

Subclasses of Analytic Multivalent Functions Associated with Generalized Mittag - Leffler type Poisson Distribution Series

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Abstract The purpose of the present paper is to derive some necessary and sufficient conditions for Generalized Mittag-Leffler type Poisson Distribution Series to be in the new subclasses of analytic multivalent functions in the open unit disk. Further, we establish an important inclusion relation for these subclasses. Furthermore an integral operator related to Generalized Mittag-Leffler type Poisson Distribution Series is also considered. We establish a significant connection between these subclasses and the Generalized Mittag-Leffler type Poisson Distribution Series. Several corollaries and consequences of the main results are also obtained. Our findings depict a clear relationship between Geometric Function Theory and Statistical Distribution Theory.

1 Introduction and Definitions

Let A_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic and multivalent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

Also let S_p be the subclass of A_p consisting of functions of the form (1.1) which are also multivalent in \mathbb{U} and let T_p be the subclass of S_p consisting of functions of the form

$$f(z) = z^p - \sum_{j=p+1}^{\infty} |a_j| z^j \tag{1.2}$$

The class $S_p^*(\delta)$ of p -valently starlike functions of order $\delta(0 \leq \delta < p)$ is defined as

$$S_p^*(\delta) = \left\{ f \in A_p : \Re \left(\frac{z f'(z)}{f(z)} \right) > \delta, z \in \mathbb{U} \right\} \tag{1.3}$$

The class $C_p^*(\delta)$ of p -valently convex functions of order $\delta(0 \leq \delta < p)$ is defined as

$$C_p^*(\delta) = \left\{ f \in A_p : \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \delta, z \in \mathbb{U} \right\} \tag{1.4}$$

The following new subclasses $S_p^*(\delta, \mu)$ and $C_p^*(\delta, \mu)$ for analytic multivalent functions were introduced by Purohit et al. [30]

Definition 1.1[30] A function $f(z)$ of the form (1.2) is said to be in the class $S_p^*(\delta, \mu)$ if it satisfies the condition

$$\left| \frac{\frac{zf'(z)}{p(1-\mu)f(z)+\mu zf'(z)} - 1}{\frac{zf'(z)}{p(1-\mu)f(z)+\mu zf'(z)} + 1} \right| < \delta. \quad (1.5)$$

for some $\mu(0 \leq \mu < 1)$, $\delta(0 \leq \delta < p)$ and for all $z \in \mathbb{U}$

Definition 1.2 [30] A function $f(z)$ of the form (1.2) is said to be in the class $C_p^*(\delta, \mu)$ if it satisfies the condition

$$\left| \frac{(1-\mu-p+p\mu)zf'(z) + (1-\mu)z^2f''(z)}{(1+\mu+p-p\mu)zf'(z) + (1+\mu)z^2f''(z)} \right| < \delta \quad (1.6)$$

for some μ ($0 \leq \mu < 1$), δ ($0 \leq \delta < p$) and for all $z \in \mathbb{U}$.

Also denote

$$S_p^*(\delta, \mu) = S_p(\delta, \mu) \cap T_p \quad \text{and} \quad C_p^*(\delta, \mu) = C_p(\delta, \mu) \cap T_p$$

Note that

$$f(z) \in C_p^*(\delta, \mu) \iff zf'(z) \in S_p^*(\delta, \mu).$$

For univalent functions i.e. for $p = 1$, the classes $S_1^*(\delta, \mu) = S(\delta, \mu)$ and $C_1^*(\delta, \mu) = C(\delta, \mu)$ were introduced by Frasin et al. [11]. These classes $S(\delta, \mu)$ and $C(\delta, \mu)$ for $\mu = 0$ further reduces to $S(\delta, 0) = S(\delta)$ and $C(\delta, 0) = C(\delta)$ respectively. The classes $S(\delta)$ and $C(\delta)$ were introduced and studied by Padmanabhan [24] (see also [16, 23]).

