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# ON A CLASS OF MULTIVARIATE HUMBERT-HERMITE POLYNOMIALS

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**Abstract** The current paper introduces a new class of multivariate Humbert-Hermite polynomials ${}_HQ_n(x_1, x_2, ..., x_r)$ . The defined class of polynomials is a generalization of Humbert-Hermite polynomials (Pathan, 2021) and generalized Humbert-Hermite polynomials of two variables (N. Khan et. al, 2023). It also includes many known and unknown polynomials like Gegenbauer, Legendre, Chebycheff, Gould, Sinha, Milovanović-Djordjević, Horadam, Horadam-Pethe, Pathan, and Khan, etc. In this paper, we will establish many properties of the considering polynomials like power series representations, generating functions, integral representation, and several expansions.

## 1 Introduction

Generating function of the Humbert polynomial is introduced in [1] in the following manner:

$$(1 - \alpha z t + t^{\alpha})^{-\lambda} = \sum_{n=0}^{\infty} h_{n,\alpha}^{\lambda}(z) t^n, \tag{1.1}$$

where the power series representation of $h_{n,\alpha}^{\lambda}(z)$ is given by Humbert [2] as follows:

$$h_{n,\alpha}^{\lambda}(z) = \sum_{k=0}^{[\frac{n}{\alpha}]} \frac{(-1)^k (\lambda)_{n+(1-\alpha)k} (\alpha z)^{n-\alpha k}}{k!(n-\alpha k)!}, \tag{1.2}$$

where $(\lambda)_n$ represents the Pochhammer symbol defined in [3] as-

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } n = 1, 2, 3... \end{cases} \tag{1.3}$$

In 1965, Gould [4] gave the generalization of generating function (1.1) as follows:

$$(c - \alpha z t + y t^{\alpha})^{p} = \sum_{n=0}^{\infty} P_n(\alpha, z, y, p, c) t^n, \tag{1.4}$$

where $\alpha \in \mathbb{N}$,

$$P_n(\alpha, z, y, p, c) = \sum_{k=0}^{[\frac{n}{\alpha}]} \binom{p}{k} \binom{p-k}{n-\alpha k} c^{p-n+(\alpha-1)k} y^k (-\alpha z)^{n-\alpha k}, \tag{1.5}$$

and other parameters are unrestricted in general.
Many more generalizations of the Humbert polynomial of one variable are available in the literature (see [5], [6], [7]). In 1997, Pathan and Khan [8] introduced generalization of the above

polynomials in one variable, shown below:

$$[a - bzt + dt^{\alpha}(2z-1)^{g}]^{-p} = \sum_{n=0}^{\infty} \Theta_n(z)t^n, \tag{1.6}$$

where

$$\Theta_n(z) = \sum_{k=0}^{[\frac{n}{\alpha}]} \frac{a^{-p-n-k+\alpha k}(p)_{n+(1-\alpha)k}(bz)^{n-\alpha k}\{-d(2z-1)^g\}^k}{k!(n-\alpha k)!}. \tag{1.7}$$

Djordjević [9] introduced a polynomial of two variables in the following manner:

$$[1 - 2(z+y)t + t^{\alpha}(2zy+1)]^{-\nu} = \sum_{n=0}^{\infty} G_n^{\nu,\alpha}(y,z)t^n, \tag{1.8}$$

where

$$G_n^{\nu,\alpha}(y,z) = \sum_{k=0}^{[\frac{n}{\alpha}]} \frac{(-1)^k(\nu)_{n+(1-\alpha)k}(2y+2z)^{n-\alpha k}(2yz+1)^k}{k!(n-\alpha k)!}. \tag{1.9}$$

Recently, Pathan and Khan [10] gave a unified presentation of above mentioned various polynomials as follows:

$$[a - (bz+cy)t + dt^{\alpha}(ezy-1)^g]^{-p} = \sum_{n=0}^{\infty} Q_n^{p,\alpha}(z,y)t^n, \tag{1.10}$$

where $\alpha \in \mathbb{N}, p > 0$,

$$Q_n^{p,\alpha}(z,y) = \sum_{k=0}^{[\frac{n}{\alpha}]} \frac{a^{-p-n-k+\alpha k}(p)_{n+(1-\alpha)k}(bz+cy)^{n-\alpha k}\{-d(ezy-1)^g\}^k}{k!(n-\alpha k)!}, \tag{1.11}$$

