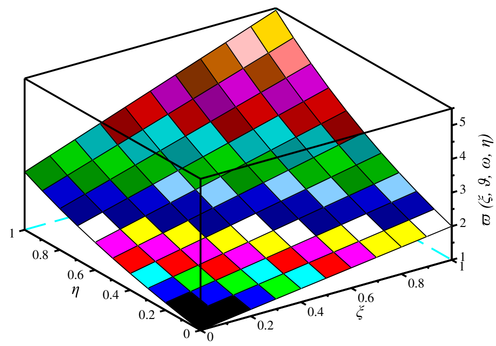
(a)



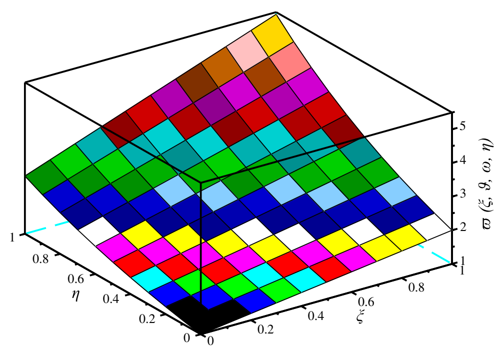
(b)

Figure 4. $3D$ nature of $\varpi(\xi, \eta)$ (a) $\zeta = 0.5$, (b) $\zeta = 0.75$ at $\vartheta = 0.5$ and $\omega = 0.5$ for application **2**.

absolute error comparison at $\zeta = 1$, obtained using the Shehu decomposition method (SDM) and the fractional variational iteration method (FVIM) for Application 1. Figure 1 illustrates the three-dimensional (3D) profiles of the approximate solutions corresponding to $\zeta = 0.5$ and $\zeta = 0.8$. A comparison between the approximate and exact solutions for Application 1 is shown



(a)



(b)

Figure 5. $3D$ nature of $\varpi(\xi, \eta)$ (a) $\zeta = 1$, (b) Exact solution at $\vartheta = 0.5$ and $\omega = 0.5$ for application **2**.

in Figure 2. Additionally, Figure 3 presents two-dimensional (2D) plots of the approximate solutions for various values of ζ.

Furthermore, Table 2 provides the approximate solutions computed via SDM for $\zeta = 0.9$ and $\zeta = 1$, including a detailed comparison of absolute errors for different parameter values $\eta = 0, 0.2, 0.4, 0.6$ and $\xi = -0.8, -0.5, 0, 0.5, 1$ in Application 2. The 3D plots of the exact solution and the SDM-generated solutions to equation (4.8) for $\zeta = 0.5$, 0.75, and 1 are depicted in Figures 4 and 5.

6 Canclusion

In this study, the Shehu decomposition method (SDM) is employed to construct both analytical and numerical solutions for two cases of the time-fractional Burgers' equations. The method generates rapidly convergent series solutions with explicitly computable terms, without the need for perturbation methods, linearization, or additional restrictive assumptions. The accuracy and efficiency of the proposed scheme are validated through numerical simulations and corresponding graphical representations. These results demonstrate that SDM serves as a robust and effective technique for the solution of time-fractional partial differential equations.

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