Definition 1.3 [10] A function $f(z)$ of the form (1.1) is said to be in the class $\mathbb{R}^b(G, H, p)$, if it satisfies the inequality:

$$\left| \frac{\frac{f'(z)}{z^{p-1}} - p}{(G-H)bp - H\left(\frac{f'(z)}{z^{p-1}} - p\right)} \right| < 1 \quad (1.7)$$

where G and H are fixed numbers such that $-1 \leq H < G \leq 1$, $b \in \mathbb{C} \setminus \{0\}$, $z \in \mathbb{U}$. This class $\mathbb{R}^b(G, H, p)$ was introduced by Dixit and Pathak [10]. By giving specific values to p , b , G and H in 1.7, we obtain many important subclasses studied by various authors in earlier works. As an example for $p = 1$ we get an important class of univalent functions $\mathbb{R}^b(G, H)$ studied by Dixit and Pal [9].

In 1903, a Swedish mathematician G. M. Mittag-Leffler [15] discovered a celebrated function E_u defined by

$$E_u(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(uj+1)}, \quad (z \in \mathbb{C}; \Re(u) > 0) \quad (1.8)$$

where $\Gamma(\cdot)$ denotes the Gamma function. In 1905, Wiman [34] studied the generalized Mittag-Leffler function $E_{u,v}$ and is given by

$$E_{u,v}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(uj+v)}, \quad (z, u, v \in \mathbb{C}; \Re(u) > 0, \Re(v) > 0) \quad (1.9)$$

The Mittag-Leffler function has gained great importance due to its various applications in many fields of science such as physics, chemistry, engineering and mathematics. It can be especially used in finding the solutions of many problems of fractional calculus which is a recent topic of research. Note that the Generalized Mittag-Leffler function $E_{u,v}$ is not a member of the function of normalized family. Therefore, some researchers normalized this function and obtained various results on the geometric properties of analytic functions. For some details, refer [1, 2, 12, 13, 22]. By following a similar method, the normalization of the Generalized Mittag-Leffler function is considered as follows

$$\mathcal{E}_{p,u,v}(z) = \Gamma(v)z^p E_{u,v}(z) = z^p + \sum_{j=p+1}^{\infty} \frac{\Gamma(v)}{\Gamma(u(j-p)+v)} z^j \quad (1.10)$$

Observe that for $p=1$ this reduces to

$$\mathcal{E}_{1,u,v}(z) = \mathcal{E}_{u,v}(z) = \Gamma(v)zE_{u,v}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(v)}{\Gamma(u(j-1) + v)} z^j \tag{1.11}$$

which for specific values of u and v further reduces to the exponential function, hyperbolic and trigonometric functions and many other important functions as its special cases. In the present paper, we shall limit our attention to the case of real-valued u, v and $z \in \mathbb{U}$.

The probability mass function of Mittag–Leffler-type Poisson distribution [27] is defined as

$$P(x = j) = \frac{\lambda^j}{\Gamma(uj + v)\mathcal{E}_{u,v}(\lambda)}, \quad j = 0, 1, 2, 3, \dots \tag{1.12}$$

where $\lambda > 0$, $u > 0$, and $v > 0$, and $\mathcal{E}_{u,v}(\lambda)$ is given by (1.11).

Using the normalized form of Mittag-Leffler function in (1.10) a power series whose coefficients are probabilities of Generalized Mittag-Leffler-type Poisson distribution is defined, as follows:

$$\Psi_{p,u,v}^{\lambda}(z) = z^p + \sum_{j=p+1}^{\infty} \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} z^j, \quad z \in \mathbb{U} \tag{1.13}$$

Also the series is defined

$$\Phi_{p,u,v}^{\lambda}(z) = 2z^p - \Psi_{p,u,v}^{\lambda}(z) = z^p - \sum_{j=p+1}^{\infty} \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} z^j, \quad z \in U. \tag{1.14}$$