and other parameters are unrestricted in general.
Recently, Jaimini et. al defined and studied multivariate Humbert polynomials [11] in the following manner:

$$[a - (\sum_{i=1}^{r} b_i x_i)t + dt^{\alpha}(c\Pi_{i=1}^{r} x_i - 1)^g]^{-p} = \sum_{n=0}^{\infty} Q_n^{p,\alpha}(x_1, x_2, ..., x_r)t^n, \tag{1.12}$$

where $\alpha, r \in \mathbb{N}, p > 0$, and other parameters are unrestricted in general.
The Gould-Hopper polynomial $H_n^{\alpha}(y,z)$ was introduced by Dattoli et al. [12] as follows-

$$e^{yt+zt^{\alpha}} = \sum_{n=0}^{\infty} H_n^{\alpha}(y,z)\frac{t^n}{n!}. \tag{1.13}$$

Further, a new generalization of Hermite polynomial [8] was defined as follows-

$$e^{\mu(y+z)t-(yz+1)t^{\alpha}} = \sum_{n=0}^{\infty} H_{n,\alpha,\mu}(y,z)\frac{t^n}{n!}. \tag{1.14}$$

In 1996, Dattoli et. al gave generating function of $N$-variable generalized Hermite polynomials $H_n\{(z)_1^N\}$[13] as-

$$exp\sum_{r=1}^{N} z_r t^r = \sum_{n=0}^{\infty} H_n(\{z\}_1^N)\frac{t^n}{n!}, \tag{1.15}$$

where $(\{z\}_1^N) = z_1, z_2, ..., z_N$.
In 2021, Khan and Pathan [14] introduced a new concept of Humbert-Hermite polynomials ${}_HG_n^{p,\mu,\alpha}(y,z)$ in two variables as follows-

$$[1 - 2(z+y)t + t^{\alpha}(2zy+1)]^{-p}e^{\mu(y+z)t-(yz+1)t^{\alpha}} = \sum_{n=0}^{\infty} {}_HG_n^{p,\mu,\alpha}(y,z)t^n, \tag{1.16}$$

where $\alpha \in \mathbb{N}, p > 0, \mu > 0$, and other parameters are unrestricted in general.
Recently, Khan et. al defined a new class of generalized Humbert-Hermite polynomials in two variables [15] such as-

$$[a-(bz+cy)t+dt^{\alpha}(ezy-1)^{g}]^{-p}e^{\mu(y+z)t-(yz+1)t^{r}}=\sum_{n=0}^{\infty}{}_{H}Q_{n}^{p,\mu,\alpha,r}(y,z)t^{n}, \tag{1.17}$$

where, $r \in \mathbb{N}, g, \mu, p > 0$, and the other parameters are unrestricted in general. Here, in this paper, we consider a generalized form of Hermite polynomial in $r$ variables such as-

$$e^{\mu(\sum_{i=1}^{r}x_i)t-\nu(\Pi_{i=1}^{r}x_i+1)t^{\eta}}=\sum_{s=0}^{\infty}H_{s,\mu,\eta}^{\nu}(x_1,x_2,...,x_r)\frac{t^s}{s!}, \tag{1.18}$$

where $\mu, \nu \geq 0$, and $r \in \mathbb{N}, \eta > 0$. The considering Hermite polynomial converts into ordinary Hermite polynomial $H_n(x)$ at $r = 2, x_2 = 0, \mu = 2, \nu = 1, \eta = 2$.

## 2 On a class of multivariate Humbert-Hermite polynomials

In an attempt to extend the definition of Humbert-Hermite polynomials, we consider the following form of Humbert-Hermite polynomials for several variables.

**Definition 2.1.** As a generalization of the class of generalized Humbert-Hermite polynomials in two variables, we define multivariate Humbert-Hermite polynomials for $\eta \in \mathbb{N}, \mu, \nu, p \geq 0$, as follows:

$$[a-(\sum_{i=1}^{r}b_ix_i)t+dt^{\alpha}(c\Pi_{i=1}^{r}x_i-1)^{g}]^{-p}e^{\mu(\sum_{i=1}^{r}x_i)t-\nu(\Pi_{i=1}^{r}x_i+1)t^{\eta}}$$
$$=\sum_{n=0}^{\infty}{}_{H}Q_{n,\mu,\eta}^{p,\alpha,\nu}(x_1,x_2,...,x_r)t^n, \tag{2.1}$$

and other parameters are unrestricted in general.

