Three-variable higher-order Gould-Hopper based Fubini numbers and polynomials

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Abstract In this paper, we introduce a new class of Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials together with exponential polynomials, extended r-Bell polynomials, r-derangement polynomials and numbers and the r-extended Lah-Bell polynomials by using series rearrangement and integral representation techniques. Furthermore, we derive theorems on the expansions of Hermite-Fubini polynomials of three variables by using generating functions.

1 Introduction

The two variable Hermite Kampé de Fériet polynomials $H_n(x, y)$ [1, 3] are defined by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \tag{1.1}$$

The Gould-Hopper polynomials $g_n^m(x, y)$ (see [10]) is a generalization of (1.1). The notation $H_n^m(x, y)$ or $g_n^m(x, y)$ was given by Dattoli *et al.* [7, 8]. These are specified by

$$e^{xt+yt^m} = \sum_{n=0}^{\infty} H_n^m(x, y) \frac{t^n}{n!}, (m \geq 2), \tag{1.2}$$

which reduces to the ordinary Hermite polynomials $H_n(x)$ when $x \to 2x$ and $m = 2$, $y = -1$.

In (2018), Kim *et al.* [13] introduced 2-variable higher-order Fubini polynomials are defined by

$$\frac{e^{xt}}{(1 - y(e^t - 1))^r} = \sum_{n=0}^{\infty} \mathbb{F}_n^{(r)}(x; y) \frac{t^n}{n!}, \tag{1.3}$$

where r is a positive integer. If $r = 1$, $\mathbb{F}_n^{(1)}(x; y) = \mathbb{F}_n(x; y)$ are called the two variable Fubini polynomials. For $x = 0$, $\mathbb{F}_n^{(r)}(0; y) = \mathbb{F}_n^{(r)}(y)$ and $\mathbb{F}_n^{(r)}(0; 1) = F_n^{(r)}(1)$ are respectively, called the Fubini polynomials of order r and the Fubini numbers of order r. Further, on taking if $y = 1$, $\mathbb{F}_n^{(r)}(x; 1)$ are called the ordered Bell polynomials of order r and they are denoted by $Ob_n^{(r)}(x)$ as follows

$$\frac{e^{xt}}{(2 - e^t)^r} = \sum_{n=0}^{\infty} Ob_n^{(r)}(x). \tag{1.4}$$

When $x = 0$, $Ob_n^{(r)}(0) = Ob_n^{(r)}$ are called the ordered Bell numbers (see [17]).

Let n, k, r be non negative integers, with $n \geq k$. The unsigned Lah number $L(n, k)$ counts the number of ways of all distributions of n balls, labelled $1, \cdots, n$. among k unlabelled, contents-ordered boxes, with no box left empty. An explicit formula and the generating function of $L(n, k)$ respectively are given by (see [21, 22]):

$$L(n, k) = \binom{n - 1}{k - 1} \frac{n!}{k!}, (k \geq 0), \tag{1.5}$$

and

$$\frac{1}{k!} \left(\frac{1}{1 - t} - 1 \right)^k = \sum_{n=k}^{\infty} L(n, k) \frac{t^n}{n!}, (k \geq 0). \tag{1.6}$$

Furthermore, the r-Lah number $L_r(n, k)$ counts the number of partitions of a set with $n + r$ elements into $n + r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks and explicit formula of $L_r(n, k)$ (see [21, 22]) given by

$$L_r(n, k) = \binom{n + 2r - 1}{k + 2r - 1} \frac{n!}{k!}, (k \geq 0). \tag{1.7}$$

From (1.7), we have the generating function of $L_r(n, k)$ given by

$$\frac{1}{k!} \left(\frac{1}{1 - t} \right)^{2r} \left(\frac{t}{1 - t} \right)^k = \sum_{n=k}^{\infty} L_r(n, k) \frac{t^n}{n!}, (k \geq 0), (\text{see } [21, 22]).$$

Recently, Kim-Kim [18, 19] introduced the generating functions of the Lah-Bell polynomials and the r-extended Lah-Bell polynomials, respectively as follows

$$e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=0}^{\infty} B_n^L(x)\frac{t^n}{n!} \tag{1.8}$$

and

$$\left(\frac{1}{1-t}\right)^{2r} e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=0}^{\infty} B_{r,n}^L(x)\frac{t^n}{n!}, (k \geq 0). \tag{1.9}$$

When $x = 1$, $B_n^L(1) = B_n^L$ and $B_{r,n}^L(1) = B_{r,n}^L$ are called the Lah-Bell numbers and r-extended Lah-Bell numbers, respectively.

Kim *et al.* [14] introduced the r-derangement polynomials are defined by means of the following generating function as

$$\frac{t^r}{(1-t)^{r+1}} e^{xt} = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, (0 \leq r \leq n), (\text{see } [19]). \tag{1.10}$$

When $x = -1$, $D_n^{(r)}(-1) = D_n^{(r)}$ are the r-derangement numbers. Further, on taking $r = 0$, we get

$$\frac{1}{1-t} e^{xt} = \sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!}, (\text{see } [22]). \tag{1.11}$$

In (2018), Kim *et al.* [16] considered the derangement polynomials defined by

$$\frac{e^{-t}}{1-t} e^{xt} = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}. \tag{1.12}$$

When $x = 0$, $D_n(0) = D_n$ are called the derangement numbers.

It is well known that the Charlier polynomials $C_n(x; \alpha)$ are defined by (see [15])

$$e^{-\alpha t}(1+t)^x = \sum_{n=0}^{\infty} C_n(x; \alpha)\frac{t^n}{n!}, \tag{1.13}$$

where $x, t, \alpha \in \mathbb{R}$.

The falling factorial sequence is defined by

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), (n \geq 1). \tag{1.14}$$

The Stirling numbers of the first kind are defined by (see [1-27])

$$(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k, (n \geq 0). \tag{1.15}$$

and as an inversion formula of (1.15), the Stirling numbers of the second kind are given by

$$x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k. \tag{1.16}$$

From (1.15) and (1.16), we note that the generating function of Stirling numbers of the first kind and that of the second kind are respectively given by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k)\frac{t^n}{n!}, \tag{1.17}$$

and

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k)\frac{t^n}{n!}, (k \geq 0), (\text{see } [22]). \tag{1.18}$$

It is known that the Stirling polynomials of the second kind are defined by

$$\frac{1}{k!}e^{xt}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k|x)\frac{t^n}{n!}, (k \geq 0), (\text{see } [25]). \tag{1.19}$$

The Bell polynomials are given by (see [18])

$$e^{x(e^t - 1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}. \tag{1.20}$$

When $x = 1$, $Bel_n(1) = Bel_n$ are called the Bell numbers.