For $f(z) \in A_p$ is given by (1.1) and $g(z) \in A_p$ is given by

$$g(z) = z^p + \sum_{j=p+1}^{\infty} b_j z^j \tag{1.15}$$

Then

$$(f * g)(z) = z^p + \sum_{j=p+1}^{\infty} a_j b_j z^j \tag{1.16}$$

Here $*$ denotes the Hadamard Product or Convolution of two analytic multivalent functions

Applying this process the Convolution Operator is defined as

$$\mathcal{I}_{p,u,v}^{\lambda} f(z) = \Psi_{p,u,v}^{\lambda}(z) * f(z) = z^p + \sum_{j=p+1}^{\infty} \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} a_j z^j, \quad z \in \mathbb{U}. \tag{1.17}$$

where the function $\Psi_{p,u,v}^{\lambda}(z)$ is defined by (1.13) and $f(z)$ is defined by (1.1). Several Eminent researchers have derived many major outcomes on connections between various subclasses of analytic univalent and multivalent functions by using various distribution series. They applied efficiently various distribution series such as Hypergeometric distribution series [6, 8, 31, 32], Generalized Bessel functions [3, 14, 17, 26], Poisson distribution series [7, 18, 19, 20, 25, 28, 29, 30, 33], Generalized distribution series [4], Generalized Pascal distribution [5, 11] and Binomial distribution [21] in their work. Inspired by several earlier results, in the present paper we obtain the necessary and sufficient conditions for functions $\Phi_{p,u,v}^{\lambda}(z)$ to be in the classes $S_p^*(\delta, \mu)$ and $C_p^*(\delta, \mu)$. Furthermore we express a certain inclusion relation between the classes $\mathbb{R}^b(G, H, p)$ and $C_p^*(\delta, \mu)$. Finally, we give a condition for an integral operator

$\mathcal{Q}(\lambda, u, v, z, p) = p \int_0^z \frac{\Phi_{p,u,v}^{\lambda}(t)}{t^j} dt$ to be in the class $C_p^*(\delta, \mu)$.

2 Preliminary lemmas

To establish our main results, we need the following lemmas.

If $\mu(0 \leq \mu < 1)$, $\delta(0 \leq \delta < p)$, $-1 \leq H < G \leq 1$, $b \in \mathbb{C} \setminus \{0\}$, and for all $z \in \mathbb{U}$. Then

Lemma 2.1. [30] A function $f(z)$ of the form (1.2) is in the class $S_p^*(\delta, \mu)$ if and only if

$$\sum_{j=p+1}^{\infty} [j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta)] |a_j| \leq 2p\delta \quad (2.1)$$

Lemma 2.2. [30] A function $f(z)$ of the form (1.2) is in the class $C_p^*(\delta, \mu)$ if and only if

$$\sum_{j=p+1}^{\infty} j [j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta)] |a_j| \leq 2p^2\delta \quad (2.2)$$

Lemma 2.3. [10] A function $f(z)$ of the form (1.1) is in the class $\mathbb{R}^b(G, H, p)$, then

$$|a_j| \leq \frac{|b|p(G - H)}{j} \quad (2.3)$$

3 The Necessary and Sufficient Conditions

In this section, we will obtain necessary and sufficient conditions for $\Phi_{p,u,v}^\lambda(z)$ to be in the classes $S_p^*(\delta, \mu)$ and $C_p^*(\delta, \mu)$.