**Special Cases**:
(i) If, we consider $r = 2, \nu = 1$ in (2.1), then we get generalized Humbert-Hermite polynomials (1.17).
(ii) Let $r = 2, \eta = \alpha, \nu = a = g = 1, b_1 = b_2 = 2, d = -1, c = -2$ in definition 2.1, then the definition convert into relation (1.16).
(iii) If in relation (2.1), we substitute $r = c = 2, b_2 = 0, \nu = \mu = \eta = x_2 = 1$, then we get a generating function (1.6) studied by Pathan and Khan.
Similarly, by substitution of appropriate values of parameters and variables, we obtain many known and unknown polynomials.

**Theorem 2.2.** *Let $\eta \in \mathbb{N}; \nu, \mu, p \geq 0$, and other parameters are unrestricted in general, then the polynomials ${}_{H}Q_{n,\mu,\eta}^{p,\alpha,\nu}(x_1, x_2, ..., x_r)$ can be expressed in product of multivariate Humbert and Hermite polynomial such as-*

$${}_{H}Q_{n,\mu,\eta}^{p,\alpha,\nu}(x_1,x_2,...,x_r)=\sum_{s=0}^{n}\frac{Q_{n-s}^{p,\alpha}(x_1,x_2,...,x_r)H_{s,\mu,\eta}^{\nu}(x_1,x_2,...,x_r)}{s!}. \tag{2.2}$$

*Proof.* In view of (1.12), and (1.18), L. H. S. of equ. (2.1) can be written as-

$$[a-(\sum_{i=1}^{r}b_ix_i)t+dt^{\alpha}(c\Pi_{i=1}^{r}x_i-1)^{g}]^{-p}e^{\mu(\sum_{i=1}^{r}x_i)t-\nu(\Pi_{i=1}^{r}x_i+1)t^{\eta}}$$
$$=\sum_{n=0}^{\infty}Q_{n}^{p,\alpha}(x_1,x_2,...,x_r)t^n\sum_{s=0}^{\infty}H_{s,\mu,\eta}^{\nu}(x_1,x_2,...,x_r)\frac{t^s}{s!}. \tag{2.3}$$

Replacing $n \to n-s$, and by series manipulation [16], we get

$$[a-(\sum_{i=1}^{r} b_i x_i)t + dt^{\alpha}(c\Pi_{i=1}^{r} x_i - 1)^g]^{-p} e^{\mu(\sum_{i=1}^{r} x_i)t - \nu(\Pi_{i=1}^{r} x_i + 1)t^{\eta}}$$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{n} Q_{n-s}^{p,\alpha}(x_1, x_2, ..., x_r) H_{s,\mu,\eta}^{\nu}(x_1, x_2, ..., x_r)\frac{t^n}{s!}. \quad (2.4)$$

Now comparing the coefficient of $t^n$ with R. H. S. of (2.1), we get the desired relation (2.2). □

**Remark 2.3.** For $r = 2, \nu = 1$ Theorem 2.1 reduces to the similar results of Khan et. al [15, pg.4, equ.(2.2)].

**Remark 2.4.** If, we put $r = 2, \eta = \alpha, \nu = a = g = 1, b_1 = b_2 = 2, d = -1, c = -2$ in (2.2), then we get the corrected form of Humbert-Hermite polynomials in two variables, such as-

$${}_H G_{n,\mu}^{p,\alpha}(x_1, x_2) = \sum_{s=0}^{n} \frac{H_{s,\mu,\alpha}(x_1, x_2) G_{n-s}^{p,\alpha}(x_1, x_2)}{s!}. \quad (2.5)$$

**Remark 2.5.** Substitute $r = g = 2, \nu = a = d = 1, b_2 = c = 0, b_1 = \alpha = 2$ in (2.2), we have Hermite-Gegenbaurer polynomials of two variables[15, pg.4, equ.2.5] which turned into Hermite-Chebychev, and Hermite-Legendre polynomials of two variables at $p = 1$, and $p = \frac{1}{2}$, respectively.