From (1.18) and (1.20), we get

$$Bel_n(x) = \sum_{k=0}^{n} S_2(n, k)x^k, (n \geq 0), (\text{see } [17]). \tag{1.21}$$

For $u \in \mathbb{C}$ with $u \neq 1$, the classical Frobenius-Euler polynomials $\mathbb{H}_n^{(\alpha)}(x; u)$ of order α are defined by means of the following generating function (see [25])

$$\left(\frac{1-u}{e^t - u}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathbb{H}_n^{(\alpha)}(x; u) \frac{t^n}{n!}. \tag{1.22}$$

In the special case when $x = 0$, $\mathbb{H}_n^{(\alpha)}(u) = \mathbb{H}_n^{(\alpha)}(0; u)$ are called n^{th} Frobenius-Euler numbers of order α. For $\alpha = 1$ into (1.22), $\mathbb{H}_n^{(1)}(x, u) = \mathbb{H}_n(x, u)$, are called the Frobenius-Euler polynomials and $\mathbb{H}_n^{(\alpha)}(0; u) = h_n^{(\alpha)}(u)$, are called the Frobenius-Euler numbers of order α. Substituting $u = -1$ into (1.22), $\mathbb{H}_n(x; -1) = E_n(x)$, are called the Euler polynomials.

Pathan and Khan [25] introduced the generalized Apostol-type Hermite-Euler polynomials of order α are defined by

$$\left(\frac{1-u}{e^t - u}\right)^\alpha e^{xt + yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u) \frac{t^n}{n!}, \tag{1.23}$$

so that

$${}_H E_n^{(\alpha)}(x, y; u) = \sum_{m=0}^{n} \binom{n}{m} \mathbb{H}_{n-m}(u) H_m(x, y).$$

The manuscript of this paper is arranged as follows: In section 2, we consider generating functions for Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of Hermite-Fubini numbers and polynomials, extended r-Bell polynomials, r-derangement polynomials and numbers and the r-extended Lah-Bell polynomials, by using integral representations and series rearrangement techniques. In Section m4, we prove several theorems on the expansions of Hermite-Fubini polynomials of three variables by using generating functions.

2. A new class of higher-order Hermite-Fubini numbers and polynomials

In this section, we introduce three-variable higher-order Gould-Hopper based Fubini polynomials and obtain some basic properties which gives us new formula for ${}_H \mathbb{F}_n^{(r,j)}(x, y; z)$. We start by the following definition as

Definition 2.1. We introduce 3-variable higher-order Hermite-Fubini polynomials by means of the following generating function:

$$\frac{e^{xt + yt^j}}{(1 - z(e^t - 1))^r} = \sum_{n=0}^{\infty} {}_H \mathbb{F}_n^{(r,j)}(x, y; z) \frac{t^n}{n!}. \tag{2.1}$$

From (2.1), we see that

$${}_H \mathbb{F}_n^{(r,j)}(0, 0; z) = \mathbb{F}_n^{(r,j)}(z), \quad {}_H \mathbb{F}_n^{(r,j)}(0, 0; 1) = \mathbb{F}_n^{(r,j)}.$$

For $r = 1$ and $j = 2$, the result reduces to the known result of Khan *et al.* [12] and further on taking $y = 0$, $r = 1$ in (2.1), we get 2-variable Fubini polynomials which is defined by Kargin [20].

When investigating the connection between Gould-Hopper polynomials $H_n^{(j)}(x, y)$ and higher-order Fubini polynomials $\mathbb{F}_n^{(r)}(z)$ of importance is the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$${}_H \mathbb{F}_n^{(r,j)}(x, y; z) = \sum_{m=0}^{n} \binom{n}{r} \mathbb{F}_{n-m}^{(r)}(z) H_m^{(j)}(x, y). \tag{2.3}$$

Proof. Using (1.2) and (2.1), we have

$$\sum_{n=0}^{\infty} {}_H \mathbb{F}_n^{(r,j)}(x, y; z) \frac{t^n}{n!} = \frac{e^{xt + yt^j}}{(1 - z(e^t - 1))^r}$$

$$= \sum_{n=0}^{\infty} \mathbb{F}_n^{(r)}(z) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m^{(j)}(x, y) \frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{r} \mathbb{F}_{n-m}^{(r)}(z) H_m^{(j)}(x, y) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (2.3).

$\square$

Theorem 2.2. The following result holds true

$${}_H \mathbb{F}_{q+l}^{(r,j)}(w, y; z) = \sum_{n,p=0}^{q,l} \binom{q}{n} \binom{l}{p} (w - y)^{n+p} {}_H \mathbb{F}_{q+l-n-p}^{(r,j)}(x, y; z).$$

Proof. Changing t by $t + u$ in (2.1) and then using the formula [25]:

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n}{n!} \frac{y^m}{m!}. \tag{2.4}$$

Now using (2.1), we find

$$\frac{1}{(1 - z(e^{t+u} - 1))^r} e^{y(t+u)^j} = e^{-x(t+u)} \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(x, y; z) \frac{t^q}{q!} \frac{u^l}{l!}.$$

Again changing x by w in the above equation, we get

$$\exp((w - x)(t + u)) \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(x, y; z) \frac{t^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(w, y; z) \frac{t^q}{q!} \frac{u^l}{l!}$$

$$\sum_{N=0}^{\infty} \frac{[(w - x)(t + u)]^N}{N!} \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(x, y; z) \frac{t^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(w, y; z) \frac{t^q}{q!} \frac{u^l}{l!}. \tag{2.5}$$

From (2.5), we see

$$\sum_{n,p=0}^{\infty} \frac{(w - x)^{n+p} t^n u^p}{n! p!} \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(x, y; z) \frac{t^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(w, y; z) \frac{t^q}{q!} \frac{u^l}{l!}$$

$$\sum_{q,l=0}^{\infty} \sum_{n,p=0}^{q,l} \frac{(w - x)^{n+p}}{n! p!} {}_H\mathbb{F}_{q+l-n-p}^{(r,j)}(x, y; z) \frac{t^q}{(q - n)!} \frac{u^l}{(l - p)!}$$

$$= \sum_{q,l=0}^{\infty} {}_H\mathbb{F}_{q+l}^{(r,j)}(w, y; z) \frac{t^q}{q!} \frac{u^l}{l!}. \tag{2.6}$$