Theorem 3.1. Let $v > 1$, $\lambda > 0$, $0 \leq \mu < 1$, $0 \leq \delta < p$. Then, the function $\Phi_{p,u,v}^\lambda(z)$ defined by (1.14) belongs to the class $S_p^*(\delta, \mu)$ if and only if

$$\begin{aligned} \frac{\Gamma(v)}{u\mathcal{E}_{p,u,v}(\lambda)} \left[(1 - \mu + \delta + \mu\delta) (E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)}) \right. \\ \left. + ((1-v)(1 - \mu + \delta + \mu\delta) + 2p\delta u) (E_{u,v}(\lambda) - \frac{1}{\Gamma(v)}) \right] \leq 2p\delta \quad (3.1) \end{aligned}$$

Proof. Since,

$$\Phi_{p,u,v}^\lambda(z) = z^p - \sum_{j=p+1}^{\infty} \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} z^j$$

according to Lemma 2.1, we must show that

$$\sum_{j=p+1}^{\infty} [j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta)] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} \leq 2p\delta$$

Let $M_1(\lambda, u, v, \delta, \mu)$

$$= \sum_{j=p+1}^{\infty} [j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta)] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)}$$

$$= \frac{\Gamma(v)}{\mathcal{E}_{p,u,v}(\lambda)} \sum_{j=1}^{\infty} [(j+p)(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta)] \frac{\lambda^j}{\Gamma(uj + v)}$$

Writing

$$j = \frac{1}{u} [(uj + v - 1) + (1 - v)]$$

and by simple computation we get

$M_1(\lambda, u, v, \delta, \mu)$

$$\begin{aligned} &= \frac{\Gamma(v)}{\mathcal{E}_{p,u,v}(\lambda)} \left[\frac{(1 - \mu + \delta + \mu\delta)}{u} \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj + v - 1)} \right. \\ &\quad + \frac{(1 - v)((1 - \mu + \delta + \mu\delta))}{u} \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj + v)} \\ &\quad \left. + 2p\delta \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj + v)} \right]. \\ &= \frac{\Gamma(v)}{u\mathcal{E}_{p,u,v}(\lambda)} \left[(1 - \mu + \delta + \mu\delta) \left(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)} \right) \right] \\ &\quad + \frac{\Gamma(v)}{u\mathcal{E}_{p,u,v}(\lambda)} \left[((1 - v)(1 - \mu + \delta + \mu\delta)) + 2p\delta u \right] \left(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)} \right). \end{aligned}$$

But this last expression is bounded above by $2p\delta$ if and only if (3.1) holds. □

Theorem 3.2. *Let $v > 2, \lambda > 0, 0 \leq \mu < 1, 0 \leq \delta < p$. Then, the function $\Phi_{p,u,v}^\lambda(z)$ defined by (1.14) belongs to the class $C_p^*(\delta, \mu)$ if and only if*

$$\begin{aligned} &\frac{\Gamma(v)}{\mathcal{E}_{p,u,v}(\lambda)} \left[\left(\frac{(1 - \mu + \delta + \mu\delta)}{u^2} \right) \left(E_{u,v-2}(\lambda) - \frac{1}{\Gamma(v-2)} \right) \right. \\ &\quad + \left(\frac{(3 - 2v)}{u^2} (1 - \mu + \delta + \mu\delta) + \frac{p}{u} (1 - \mu + 3\delta + \mu\delta) \right) \\ &\quad \left(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)} \right) + \left(\frac{(1 - v)^2}{u^2} (1 - \mu + \delta + \mu\delta) \right. \\ &\quad \left. + \frac{p(1 - v)}{u} (1 - \mu + 3\delta + \mu\delta) + 2p^2\delta \right) \left(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)} \right) \right] \leq 2p^2\delta \quad (3.2) \end{aligned}$$

Proof. Since

$$\Phi_{p,u,v}^\lambda(z) = z^p - \sum_{j=p+1}^{\infty} \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} z^j$$

according to Lemma 2.2, we must show that

$$\sum_{j=p+1}^{\infty} j [j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta)] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} \leq 2p^2\delta$$