**Theorem 2.6.** *Let $j \in \mathbb{N}, \mu, \nu, p \geq 0$, and other parameters are unrestricted in general, then the multivariate Humbert-Hermite polynomials hold the following result:*

$$\sum_{s=0}^{n} \frac{H_s^{\eta}(\mu j(\sum_{i=1}^{r} x_i), -j\nu(\Pi_{i=1}^{r} x_i + 1)) Q_{n-s}^{pj,\alpha}(x_1, x_2, ..., x_r)}{s!}$$

$$= \sum_{n_1+n_2+...+n_j=n} {}_H Q_{n_1,\mu,\eta}^{p,\alpha,\nu}(x_1, x_2, ..., x_r) \cdots {}_H Q_{n_j,\mu,\eta}^{p,\alpha,\nu}(x_1, x_2, ..., x_r). \quad (2.6)$$

*Proof.* According to definition 2.1, we can obtain

$$\Big[\sum_{n=0}^{\infty} {}_H Q_{n,\mu,\eta}^{p,\alpha,\nu}(x_1, x_2, ..., x_r) t^n\Big]^j$$

$$= \Big[[a-(\sum_{i=1}^{r} b_i x_i)t + dt^{\alpha}(c\Pi_{i=1}^{r} x_i - 1)^g]^{-p} e^{\mu(\sum_{i=1}^{r} x_i)t - \nu(\Pi_{i=1}^{r} x_i + 1)t^{\eta}}\Big]^j$$

$$= [a-(\sum_{i=1}^{r} b_i x_i)t + dt^{\alpha}(c\Pi_{i=1}^{r} x_i - 1)^g]^{-pj} e^{\mu j(\sum_{i=1}^{r} x_i)t - j\nu(\Pi_{i=1}^{r} x_i + 1)t^{\eta}}. \quad (2.7)$$

In view of (1.13), we have

$$e^{\mu j(\sum_{i=1}^{r} x_i)t - j\nu(\Pi_{i=1}^{r} x_i + 1)t^{\eta}} = \sum_{s=0}^{\infty} \frac{H_s^{\eta}(\mu j(\sum_{i=1}^{r} x_i), -j\nu(\Pi_{i=1}^{r} x_i + 1))t^s}{s!}. \quad (2.8)$$

Now, using (1.12), and (2.8) in R. H. S. of (2.7)

$$\Big[\sum_{n=0}^{\infty} {}_H Q_{n,\mu,\eta}^{p,\alpha,\nu}(x_1, x_2, ..., x_r) t^n\Big]^j$$

$$= \sum_{n=0}^{\infty} Q_n^{pj,\alpha}(x_1, x_2, ..., x_r) t^n \sum_{s=0}^{\infty} \frac{H_s^{\eta}(\mu j(\sum_{i=1}^{r} x_i), -j\nu(\Pi_{i=1}^{r} x_i + 1))t^s}{s!}. \quad (2.9)$$

So,

$$\sum_{n=0}^{\infty}\sum_{n_1+n_2+...+n_j=n} {}_HQ^{p,\alpha,\nu}_{n_1,\mu,\eta}(x_1,x_2,...,x_r)\cdots {}_HQ^{p,\alpha,\nu}_{n_j,\mu,\eta}(x_1,x_2,...,x_r)t^n$$
$$=\sum_{n=0}^{\infty} Q^{pj,\alpha}_n(x_1,x_2,...,x_r)t^n \sum_{s=0}^{\infty}\frac{H^{\eta}_s(\mu j(\sum_{i=1}^r x_i), -j\nu(\Pi_{i=1}^r x_i+1))t^s}{s!}. \quad (2.10)$$

Replacing $n \to n-s$ and by series manipulation, the equation becomes

$$\sum_{n=0}^{\infty}\sum_{n_1+n_2+...+n_j=n} {}_HQ^{p,\alpha,\nu}_{n_1,\mu,\eta}(x_1,x_2,...,x_r)\cdots {}_HQ^{p,\alpha,\nu}_{n_j,\mu,\eta}(x_1,x_2,...,x_r)t^n$$
$$=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{H^{\eta}_s(\mu j(\sum_{i=1}^r x_i), -j\nu(\Pi_{i=1}^r x_i+1))Q^{pj,\alpha}_{n-s}(x_1,x_2,...,x_r)t^n}{s!}. \quad (2.11)$$