On equating the coefficients of the like powers of t and u in the above equation, we get at the required result. □

Remark 2.1. On taking $l = 0$ in Theorem 2.2, we obtain

$$_H\mathbb{F}_q^{(r,j)}(w, y; z) = \sum_{n=0}^{q} \binom{q}{n} (w - x)^n \, {}_H\mathbb{F}_{q-n}^{(r,j)}(x, y; z). \tag{2.7}$$

Remark 2.2. Replacing w by $w + x$ in (2.7), we find

$$_H\mathbb{F}_q^{(r,j)}(x + w, y; z) = \sum_{n=0}^{q} \binom{q}{n} w^n \, {}_H\mathbb{F}_{q-n}^{(r,j)}(x, y; z). \tag{2.8}$$

Theorem 2.3. For $n \geq 0$ and $z \neq 0$, we have

$$_H\mathbb{F}_n^{(r,j)}(x, y; z) = {}_HE_n^{(r,j)}\left(x, y; \frac{1 + z}{z}\right). \tag{2.9}$$

Proof. Using (2.1) and (1.23), we can write (2.1) as

$$\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z) \frac{t^n}{n!} = \frac{1}{(1 - z(e^t - 1))^r} e^{xt+yt^j}$$

$$= \left(\frac{1 - \frac{1+z}{z}}{e^t - \frac{1+z}{z}}\right)^r e^{xt+yt^j}$$

$$= \sum_{n=0}^{\infty} {}_HE_n^{(r,j)}\left(x, y; \frac{1 + z}{z}\right) \frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.9). □

Theorem 2.4. For $n, r, s \geq 0$, the following formula holds true:

$$_H\mathbb{F}_n^{(r,j)}(x, y; z) = \sum_{k=0}^{n} \binom{n}{k} {}_H\mathbb{F}_{n-k}^{(r-s,j)}(x, y; z) \mathbb{F}_k^{(s)}(z). \tag{2.10}$$

Proof. Equation (2.1) can be written as

$$\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z) \frac{t^n}{n!} = \left(\frac{1}{1 - z(e^t - 1)}\right)^{r-s} \left(\frac{1}{1 - z(e^t - 1)}\right)^s e^{xt+yt^j}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H\mathbb{F}_n^{(r-s,j)}(x, y; z) \frac{t^n}{n!} \mathbb{F}_k^{(s)}(z) \frac{t^k}{k!}$$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} {}_H\mathbb{F}_{n-k}^{(r-s,j)}(x, y; z) \mathbb{F}_k^{(s)}(z)\right) \frac{t^n}{n!}.$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides, we get (2.10). □

Theorem 2.5. For $n \geq 0$, $p, q \in \mathbb{R}$, the following formula for Hermite-Fubini polynomials holds true:

$$_H\mathbb{F}_n^{(r,j)}(px, qy; z) = n! \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{m} \rfloor} {}_H\mathbb{F}_{n-k}^{(r,j)}(px, qy; z) \frac{((p-1)x)^k((q-1)y)^m}{(n-k)!(k-mj)!m!}. \tag{2.11}$$

Proof. Rewrite the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(px, qy; z)\frac{t^n}{n!} = \frac{1}{(1-z(e^t-1))^r}e^{xt+yt^j}e^{(p-1)xt}e^{(q-1)yt^j}$$

$$= \left(\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(px, qy; z)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x)^k\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}((q-1)y)^m\frac{t^{mj}}{m!}\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(px, qy; z)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x)^k((q-1)y)^m\frac{t^{k+mj}}{n!k!m!}\right).$$

Replacing k by $k - mj$ in above equation, we have

$$L.H.S. = \left(\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(px, qy; z)\frac{t^n}{n!}\right)\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\lfloor \frac{k}{m} \rfloor}((p-1)x)^{k-mj}((q-1)y)^m\frac{t^k}{(k-mj)!m!}\right)$$

Again replacing n by $n - k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor \frac{k}{m} \rfloor} {}_H\mathbb{F}_{n-k}^{(r,j)}(px, qy; z)((p-1)x)^{k-mj}((q-1)y)^m\frac{t^n}{(n-k)!(k-mj)!m!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result (2.11). $\square$

Theorem 2.6. For $n \geq 0$, the following formula holds true:

$$_H\mathbb{F}_n^{(r,j)}(x, y; z) = \sum_{l=0}^{n}\binom{n}{l}H_{n-l}^{(j)}(x, y)\sum_{k=0}^{l}(r)_k z^k S_2(l, k). \tag{2.12}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z)\frac{t^n}{n!} = \frac{e^{xt+yt^j}}{(1-z(e^t-1))^r}$$

$$= e^{xt+yt^j}\sum_{k=0}^{\infty}(r)_k z^k\frac{1}{k!}(e^t-1)^k = e^{xt+yt^j}\sum_{k=0}^{\infty}(r)_k z^k\sum_{l=k}^{\infty}S_2(l, k)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty}H_n^{(j)}(x, y)\frac{t^n}{n!}\sum_{l=0}^{\infty}(r)_k z^k\sum_{k=0}^{l}S_2(l, k)\frac{t^l}{l!}$$

$$\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}H_{n-l}^{(j)}(x, y)\sum_{k=0}^{l}(r)_k z^k S_2(l, k)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.12). $\square$

Remark 2.3. For $y = 0$ in Theorem 2.6 yields to the correct result of known result of Kim *et al.*[13] as follows

$$\mathbb{F}_n^{(r)}(x; z) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(r)_k z^k S_2(l, k)x^{n-l}.$$

Theorem 2.7. For $n \geq 0$, we have

$$H_n^{(j)}(x, y) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(-r)_l z^l S_2(k, l) {}_H\mathbb{F}_{n-k}^{(r,j)}(x, y; z). \tag{2.14}$$

Proof. From (2.1) and (1.18), we note that

$$e^{xt+yt^j} = (1-z(e^t-1))^r\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}H_n^{(j)}(x, y)\frac{t^n}{n!} = \left(\sum_{l=0}^{\infty}(-r)_l z^l\frac{(e^t-1)^l}{l!}\right)\left(\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z)\frac{t^n}{n!}\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}(-r)_l S_2(k, l)z^l\right)\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x, y; z)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(-r)_l S_2(k, l)z^l {}_H\mathbb{F}_{n-k}^{(r,j)}(x, y; z)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get (2.14). $\square$