Let

$$M_2(\lambda, u, v, \delta, \mu) = \sum_{j=p+1}^{\infty} [j^2(1 - \mu + \delta + \mu\delta) + jp(\mu - 1 + \delta - \mu\delta)] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)}$$

$$\begin{aligned}
 &= \sum_{j=1}^{\infty} [(j+p)^2(1-\mu+\delta+\mu\delta) + p(j+p)(\mu-1+\delta-\mu\delta)] \frac{\Gamma(v)\lambda^j}{\Gamma(uj+v)\mathcal{E}_{p,u,v}(\lambda)} \\
 &= \sum_{j=1}^{\infty} [j^2(1-\mu+\delta+\mu\delta) + jp(1-\mu+3\delta+\mu\delta) + 2p^2\delta] \frac{\Gamma(v)\lambda^j}{\Gamma(uj+v)\mathcal{E}_{p,u,v}(\lambda)}
 \end{aligned}$$

writing

$$j^2 = \frac{1}{u^2} [(uj+v-1)(uj+v-2) + (3-2v)(uj+v-1) + (1-v)^2]$$

and

$$j = \frac{1}{u} [(uj+v-1) + (1-v)]$$

and by simple computation we get

$$\begin{aligned}
 M_2(\lambda, u, v, \delta, \mu) &= \frac{\Gamma(v)}{\mathcal{E}_{p,u,v}(\lambda)} \left[\left[\frac{(1-\mu+\delta+\mu\delta)}{u^2} \right] \sum_{j=1}^{\infty} [(uj+v-1)(uj+v-2)] \frac{\lambda^j}{\Gamma(uj+v)} \right. \\
 &+ \left[\frac{(3-2v)}{u^2} (1-\mu+\delta+\mu\delta) + \frac{p}{u} (1-\mu+3\delta+\mu\delta) \right] \sum_{j=1}^{\infty} [uj+v-1] \frac{\lambda^j}{\Gamma(uj+v)} + \left[\frac{(1-v)^2}{u^2} (1-\mu+\delta+\mu\delta) \right. \\
 &\quad \left. + \frac{p(1-v)}{u} (1-\mu+3\delta+\mu\delta) + 2p^2\delta \right] \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj+v)} \Big] \\
 &= \frac{\Gamma(v)}{\mathcal{E}_{p,u,v}(\lambda)} \left[\left[\frac{(1-\mu+\delta+\mu\delta)}{u^2} \right] \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj+v-2)} + \left[\frac{(3-2v)}{u^2} (1-\mu+\delta+\mu\delta) + \frac{p}{u} (1-\mu+3\delta+\mu\delta) \right] \right. \\
 &\quad \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj+v-1)} + \left[\frac{(1-v)^2}{u^2} (1-\mu+\delta+\mu\delta) + \frac{p(1-v)}{u} (1-\mu+3\delta+\mu\delta) \right. \\
 &\quad \left. \left. + 2p^2\delta \right] \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(uj+v)} \right] \\
 &= \frac{\Gamma(v)}{\mathcal{E}_{p,u,v}(\lambda)} \left[\left(\frac{(1-\mu+\delta+\mu\delta)}{u^2} \right) \left(E_{u,v-2}(\lambda) - \frac{1}{\Gamma(v-2)} \right) \right. \\
 &\quad + \left(\frac{(3-2v)}{u^2} (1-\mu+\delta+\mu\delta) + \frac{p}{u} (1-\mu+3\delta+\mu\delta) \right) \\
 &\quad \left(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)} \right) + \left(\frac{(1-v)^2}{u^2} (1-\mu+\delta+\mu\delta) \right. \\
 &\quad \left. + \frac{p(1-v)}{u} (1-\mu+3\delta+\mu\delta) + 2p^2\delta \right) \left(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)} \right) \Big]
 \end{aligned}$$

But this last expression is bounded above by $2p^2\delta$ if and only if (3.2) holds. □

4 Inclusion Property

Applying Lemma 2.3, we will observe the effect of the Generalized Mittag–Leffler-type Poisson distribution series on the class $C_p^*(\delta, \mu)$