Now comparison of coefficients of $t^n$, provides the desired result (2.6). □

**Corollary 2.7.** *Consider $r=2,\nu=1$ in theorem 2.2, the resulting theorem will give the correct result of the theorem studied by Khan et. al [15, Theorem 2.1, pg.4] as follows:*

$$\sum_{s=0}^{n}\frac{H^{\eta}_s(\mu j(x_1+x_2), -j(x_1x_2+1))Q^{pj,\alpha}_{n-s}(x_1,x_2)}{s!}$$
$$=\sum_{n_1+n_2+...+n_j=n} {}_HQ^{p,\alpha}_{n_1,\mu,\eta}(x_1,x_2)\cdots {}_HQ^{p,\alpha}_{n_j,\mu,\eta}(x_1,x_2). \quad (2.12)$$

**Corollary 2.8.** *For $r=2,\eta=\alpha,\nu=a=g=1,b_1=b_2=2,d=-1,c=-2$, theorem 2.2 converts into the corrected form of the theorem proved by Pathan and Khan [14], such as-*

$$\sum_{s=0}^{n}\frac{H^{\alpha}_s(\mu j(x_1+x_2), -j(x_1x_2+1))G^{pj,\alpha}_{n-s}(x_1,x_2)}{s!}$$
$$=\sum_{n_1+n_2+...+n_j=n} {}_HG^{p,\alpha}_{n_1,\mu}(x_1,x_2)\cdots {}_HG^{p,\alpha}_{n_j,\mu}(x_1,x_2). \quad (2.13)$$

**Remark 2.9.** For $r=\mu=\eta=2, p=x_2=0, \nu=1$ Theorem 2.2 reduces to a former result studied by Batahan and Shehata[17, pg. 50, equ. (2.1)].

## 3 Power Series Representations

**Theorem 3.1.** *Let $\alpha\geq 2, r,\eta\in\mathbb{N}$, and $\nu,\mu,p\geq 0$, then*

$${}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)$$
$$=\sum_{s=0}^{n}\sum_{k=0}^{[\frac{n-s}{\alpha}]}\frac{(p)_{n-s-(\alpha-1)k}}{k!s!(n-s-\alpha k)!}a^{-p-n+s+(\alpha-1)k}\{-d(c\Pi_{i=1}^r x_i-1)^g\}^k$$
$$\times\left(\sum_{i=1}^{r} b_i x_i\right)^{n-s-\alpha k} H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r), \quad (3.1)$$

$$
\begin{aligned}
&{}_{H}Q^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)\\
&= \sum_{q=0}^{n}\sum_{k=0}^{[\frac{n-q-(\alpha-2)s}{2}]}\sum_{s=0}^{k}\frac{a^{-p-n+q+(\alpha-2)s}(p)_k(-k)_s(2p+2k)_{n-q-(\alpha-2)s-2k}}{(n-q-(\alpha-2)s-2k)!k!s!q!}\\
&\times\left(\frac{(\sum_{i=1}^{r}b_ix_i)}{2a}\right)^{n-q+(2-\alpha)s}\left(\frac{4ad(c\Pi_{i=1}^{r}x_i-1)^g}{(\sum_{i=1}^{r}b_ix_i)^2}\right)^{s}H^{\nu}_{q,\mu,\eta}(x_1,x_2,...,x_r). \quad (3.2)
\end{aligned}
$$

*Proof.* Let

$$
\begin{aligned}
\Phi &= [a-(\sum_{i=1}^{r}b_ix_i)t+dt^{\alpha}(c\Pi_{i=1}^{r}x_i-1)^g]^{-p}e^{\mu(\sum_{i=1}^{r}x_i)t-\nu(\Pi_{i=1}^{r}x_i+1)t^{\eta}}\\
&= a^{-p}\sum_{n=0}^{\infty}\frac{(p)_n}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{(\sum_{i=1}^{r}b_ix_i)t}{a}\right)^{n-k}\left(\frac{-dt^{\alpha}(c\Pi_{i=1}^{r}x_i-1)^g}{a}\right)^{k}\\
&\times\sum_{s=0}^{\infty}H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)\frac{t^s}{s!}. \quad (3.3)
\end{aligned}
$$