Remark 2.4. Letting $y = 0$ in Theorem 2.7 yields to the correct result of known result of Kim *et al.*[13] as follows

$$x^n = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k}(-r)_l z^l S_2(k,l) \mathbb{F}_{n-k}^{(r)}(x;z).$$

Theorem 2.8. For m and n nonnegative integer, we have

$$_H\mathbb{F}_n^{(r,j)}(x+u,y;z) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} u^{(m)} S_2(l,m) \,_H\mathbb{F}_{n-l}^{(r,j)}(x-m,y;z), \tag{2.15}$$

$$_H\mathbb{F}_n^{(r,j)}(x+u,y;z) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (u)_m S_2(l,m) \,_H\mathbb{F}_{n-l}^{(r,j)}(x,y;z), \tag{2.16}$$

where $(u)_m$ and $u^{(m)}$ are the falling and rising factorials respectively defined as $(u)_m = u(u-1)\cdots(u-m+1)$ and $u^{(m)} = u(u+1)\cdots(u+m-1)$ for $m \geq 1$, and $(u)_0 := u^{(0)} := 1$.

Proof. In view of (2.1) and applying Newton binomial's theorem, we find

$$\sum_{n=0}^{\infty} \,_H\mathbb{F}_n^{(r,j)}(x+u,y;z)\frac{t^n}{n!} = \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j} e^{ut}$$

$$= \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j} (1-(1-e^{-t}))^{-u}$$

$$= \sum_{m=0}^{\infty} \binom{u+m-1}{m}(1-e^{-t})^m \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j}$$

$$= \sum_{m=0}^{\infty} u^{(m)} \frac{(e^t-1)^m}{m!} \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j} e^{-mt}$$

$$= \sum_{l=0}^{\infty} \sum_{m=0}^{l} u^{(m)} S_2(l,m) \frac{t^l}{l!} \sum_{n=0}^{\infty} \,_H\mathbb{F}_n^{(r,j)}(x-m,y;z)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} u^{(m)} S_2(l,m) \,_H\mathbb{F}_{n-l}^{(r,j)}(x-m,y;z) \right) \frac{t^n}{n!}. \tag{2.17}$$

Comparing the coefficients of t, we obtain the result (2.15).

Again, we consider (2.1) that

$$\sum_{n=0}^{\infty} \,_H\mathbb{F}_n^{(r,j)}(x+u,y;z)\frac{t^n}{n!} = \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j} e^{ut}$$

$$= \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j} ((e^t-1)+1)^u$$

$$= \sum_{m=0}^{\infty} \binom{u}{m}(e^t-1)^m \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j}$$

$$= \sum_{m=0}^{\infty} (u)_m \frac{(e^t-1)^m}{m!} \frac{1}{(1-z(e^t-1))^r} e^{xt+yt^j}$$

$$= \sum_{l=0}^{\infty} \sum_{m=0}^{l} (u)_m S_2(l,m) \frac{t^l}{l!} \sum_{n=0}^{\infty} \,_H\mathbb{F}_n^{(r,j)}(x,y;z)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (u)_m S_2(l,m) \,_H\mathbb{F}_{n-l}^{(r,j)}(x,y;z) \right) \frac{t^n}{n!}. \tag{2.18}$$

Equating the coefficients of t, we get the result (2.16). □

3. Integrals involving polyexponential polynomials and functions

First, we prove the following result involving the Hermite-Fubini polynomials $_H\mathbb{F}_n(x,y;z)$ by using series rearrangement technique and consider its special case:

The polylogarithmic function is defined by

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}, \text{ (see [27])} \tag{3.1}$$

is a special case of the more general Lerch Transcendent (or Lerch zeta function) (see [11])

$$\Phi(x,s,\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{(n+\lambda)^s}, \, (\Re(\lambda) > 0). \tag{3.2}$$

A special function discussed in [5, 6]

$$e_s(x, \lambda) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\lambda)^s}, \tag{3.3}$$

extends the natural exponential function and also the exponential integral. The function $e_s(x, \lambda)$ is relevant also to the theory of the Lerch Transcendent (3.2) and the Hurwitz zeta function.

$$\zeta(s, \lambda) = \sum_{n=0}^{\infty} \frac{1}{(n+\lambda)^s} = \frac{1}{s} \int_0^{\infty} \frac{e^{-\lambda x} x^{s-1}}{1 - e^{-x}} dx, \, (\Re(s) > 1, \Re(\lambda) > 0). \tag{3.4}$$

The Laplace transform in the variable $y > 0$, is

$$L[e_s(x, \lambda)](y) = \frac{1}{y} \Phi\left(\frac{1}{y}, s, \lambda\right), \tag{3.5}$$

where Φ is the Lerch Transcendent (3.2). For $\Re(s) > 0$ and $y = 1$

$$L[e_s(x, \lambda)](1) = \zeta(s, \lambda). \tag{3.6}$$

Comparison gives the following identification

$$\Phi(x, s, \lambda) = \int_0^{\infty} e_s(tx, \lambda)e^{-t} dt, \tag{3.7}$$

and therefore for $x = 1$, we have

$$\zeta(s, \lambda) = \int_0^{\infty} e_s(t, \lambda)e^{-t} dt. \tag{3.8}$$

When $\lambda = 1$, it reduces to

$$\zeta(s) = \int_0^{\infty} e_s(t)e^{-t} dt. \tag{3.9}$$

The function $\Gamma(s)e_s(x, \lambda)$ is a Mellin-Transform in the variable s (see [4]):

$$\Gamma(s)e_s(x, \lambda) = \int_0^{\infty} t^{s-1} e^{-\lambda t} e^{xe^{-t}} dt. \tag{3.10}$$

Comparing results (3.7) and (3.10), we get

$$e_s(xz, \lambda) = z\phi(xz, s, \lambda),$$

and we can write

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-\lambda t} e^{xze^{-t}} dt = z \int_0^{\infty} e_s(txz, \lambda)e^{-t} dt.$$