Theorem 4.1. *Let $v > 1, \lambda > 0, 0 \leq \mu < 1, 0 \leq \delta < p, -1 \leq H < G \leq 1, b \in \mathbb{C} \setminus \{0\}$. If $f \in \mathbb{R}^b(G, H, p)$, then $\mathcal{I}_{p,u,v}^\lambda f \in C_p^*(\delta, \mu)$ if*

$$\frac{|b| p(G - H)\Gamma(v)}{u\mathcal{E}_{p,u,v}(\lambda)} \left[(1 - \mu + \delta + \mu\delta)(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)}) + ((1 - v)(1 - \mu + \delta + \mu\delta) + 2p\delta u)(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)}) \right] \leq 2p\delta \quad (4.1)$$

Proof. In view of Lemma 2.2 it suffices to show that

$$\sum_{j=p+1}^\infty \left[j \left(j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta) \right) \right] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} |a_j| \leq 2p^2\delta$$

Since $f \in \mathbb{R}^b(G, H, p)$, using Lemma 2.3, we have

$$|a_j| \leq \frac{|b| p(G - H)}{j}$$

Therefore, it is enough to show that

$$|b| p(G - H) \sum_{j=p+1}^\infty \left[j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta) \right] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} \leq 2p^2\delta$$

or

$$|b| (G - H) \sum_{j=p+1}^\infty \left[j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta) \right] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} \leq 2p\delta$$

By a similar proof like those to Theorem 3.1, we get that $\mathcal{I}_{p,u,v}^\lambda f \in C_p^*(\delta, \mu)$ if inequality (4.1) holds. □

5 An Integral Operator

Here we introduce, an integral operator $\mathcal{Q}(\lambda, u, v, z, p)$ as follows:

$$\mathcal{Q}(\lambda, u, v, z, p) = p \int_0^z \frac{\Phi_{p,u,v}^\lambda(t)}{j} dt \quad (5.1)$$

where $\Phi_{p,u,v}^\lambda(z)$ is defined by (1.14). In this section we obtain the necessary and sufficient conditions for the integral operator $\mathcal{Q}(\lambda, u, v, z, p)$ defined by (5.1) belonging to the class $C_p^*(\delta, \mu)$.

Theorem 5.1. *Let $v > 1, \lambda > 0, 0 \leq \mu < 1, 0 \leq \delta < p$. Then, the integral operator $\mathcal{Q}(\lambda, u, v, z, p)$ defined by (5.1) belongs to the class $C_p^*(\delta, \mu)$ if and only if inequality (3.1) is satisfied.*

Proof. Since

$$\mathcal{Q}(\lambda, u, v, z, p) = z^p - \sum_{j=p+1}^\infty \frac{p\Gamma(v)\lambda^{j-p}}{j\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} z^j.$$

Then by Lemma 2.2, we need only to show that

$$\sum_{j=p+1}^\infty j \left[j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta) \right] \frac{p\Gamma(v)\lambda^{j-p}}{j\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} \leq 2p^2\delta$$

Or, equivalently

$$\sum_{j=p+1}^{\infty} \left[j(1 - \mu + \delta + \mu\delta) + p(\mu - 1 + \delta - \mu\delta) \right] \frac{\Gamma(v)\lambda^{j-p}}{\Gamma(u(j-p) + v)\mathcal{E}_{p,u,v}(\lambda)} \leq 2p\delta$$

By a similar proof like those to Theorem 4.1, we get that $\mathcal{Q}(\lambda, u, v, z, p) \in C_p^*(\delta, \mu)$ if inequality (3.1) holds. \square