Replacing $n\to n+k$; $n\to n-\alpha k$, and series manipualtion, we have

$$
\begin{aligned}
\Phi &= a^{-p}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{[\frac{n}{\alpha}]}\frac{(p)_{n-(\alpha-1)k}a^{-(n-(\alpha-1)k)}}{k!(n-\alpha k)!}(-d(c\Pi_{i=1}^{r}x_i-1)^g)^k\\
&\times(\sum_{i=1}^{r}b_ix_i)^{n-\alpha k}t^nH^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)\frac{t^s}{s!}. \quad (3.4)
\end{aligned}
$$

Now, replacing $n\to n-s$, and comparing the coefficient of $t^n$ with R. H. S. of (2.1), we at once arrive at our required result (3.1).
Again, L. H. S. of (2.1) can be written as

$$
\begin{aligned}
\Phi &= [a-(\sum_{i=1}^{r}b_ix_i)t+dt^{\alpha}(c\Pi_{i=1}^{r}x_i-1)^g]^{-p}e^{\mu(\sum_{i=1}^{r}x_i)t-\nu(\Pi_{i=1}^{r}x_i+1)t^{\eta}}\\
&= a^{-p}\left[\left(1-\frac{(\sum_{i=1}^{r}b_ix_i)t}{2a}\right)^2-\left(\frac{(\sum_{i=1}^{r}b_ix_i)t}{2a}\right)^2+\frac{dt^{\alpha}(c\Pi_{i=1}^{r}x_i-1)^g}{a}\right]^{-p}\\
&\times\sum_{q=0}^{\infty}H^{\nu}_{q,\mu,\eta}(x_1,x_2,...,x_r)\frac{t^q}{q!}. \quad (3.5)
\end{aligned}
$$

Let us consider $\frac{(\sum_{i=1}^{r}b_ix_i)}{2a}=A$, $\frac{d(c\Pi_{i=1}^{r}x_i-1)^g}{a}=B$.
So, the above equation reduces as

$$
\begin{aligned}
\Phi &= a^{-p}(1-At)^{-2p}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\frac{(p)_k}{k!}\left(\frac{A^2t^2-Bt^{\alpha}}{(1-At)^2}\right)^kH^{\nu}_{q,\mu,\eta}(x_1,x_2,...,x_r)\frac{t^q}{q!}\\
&= a^{-p}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(2p+2k)_n}{k!n!}(At)^n\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}(A^2t^2)^{k-s}(-Bt^{\alpha})^s\\
&\times H^{\nu}_{q,\mu,\eta}(x_1,x_2,...,x_r)\frac{t^q}{q!}. \quad (3.6)
\end{aligned}
$$

Replacing $n\to n-(\alpha-2)s-2k$; using the identity [18, pg. 36, equ. (4.4)]; replacing $n\to n-q$, and then equating coefficient of $t^n$ with R. H. S. of (2.1), we at once arrive at our required result (3.2). □

## 4 Generating Functions

**Theorem 4.1.** *For an arbitrary number* $\omega, \nu, \mu, p \geq 0$*, and* $\eta \in \mathbb{N}$ *there exist some generating functions:*

$$\sum_{n=0}^{\infty} \frac{{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,...,x_r)t^n}{(p)_n} = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{a^{-p-n}\{(\sum_{i=1}^r b_i x_i)t\}^n t^s H^{\nu}_{s,\mu,\eta}(x_1,...,x_r)}{(p+n)_s n! s!} \times {}_1F_{\alpha}\left[\begin{array}{c} p+n; \\ \frac{p+n+s}{\alpha}, \frac{p+n+s+1}{\alpha}, ..., \frac{p+n+s+\alpha-1}{\alpha}; \end{array} ; -\frac{dt^{\alpha}(c\Pi_{i=1}^r x_i - 1)^g}{a\alpha^{\alpha}}\right], \quad (4.1)$$

$$\sum_{n=0}^{\infty} \frac{{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,...,x_r)(\omega)_n t^n}{(p)_n} = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{\{(\sum_{i=1}^r b_i x_i)t\}^n H^{\nu}_{s,\mu,\eta}(x_1,...,x_r)(\omega)_{n+s}}{a^{p+n}(p+n)_s s! n!} \times t^s{}_{\alpha+1}F_{\alpha}\left[\begin{array}{c} p+n, \frac{\omega+n+s}{\alpha}, \frac{\omega+n+s+1}{\alpha}, ..., \frac{\omega+n+s+\alpha-1}{\alpha} \\ \frac{p+n+s}{\alpha}, \frac{p+n+s+1}{\alpha}, ..., \frac{p+n+s+\alpha-1}{\alpha}; \end{array} ; -\frac{dt^{\alpha}(c\Pi_{i=1}^r x_i - 1)^g}{a}\right], \quad (4.2)$$