Theorem 3.1. For $m, n > 0$, we have

$$\int_0^{\infty} \frac{t^{s-1} e^{-\lambda t + xe^{-t} + yt + zt^j}}{(1 - x(e^t - 1))^r} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(s+m+n)}{m!n!} \mathbb{F}_m^{(r)}(x) e_{s+m+n}(x, \lambda) H_n^{(j)}(y, z). \tag{3.11}$$

Proof. Using the generating function

$$(1 - x(e^t - 1))^{-r} = \sum_{m=0}^{\infty} \mathbb{F}_m^{(r)}(x) \frac{t^m}{m!}$$

and the definition (1.2) of Gould-Hopper polynomials, we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathbb{F}_m^{(r)}(x) H_n^{(j)}(y, z)}{m!n!} \int_0^{\infty} t^{s+m+n-1} e^{-\lambda t + xe^{-t}} dt.$$

Finally, by using (3.10), we get the theorem.   □

Remark 3.1. On setting $r = 0$ in Theorem 3.1, we get

$$\int_0^{\infty} t^{s-1} e^{-\lambda t + xe^{-t} + yt + zt^j} dt = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} e_{s+n}(x, \lambda) H_n^{(j)}(y, z).$$

On the other hand, setting $y = z = 0$ in Theorem 3.1, we get

Corollary 3.1. For $n > 0$, we have

$$\int_0^{\infty} \frac{t^{s-1} e^{-\lambda t + xe^{-t}}}{(1 - x(e^t - 1))^r} dt = \sum_{n=0}^{\infty} \mathbb{F}_n^{(r)}(x) e_{s+n}(x, \lambda) \frac{1}{n!}.$$

Theorem 3.2. For $n \geq 0$, we have

$$e_s(x, \lambda) = \sum_{n=0}^{\infty} \frac{\mathbb{F}_n(x)}{n!} [(s)_n e_{s+n}(x, \lambda) - x(s)_{n+1} e_{s+n+1}(x, \lambda) + x(s)_n e_{s+n}(x, \lambda)]. \tag{3.12}$$

Proof. Since we have

$$1 = (1 - x(e^t - 1)) \sum_{n=0}^{\infty} \mathbb{F}_n(x) \frac{t^n}{n!}. \tag{3.13}$$

Multiplying both the sides by $t^{s-1} e^{-\lambda t + x e^{-t}}$ and integrating with respect to t between the limits 0 and ∞, we get desired theorem.

□

Theorem 3.3. For $n \geq 0$, we have

$$\sum_{n=0}^{\infty} (s)_n e_{s+n}(x, \lambda) \frac{H_n^{(j)}(y, z)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{_H \mathbb{F}_n^{(r,j)}(x, y, z)}{n!} \left[(s)_n e_{s+n}(x, \lambda) - x(s)_{n+1} e_{s+n+1}(x, \lambda) + x(s)_n e_{s+n}(x, \lambda) \right]. \tag{3.14}$$

Proof. Multiplying both the sides by $t^{s-1} e^{-\lambda t + x e^{-t}} e^{yt + zt^j}$ and integrating with respect to t between the limits 0 and ∞. Now use Theorem 3.2 with $r = 0$ to get the required result.

□

3.1. Exponential polynomials

The evaluation of the series

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, n = 0, 1, 2, \cdots \tag{3.15}$$

has a long and interesting history. Clearly, $S_0 = e$, with the agreement that $0^0 = 1$. Continuing like that, for every $n = 0, 1, 2, \cdots$, we find the relation

$$\left(x \frac{d}{dx} \right)^n e^x = \Phi_n(x) e^x = \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}, \tag{3.16}$$

where $\Phi_n(x)$ are polynomials of degree n.

Thus, we have

$$S_n = \Phi_n(1) e, \forall n \geq 0.$$

.

The polynomials $\Phi_n(x)$ can be defined also by the exponential generating function (extending Ligowski's formula)

$$e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \Phi_n(x) \frac{z^n}{n!}. \tag{3.17}$$

Comparing coefficient for z in the equation

$$e^{(x+y)e^z} = e^{xe^z} e^{ye^z},$$

yields the binomial identity

$$\Phi_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \Phi_k(x) \Phi_{n-k}(y). \tag{3.18}$$

The numbers

$$Bel_n = \Phi_n(1) = \frac{1}{e} S_n \tag{3.19}$$

are sometimes called exponential numbers, but a more established name is Bell numbers. They have interesting combinatorial and analytical applications (see [3], [4], [5]). An extensive list of 202 references for Bell numbers is given in [6].

The extended r-Bell polynomials are defined by (see [15])

$$e^{rt} e^{x(e^t - 1)} = \sum_{n=0}^{\infty} \Phi_n^{(r)}(x) \frac{t^n}{n!}, \tag{3.20}$$

so that

$$e^{rt} e^{(x+y)(e^t - 1)} = \sum_{n=0}^{\infty} \Phi_n^{(r)}(x + y) \frac{t^n}{n!}, \tag{3.21}$$

Next we write

$$e^{x(1-y)(e^t - 1) + rt} e^{-y(1-x)(e^t - 1) + rt} = e^{x(e^t - 1) + rt} e^{-y(e^t - 1) + rt}$$

and use (3.21) to get

$$\sum_{m,n=0}^{\infty} \Phi_m^{(r)}(x(1 - y)) \Phi_n^{(r)}(-y(1 - x)) \frac{t^{m+n}}{m! n!} = \sum_{m,n=0}^{\infty} \Phi_m^{(r)}(x) \Phi_n^{(r)}(-y) \frac{t^{m+n}}{m! n!}, \tag{3.22}$$

which on comparing the coefficients of t gives the following theorem.

Theorem 3.4. For $m, n > 0$, we have

$$\sum_{m=0}^{n} \binom{n}{m} \Phi_m^{(r)}(x(1-y)) \Phi_{n-m}^{(r)}(-y(1-x)) = \sum_{m=0}^{n} \binom{n}{m} \Phi_m^{(r)}(x) \Phi_{n-m}^{(r)}(-y).$$

Theorem 3.5. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1} e^{x(e^t-1)+rt+yt+zt^j}}{(1-x(e^t-1))^r} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(s+m+n)}{n!} F_m^{(r)}(x) \Phi_{s+m+n}^{(r)}(x) H_n^{(j)}(y,z). \tag{3.23}$$

Proof. Theorem 3.5 can be easily obtained in the usual way by following the proof of Theorem 3.2.