6 Corollaries and Consequences

In the present section, assigning particular values $p = 1$ and $\mu = 0$ in our main results the following corollaries and consequences are obtained. Firstly if $p = 1$, then from (1.14), (1.17) and (5.1) we have

$$\Phi_{1,u,v}^\lambda(z) = \Phi_{u,v}^\lambda(z) = z - \sum_{j=2}^{\infty} \frac{\Gamma(v)\lambda^{j-1}}{\Gamma(u(j-1) + v)\mathcal{E}_{u,v}(\lambda)} z^j \quad (6.1)$$

$$\mathcal{I}_{1,u,v}^\lambda(z) = \mathcal{I}_{u,v}^\lambda(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(v)\lambda^{j-1}}{\Gamma(u(j-1) + v)\mathcal{E}_{u,v}(\lambda)} a_j z^j, \quad (6.2)$$

$$\mathcal{Q}(\lambda, u, v, z, 1) = \mathcal{Q}(\lambda, u, v, z) = \int_0^z \frac{\Phi_{u,v}^\lambda(t)}{j} dt \quad (6.3)$$

and $\mathcal{E}_{u,v}(\lambda)$ is given by (1.11).

Corollary 6.1. Let $v > 1$, $\lambda > 0$ and $0 \leq \delta < 1$. Then, the function $\Phi_{u,v}^\lambda(z)$ defined by (6.1) belongs to the class $S(\delta)$ if and only if

$$\frac{\Gamma(v)}{u\mathcal{E}_{u,v}(\lambda)} \left[(1 + \delta) \left(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)} \right) + ((1-v)(1+\delta) + 2\delta u) \left(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)} \right) \right] \leq 2\delta \quad (6.4)$$

Corollary 6.2. Let $v > 2$, $\lambda > 0$ and $0 \leq \delta < 1$. Then, the function $\Phi_{u,v}^\lambda(z)$ defined by (6.1) belongs to the class $C(\delta)$ if and only if

$$\begin{aligned} & \frac{\Gamma(v)}{\mathcal{E}_{u,v}(\lambda)} \left[\left(\frac{(1+\delta)}{u^2} \right) \left(E_{u,v-2}(\lambda) - \frac{1}{\Gamma(v-2)} \right) \right. \\ & + \left(\frac{1}{u^2} (3-2v)(1+\delta) + \frac{1}{u} (3\delta+1) \right) \left(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)} \right) \\ & \left. + \left(\frac{1}{u^2} (1+\delta)(1-v)^2 + \frac{1}{u} (3\delta+1)(1-v) + 2\delta \right) \left(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)} \right) \right] \leq 2\delta. \quad (6.5) \end{aligned}$$

Corollary 6.3. Let $v > 1$, $\lambda > 0$, $0 \leq \delta < 1$, $-1 \leq H < G \leq 1$, $b \in \mathbb{C} \setminus \{0\}$. If $f \in \mathbb{R}^b(G, H)$, then $\mathcal{I}_{u,v}^\lambda f$ defined by (6.2) belongs to the class $C(\delta)$ if

$$\begin{aligned} & \frac{|b|(G-H)\Gamma(v)}{u\mathcal{E}_{u,v}(\lambda)} \left[(1 + \delta) \left(E_{u,v-1}(\lambda) - \frac{1}{\Gamma(v-1)} \right) \right. \\ & \left. + ((1-v)(1+\delta) + 2\delta u) \left(E_{u,v}(\lambda) - \frac{1}{\Gamma(v)} \right) \right] \leq 2\delta \quad (6.6) \end{aligned}$$

Corollary 6.4. Let $v > 1$, $\lambda > 0$, $0 \leq \delta < 1$. Then, the integral operator $Q(\lambda, u, v, z)$ defined by (6.3) belongs to the class $C(\delta)$ if and only if the inequality (6.4) is satisfied.

7 Conclusion remarks

This article aims at establishing significant connection between new subclasses of analytic multivalent functions in the open unit disk and the Generalized Mittag-Leffler type Poisson Distribution Series. Some interesting properties of the functions belonging to these subclasses are derived. Some important results on these subclasses are also obtained using the inclusion relation and integral operator. Several corollaries and consequences of the main results are also derived. Our results successfully show a clear relationship between Geometric Function Theory and Statistical Distribution Theory. Therefore, the results of this work are significantly variant and hence it is interesting and capable of developing and providing opportunity for its study to the future researchers.

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