$$\sum_{n=0}^{\infty} \frac{{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)t^n}{(2p)_n} = \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{2^{-2k}a^{-p-n-2k}(-k)_s}{(p+\frac{1}{2})_k(2p+2k+n)_{q+(\alpha-2)s}} \times \left(\frac{(\sum_{i=1}^r b_i x_i)t}{2a}\right)^{n+2k}\left(\frac{4adt^{(\alpha-2)}(c\Pi_{i=1}^r x_i - 1)^g}{(\sum_{i=1}^r b_i x_i)^2}\right)^s \frac{H^{\nu}_{q,\mu,\eta}(x_1,x_2,...,x_r)t^q}{n!q!k!s!}, \quad (4.3)$$

$$\sum_{n=0}^{\infty} \frac{{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)(\omega)_n t^n}{(2p)_n} = \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{2^{-2k}a^{-p-n-2k}(-k)_s}{n!q!k!s!} \times \frac{(\omega)_{n+2k}(\omega+n+2k)_{q+(\alpha-2)s}H^{\nu}_{q,\mu,\eta}(x_1,x_2,...,x_r)t^q}{(p+\frac{1}{2})_k(2p+2k+n)_{q+(\alpha-2)s}}\left(\frac{(\sum_{i=1}^r b_i x_i)t}{2a}\right)^{n+2k} \times \left(\frac{4adt^{(\alpha-2)}(c\Pi_{i=1}^r x_i - 1)^g}{(\sum_{i=1}^r b_i x_i)^2}\right)^s. \quad (4.4)$$

*Proof.* With the help of relation (3.1), we can write

$$\sum_{n=0}^{\infty} \frac{{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)t^n}{(p)_n} = \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{k=0}^{[\frac{n-s}{\alpha}]} \frac{(p)_{n-s-(\alpha-1)k}}{k!s!(n-s-\alpha k)!(p)_n} a^{-p-k} \times \left(\frac{\sum_{i=1}^r b_i x_i}{a}\right)^{n-s-\alpha k} \{-d(c\Pi_{i=1}^r x_i - 1)^g\}^k H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)t^n. \quad (4.5)$$

Replacing $n \to n+s$; $n \to n+\alpha k$; by use of the identity [19, pg. 95, equ. 2.3]; applying Gauss's multiplication theorem, and by summation of $k$th series, we obtain our desired result (4.1). Using the same proof pattern of relation (4.1), we can get the relation (4.2). In a similar manner, relations (4.3), and (4.4) can be achieved using equ. (3.2). □

## 5 Expression of ${}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1, x_2, ..., x_r)$ in series of various polynomials

**Theorem 5.1.** *The following theorem gives expression of* ${}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)$ *in series of Legendre, and Gegenbauer polynomials:*

$$
{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)=\sum_{s=0}^{n}\sum_{k=0}^{[\frac{n-s+(\alpha-2)j}{\alpha}]}\sum_{j=0}^{k}\frac{(-1)^k(p)_{n-s-(\alpha-1)(k-j)}(-k)_j}{k!j!s!(\frac{3}{2})_{n-s-\alpha k+(\alpha-1)j}}
\times a^{-p-n+s+(\alpha-1)(k-j)}\{d(c\Pi_{i=1}^r x_i-1)^g\}^{k-j}(1+2n-2s+2j\alpha-4j-2\alpha k)
\times H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)P_{n-s-\alpha k+(\alpha-2)j}\left(\frac{(\sum_{i=1}^r b_i x_i)}{2}\right), \quad (5.1)
$$

*where $P_n(x)$ represents Legendre polynomial.*

$$
{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)=\sum_{s=0}^{n}\sum_{k=0}^{[\frac{n-s+(\alpha-2)j}{\alpha}]}\sum_{j=0}^{k}\frac{(-1)^k(p)_{n-s-(\alpha-1)(k-j)(-k)_j}}{k!j!s!(p)_{n-s+(\alpha-1)j-\alpha k+1}}
\times a^{-p-n+s+(\alpha-1)(k-j)}\{d(c\Pi_{i=1}^r x_i-1)^g\}^{k-j}(p+n-s-\alpha k+(\alpha-2)j)
\times H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)C^{p}_{n-s-\alpha k+(\alpha-2)j}\left(\frac{(\sum_{i=1}^r b_i x_i)}{2}\right), \quad (5.2)
$$