$\square$

Remark 3.2. On setting $r = 0$ in Theorem 3.5, we get the integral

$$\int_0^\infty t^{s-1} e^{x(e^t-1)+yt+zt^j} dt = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} \Phi_{s+n}(x) H_n^{(j)}(y,z), \tag{3.24}$$

which further setting $x = 1$ and using (3.19) gives

$$\int_0^\infty t^{s-1} e^{e^t-1+yt+zt^j} dt = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} Bel_{s+n} H_n^{(j)}(y,z).$$

Remark 3.3. On the other hand, setting $y = z = 0$ in Theorem 3.5, we get

Corollary 3.2. For $m > 0$, we have

$$\int_0^\infty \frac{t^{s-1} e^{x(e^t-1)+rt}}{(1-x(e^t-1))^r} dt = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{n!} \mathbb{F}_m^{(r)}(x) \Phi_{s+m}^{(r)}(x).$$

Theorem 3.6. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1} e^{x(\frac{1}{1-t}-1)+yt+zt^j}}{(1-x(e^t-1))^r} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(s+m+n)}{n!m!} \mathbb{F}_m^{(r)}(x) B_{s+m+n}^L(x) H_n^{(j)}(y,z). \tag{3.25}$$

Proof. Multiplying both the sides of (2.1) by $t^{s-1} e^{x(\frac{1}{1-t}-1)} e^{yt+zt^j}$ and integrating with respect to t between the limits 0 and ∞, we get desired theorem.

$\square$

Theorem 3.7. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1} e^{x(\frac{1}{1-t}-1)+yt+zt^j}}{(1-x(e^t-1))^r (1-t)^{2r}} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(s+m+n)}{n!m!} \mathbb{F}_m^{(r)}(x) B_{s+m+n,r}^L(x) H_n^{(j)}(y,z). \tag{3.26}$$

Proof. Multiplying both the sides of (2.1) by $t^{s-1} e^{x(\frac{1}{1-t}-1)} (\frac{1}{1-t})^{2r} e^{yt+zt^j}$ and integrating with respect to t between the limits 0 and ∞, we get desired theorem.

$\square$

Remark 3.4. On setting $r = 0$ in Theorem 3.7, we get the integral

$$\int_0^\infty t^{s-1} e^{x(\frac{1}{1-t}-1)+yt+zt^j} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(s+m+n)}{n!m!} B_{s+m+n}^L(x) H_n^{(j)}(y,z). \tag{3.27}$$

Remark 3.5. On setting $y = z = 0$ in Theorem 3.7, we obtain

Corollary 3.3. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1} e^{x(\frac{1}{1-t}-1)}}{(1-x(e^t-1))^r (1-t)^{2r}} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(s+m+n)}{m!n!} \mathbb{F}_m^{(r)}(x) B_{s+m+n,r}^L(x). \tag{3.28}$$

Theorem 3.8. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{r+s-1} e^{xt+yt+zt^j}}{(1-x(e^t-1))^r (1-t)^{r+1}} dt = \sum_{m,n=0}^{\infty} \frac{\Gamma(r+s+m+n)}{n!m!} \mathbb{F}_m^{(r)}(x) D_{s+m+n}^{(r)}(x) H_n^{(j)}(y,z). \tag{3.29}$$

Proof. Multiplying both the sides of (2.1) by $t^{s-1} \frac{t^r}{(1-t)^{r+1}} e^{xt}$ and integrating with respect to t between the limits 0 and ∞, we get desired theorem.

$\square$

Remark 3.6. On setting $r = 0$ in Theorem 3.8 and using (1.11), we get the integral

$$\int_0^\infty t^{s-1} \frac{1}{1-t} e^{xt+yt+zt^j} dt = \sum_{m,n=0}^\infty \frac{\Gamma(s+n)}{n!} d_{s+n}(x) H_n^{(j)}(y,z). \tag{3.30}$$

Remark 3.7. On setting $y = z = 0$ in Theorem 3.8, we obtain the corollary.

Corollary 3.4. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{r+s-1}e^{xt}}{(1-x(e^t-1))^r(1-t)^{r+1}} dt = \sum_{m,n=0}^\infty \frac{\Gamma(r+s+m+n)}{n!m!} \mathbb{F}_m^{(r)}(x) D_{s+m+n}^{(r)}(x). \tag{3.31}$$

Theorem 3.9. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1}e^{-t+xt+yt+zt^j}}{(1-x(e^t-1))^r(1-t)} dt = \sum_{m,n=0}^\infty \frac{\Gamma(s+m+n)}{n!m!} \mathbb{F}_m^{(r)}(x) D_{s+m+n}(x) H_n^{(j)}(y,z). \tag{3.32}$$

Proof. Multiplying both the sides of (2.1) by $t^{s-1} \frac{e^{-t}}{1-t} e^{xt}$ and integrating with respect to t between the limits 0 and ∞, we get desired theorem. $\square$

Remark 3.8. On setting $r = 0$ in Theorem 3.9, we get the corollary.

Corollary 3.5. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1}e^{-t+xt+yt+zt^j}}{1-t} dt = \sum_{m,n=0}^\infty \frac{\Gamma(s+m+n)}{m!n!} D_{s+m+n}(x) H_n^{(j)}(y,z). \tag{3.33}$$

Remark 3.9. On setting $y = z = 0$ in Theorem 3.9, we obtain the corollary.

Corollary 3.6. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1}e^{-t+xt}}{(1-x(e^t-1))^r(1-t)} dt = \sum_{m=0}^\infty \frac{\Gamma(s+m)}{m!} \mathbb{F}_m^{(r)}(x) D_{s+m}(x). \tag{3.34}$$

Theorem 3.10. For $m, n > 0$, we have

$$\int_0^\infty \frac{t^{s-1}e^{-\alpha t+yt+zt^j}(1+t)^x}{(1-x(e^t-1))^r} dt = \sum_{m,n=0}^\infty \frac{\Gamma(s+m+n)}{n!m!} \mathbb{F}_m^{(r)}(x) C_{s+m+n}(x;\alpha) H_n^{(j)}(y,z), \tag{3.35}$$

where $C_n(x;\alpha)$ is Charlier polynomial given by (1.13).

Proof. Multiplying both the sides of (2.1) by $t^{s-1}e^{-\alpha t}(1+t)^x$ and integrating with respect to t between the limits 0 and ∞, we get desired theorem. $\square$

4. Expansions of Hermite-Fubini polynomials

In this section, we prove several theorems on the expansions of Hermite-Fubini polynomials of three variables. We start with the following theorem.