*where $C^\nu_n(x)$ is Gegenbauer polynomial.*

*Proof.* By use of (3.1), we can write

$$
\sum_{n=0}^{\infty}{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)t^n=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{k=0}^{[\frac{n-s}{\alpha}]}\frac{(p)_{n-s-(\alpha-1)k}}{k!s!(n-s-\alpha k)!}a^{-p-n+s+(\alpha-1)k}
\times\left(\sum_{i=1}^r b_i x_i\right)^{n-s-\alpha k}\{-d(c\Pi_{i=1}^r x_i-1)^g\}^k H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)t^n. \quad (5.3)
$$

Replacing $n\to n+s$; $n\to n+\alpha k$, and then using the relation [16, pg. 181, equ. (4)], we get

$$
\sum_{n=0}^{\infty}{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(p)_{n+k}}{k!s!}\frac{\{-d(c\Pi_{i=1}^r x_i-1)^g\}^k t^{n+s+\alpha k}}{a^{p+n+k}}
\times\sum_{j=0}^{[\frac{n}{2}]}\frac{(2n-4j+1)P_{n-2j}\left(\frac{(\sum_{i=1}^r b_i x_i)}{2}\right)}{j!(\frac{3}{2})_{n-j}}H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r). \quad (5.4)
$$

Now replacing $n\to n+2j$; $k\to k-j$; $n\to n+(\alpha-2)j-\alpha k$; by use of the relation [18, pg. 36, equ. (4.4)]; $n\to n-s$ and equating the coefficient of $t^n$, we at once arrive at our required result (5.1).
In a similar manner, using the identity [16, pg. 283, equ. (36)], we can prove the relations (5.2). □

## 6 Integral Representation

**Theorem 6.1.** *The integral representation of multivariable Humbert-Hermite polynomials can be expressed as:*

$$
{}_HQ^{p,\alpha,\nu}_{n,\mu,\eta}(x_1,x_2,...,x_r)=\frac{1}{\Gamma(p)}\sum_{s=0}^{n}\frac{1}{s!(n-s)!}H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r)
\times\int_0^{\infty}e^{-at}t^{p+n-s-1}H^{\alpha}_{n-s}\left(A,-\frac{B}{t^{\alpha-1}}\right)dt, \quad (6.1)
$$

*where $A=\sum_{i=1}^r b_i x_i$, $B=\{d(c\Pi_{i=1}^r x_i-1)^g\}$.*

*Proof.* The Kampé de Fériet Hermite Polynomial of order $m$ is represented by Dattoli et al. [12] as

$$H_n^m(y,z) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{y^{n-mk}z^k}{k!(n-mk)!}. \tag{6.2}$$

Let

$$\Psi = \frac{1}{\Gamma(p)}\sum_{s=0}^{n}\frac{1}{s!(n-s)!}H^{\nu}_{s,\mu,\eta}(x_1,x_2,...,x_r) \times \int_0^\infty e^{-at}t^{p+n-s-1}H^{\alpha}_{n-s}\left(A,-\frac{B}{t^{\alpha-1}}\right)dt. \tag{6.3}$$

Using equ. (6.2) in (6.3); equ. (1.3), and then in view of equ. (3.1), we can obtain the R. H. S. of (6.1). □

## 7 Conclusion

In this article, a new class of multivariate Humbert-Hermite polynomials is defined. We have proved various properties of the considering polynomials. By our theorem 2.1, and theorem 2.2 correct form of earlier studied results can be obtained that we have given as remark 2, corollary 1, and corollary 2. By choosing appropriate values of parameters we can obtain results of Pathan and Khan [8], and Jaimini et. al [11]. For $r = c = 2, b_2 = 0, \nu = \mu = \eta = x_2 = 1$ theorem 6.1, gives the result of Jaimini and Saxena [20, pg. 330, equ.(13)]. We also get the result of Batahan and Shehata that is given in remark 4. Similarly, numerous known and unknown results can be obtained which may be useful in various research fields.

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