Theorem 4.1. For $k \in \mathbb{N}$ and $x, y, z \in \mathbb{C}$, we have

$$\sum_{s=0}^n \binom{n}{s} H_s^{(j)}(kx, ky) \mathbb{F}_{n-s}^{(rk)}(z) \frac{1}{n!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_H\mathbb{F}_{n_1,m}^{(r)}(x,y;z) {}_H\mathbb{F}_{n_2}^{(r)}(x,y;z) \cdots {}_H\mathbb{F}_{n_k}^{(r)}(x,y;z)}{n_1!n_2!\cdots n_k!}. \tag{4.1}$$

Proof. Equation (2.1) can be written as

$$\left[(1-z(e^t-1))^{-r} e^{xt+yt^j}\right]^k$$

$$= (1-z(e^t-1))^{-rk} e^{kxt+kyt^j} = \left[\sum_{n=0}^\infty {}_H\mathbb{F}_n^{(r)}(x,y;z) \frac{t^n}{n!}\right]^k.$$

Using (1.2), we can write

$$e^{kxt+kyt^j} = \sum_{s=0}^\infty H_s^{(j)}(kx,ky) \frac{t^s}{s!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^\infty \mathbb{F}_n^{(rk)}(z) \frac{t^n}{n!} \sum_{s=0}^\infty H_s(kx,ky) \frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_H\mathbb{F}_{n_1}^{(r)}(x,y;z) {}_H\mathbb{F}_{n_2}^{(r)}(x,y;z) \cdots {}_H\mathbb{F}_{n_k}^{(r)}(x,y;z)}{n_1! n_2! \cdots n_k!} t^n.$$

An application of manipulation of series yields

$$\sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} H_s^{(j)}(kx,ky) \mathbb{F}_{n-s}^{(rk)}(z) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_H\mathbb{F}_{n_1,m}^{(r)}(x,y;z) {}_H\mathbb{F}_{n_2}^{(r)}(x,y;z) \cdots {}_H\mathbb{F}_{n_k}^{(r)}(x,y;z)}{n_1! n_2! \cdots n_k!} \frac{t^n}{n}.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. $\qquad\square$

Remark 4.1. On taking $r = 0$ in Theorem 4.1, we obtain

Corollary 4.1. For $k \in \mathbb{N}$ and $y, z \in \mathbb{C}$, we have

$$\frac{H_n^{(j)}(kx,ky)}{n!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^{(j)}(x,y) H_{n_2}^{(j)}(x,y) \cdots H_{n_k}^{(j)}(x,y)}{n_1! n_2! \cdots n_k!}. \tag{4.2}$$

Remark 4.2. On setting $r = 0$, $j = 2$, $y = -1$ and $x \to 2x$ in Theorem 4.1, the result reduces to known result of Batahan and Shehata [2,p.50.,Eq.(2.1)].

Corollary 4.2. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$, we have

$$\sum_{m=0}^{[\frac{n}{2}]} \frac{(-k)^m (2kx)^{n-2m}}{(n-2m)! m!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x) H_{n_2}(x) \cdots H_{n_k}(x)}{n_1! n_2! \cdots n_k!}. \tag{4.3}$$

Theorem 4.2. For $k \in \mathbb{N}$ and $X, Y \in \mathbb{C}$, we have

$$\sum_{s=0}^{n} \binom{n}{s} \mathbb{F}_{n-s}^{(rk)}(Z) H_s^{(j)}(kX,kY) \frac{1}{n!}$$

$$\sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_H\mathbb{F}_{n_1}^{(r,j)}(X,Y;Z) {}_H\mathbb{F}_{n_2}^{(r,j)}(X,Y;Z) \ldots {}_H\mathbb{F}_{n_k}^{(r,j)}(X,Y;Z)}{n_1! n_2! \cdots n_k!},$$

where $X = \sum_{i=0}^{k} x_i$ and $Y = \sum_{j=0}^{k} y_j$, $Z = \sum_{l=0}^{k} z_l$.

Proof. From (2.1), we note that

$$\left[\left(1 - Z(e^t - 1)\right)^{-r} e^{Xt + Yt^j} \right]^k$$

$$= \left(1 - Z(e^t - 1)\right)^{-rk} e^{kXt + kYt^j}$$

$$= \left[\sum_{n=0}^{\infty} {}_H\mathbb{F}_n^{(r,j)}(x_1 + x_2 + \cdots + x_k, y_1 + y_2 + \cdots + y_k; z_1 + z_2 + \cdots + z_k) \frac{t^n}{n!} \right]^k.$$

Using (1.2) and (1.3), we see that

$$\sum_{n=0}^{\infty} \mathbb{F}_n^{(rk)}(Z) \frac{t^n}{n!} \sum_{s=0}^{\infty} H_s^{(j)}(kX,kY) \frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_H\mathbb{F}_{n_1}^{(r,j)}(X,Y;Z) {}_H\mathbb{F}_{n_2}^{(r,j)}(X,Y;Z) \ldots {}_H\mathbb{F}_{n_k}^{(r,j)}(X,Y;Z)}{n_1! n_2! \cdots n_k!} t^n$$

$$\sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} \mathbb{F}_{n-s}^{(rk)}(Z) H_s^{(j)}(kX,kY) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_H\mathbb{F}_{n_1}^{(r,j)}(X,Y;Z) {}_H\mathbb{F}_{n_2}^{(r,j)}(X,Y;Z) \ldots {}_H\mathbb{F}_{n_k}^{(r,j)}(X,Y;Z)}{n_1! n_2! \cdots n_k!} t^n.$$

On comparing the coefficients of t^n on both sides of the resulting equation, we obtain at the required result. $\qquad\square$

Remark 4.3. On setting $r = 0$ in Theorem 4.2, we get

Corollary 4.3. For $k \in \mathbb{N}$ and $Y, Z \in \mathbb{C}$, we have

$$\frac{H_n^{(j)}(kX,kY)}{n!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^{(j)}(X,Y) H_{n_2}^{(j)}(X,Y) \cdots H_{n_k}^{(j)}(X,Y)}{n_1! n_2! \cdots n_k!}. \tag{4.4}$$

Remark 4.4. On setting $j = 2$, $r = 0$, $Y = -1$, and $X \to 2X$ in Theorem 4.2, the result reduces to known result of Batahan and Shehata [2,p.51.,Eq.(2.4)].

Corollary 4.4. For $k \in \mathbb{N}$ and $X \in \mathbb{C}$, then

$$\sum_{s=0}^{[\frac{n}{2}]} \frac{(-k)^s (2k(x_1 + x_2 + \cdots + x_k))^{n-2s}}{(n-2s)!s!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x_1) H_{n_2}(x_2) \cdots H_{n_k}(x_k)}{n_1! n_2! \cdots n_k!}. \tag{4.5}$$

Theorem 4.3. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$, we have

$$\sum_{m=0}^{n} (rk + m - 1)_m z^m S_2(n, m) \frac{1}{n!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{\mathbb{F}_{n_1}^{(r)}(z) \mathbb{F}_{n_2}^{(r)}(z) \cdots \mathbb{F}_{n_k}^{(r)}(z)}{n_1! n_2! \cdots n_k!}. \tag{4.6}$$

Proof. Using the power series of $[1 - z(e^t - 1)]^{-r}$ and making the necessary series arrangements gives

$$[1 - z(e^t - 1)]^{-rk} = \sum_{m=0}^{\infty} (rk + m - 1)_m z^m \frac{1}{m!} (e^t - 1)^m$$

$$= \sum_{m=0}^{\infty} (rk + m - 1)_m z^m \sum_{m=n}^{\infty} S_2(n, m) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (rk + m - 1)_m z^m S_2(n, m) \right) \frac{t^n}{n!}. \tag{4.7}$$

On the other hand,

$$[1 - z(e^t - 1)]^{-rk} = \left[[1 - z(e^t - 1)]^{-r} \right]^k = \left[\sum_{n=0}^{\infty} \mathbb{F}_n^{(r)}(z) \frac{t^n}{n!} \right]^k$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{\mathbb{F}_{n_1}^{(r)}(z) \mathbb{F}_{n_2}^{(r)}(z) \cdots \mathbb{F}_{n_k}^{(r)}(z)}{n_1! n_2! \cdots n_k!} t^n. \tag{4.8}$$

Therefore, by (4.7) and (4.8), we get the result. □

5. Concluding remarks.

In the present paper, we have discussed in Sections 2, a new class of higher-order Hermite-Fubini numbers and polynomials and some basic properties of a novel class of connected polynomials of Hermite type. In Sections 3 and 4, we have obtained operational and integral representations, involving polyexponential polynomials and functions and expansions of Hermite-Fubini polynomials.In addition, we have expressed the three variable higher-order Hermite-Fubini polynomials in terms of some families of special polynomials and vice versa.

The present paper has provided the basic elements to study the theory of pseudo trigonometric functions, pseudo Laguerre and pseudo Hermite polynomials,and pseudo-Fubini polynomials. We are now not far from having clarified their properties which can be exploited in many fields of research as in classical and quantum optics.

Ricci in [26] extended the definition of exponential by introducing a new family of exponential functions, characterized by an integer r,

$$E_0(x; r) = \sum_{m=0}^{\infty} \frac{x^{mr}}{(mr)!}.$$

One can infer directly from their definition that the functions $E_0(x; r)$, called from now on pseudo hyperbolic, can be complemented by

$$E_j(x; r) = \sum_{m=0}^{\infty} \frac{t^{rm+j}}{(rm + j)!}. \tag{5.1}$$

Dattoli *et al.* shown in [7, 8] that starting from the functions introduced in [1], we can recover a common thread linking them to non-standard forms of Hermite and Laguerre polynomials and of Bessel functions. They discussed the link with families of Bessel functions and analyzed all the associated problems from a unifying point of view, employing operational tools.

First we consider an infinite sum

$$A(j, k, t) = \sum_{m=0}^{\infty} \frac{t^{jm+k}}{(jm + k)!}, j \geq 1, k > 0, \tag{5.2}$$

and note that

$$A(1, 0, t) = e^t, \quad A(2, 0, t) = \frac{e^t + e^{-t}}{2},$$

and

$$A(3, 0, t) = \frac{d}{dt} A(3, 1, t) = \frac{d}{dt} \left[\frac{1}{3} e^t - \frac{2}{3} e^{\frac{-t}{2}} \cos \left(\frac{1}{2} \sqrt{3} t + \frac{1}{3} \pi \right) \right].$$

A recent paper of Dattoli [7] has provided the basic elements to study the theory of pseudo Laguerre and pseudo Hermite polynomials.We find their properties which can be exploited in many fields of research as in classical and quantum optics. The evolution of the optical field of a high gain free electron laser [23, 24] is governed by pseudo trigonometric functions, while polynomials of the type (5.2) play a central role in the theory of squeezed states of light [23]. They are indeed the

appropriate polynomial forms describing quantum states with reduced optical fluctuations [24].

Definition 5.1. Let $j \geq 1$ and $k > 0$. The generalized arbitrary order 2-variable Fubini polynomials $F_n^{(r,j,k)}(x;y)$ are defined by means of the generating function:

$$A(j,k,x,y.t) = \frac{A(j,k,t)e^{xt}}{(1 - y(e^t - 1))^r} = \sum_{n=0}^{\infty} \mathbb{F}_n^{(r,j,k)}(x;y)\frac{t^n}{n!}. \tag{5.3}$$

where $A(j,k,t)$ is given by (5.2).

In calculations involving coherent and squeezed states, infinite sums involving Hermite polynomials are extremely useful. Motivated by this situation, we can extend the definition 5.1 to the generalized arbitrary order 2-variable Hermite-Fubini polynomials as follows.

Definition 5.2. Let $j \geq 1$ and $k > 0$. The generalized arbitrary order 2-variable Hermite-Fubini polynomials $_H\mathbb{F}_n^{(r,j,k)}(x;y)$ are defined by means of the generating function:

$$G(j,k,x,y.t) = \frac{A(j,k,t)e^{2xt-t^2}}{(1 - y(e^t - 1))^r} = \sum_{n=0}^{\infty} {_H\mathbb{F}_n^{(r,j,k)}}(x;y)\frac{t^n}{n!}. \tag{5.4}$$

where $A(j,k,t)$ is given by (5.2).

(5.4) may play a central role in the theory of squeezed states of light [7]. Physically, Eqs. (5.2)-(5.4) are very important as they describe the decomposition of ordinary, even, and odd coherent states, respectively, into the harmonic-oscillator number states [7, 24]. This will be considered in detail in a forthcoming paper.

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