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ON q-ANALOGUES OF TARIG TRANSFORM

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Abstract In this paper, we present the q-Tarig transforms of the first and second kind and establish their relationships with the corresponding q-Laplace transforms. We derive several properties of these transforms, including linearity, convolution identity, change of scale, the transforms of the q-derivative, the q-derivative of transforms, and relevant integral properties. To illustrate the applicability, we compute the transforms of several basic functions. Furthermore, we demonstrate the usefulness of the q-Tarig transform of the first kind by solving some q-differential equations.

1 Introduction

Integral transforms are fundamental analytical tools with extensive applications in mathematics, science, and engineering. Within this framework, Elzaki and Elzaki [2] introduced the Tarig transform and explored its applications ([2], [3], [4]). Following this approach, we develop q-analogues of the Tarig transform, enhancing its applicability in q-calculus.
The Tarig transform is defined by

$$T[f(t);v] = F(v) = \frac{1}{v}\int_0^\infty e^{-\left(\frac{t}{v^2}\right)} f(t)dt, v \neq 0, k_1 < v < k_2. \tag{1.1}$$

In the set P, where P is defined by

$$P = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\left(\frac{|t|}{k_j}\right)}, \quad t \in (-1)^j \times [0,\infty) \right\}. \tag{1.2}$$

Here, M denotes a finite number, while k_1 and k_2 may take finite or infinite values.
An alternative representation of the Tarig transform is given as

$$T[f(t),v] = F(v) = \int_0^\infty e^{-\left(\frac{t}{v}\right)} f(vt)dt, v \neq 0. \tag{1.3}$$

The rising need for mathematical tools compatible with quantum computing has sparked interest in quantum calculus. Acting as a link between mathematics and physics, it is formulated as q-calculus.
The q-factorial and q-binomial coefficients are defined for $0 < |q| < 1$ as follows [5]

$$[n]_q = \frac{q^n - 1}{q-1}, \qquad [n]_{k,q} = \prod_{j=0}^{k-1} \frac{q^{n-j}-1}{q-1}, \qquad n,k \in N. \tag{1.4}$$

$$[n]_q! = \prod_{j=1}^{n} [j]_q!, \qquad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\ [n-k]_q!}, \qquad n,k \in N. \tag{1.5}$$

The q-exponential functions $E_q(t)$ and $e_q(t)$, representing the first and second q-analogues of the exponential function, are defined as in [6]

$$E_q(t) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} t^n}{[n]_q!}, \tag{1.6}$$

$$e_q(t)=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}. \tag{1.7}$$

We can note that [1]

$$E_q(it)=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{[n]_q!}i^nt^n=\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(2n-1)}}{[2n]_q!}t^{2n}+i\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(2n+1)}}{[2n+1]_q!}t^{2n+1}, \tag{1.8}$$

$$e_q(it)=\sum_{n=0}^{\infty}\frac{i^nt^n}{[n]_q!}=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{[2n]_q!}+i\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{[2n+1]_q!}. \tag{1.9}$$

These two pairs of exponential functions provide two pairs of Trigonometric functions in q-analogue as given by [5]:

$$Cos_q(at)=\frac{1}{2}\left[E_q(iat)+E_q(-iat)\right]. \tag{1.10}$$

$$Sin_q(at)=\frac{1}{2i}\left[E_q(iat)-E_q(-iat)\right]. \tag{1.11}$$

$$\cos_q(at)=\frac{1}{2}\left[e_q(iat)+e_q(-iat)\right]. \tag{1.12}$$

$$\sin_q(at)=\frac{1}{2i}\left[e_q(iat)-e_q(-iat)\right]. \tag{1.13}$$

The q-gamma functions of the first and second kind are defined as follows [6]

$$\Gamma_q(n)=\int_0^{\infty}t^{n-1}E_q(-qt)\,d_qt,(n>0), \tag{1.14}$$

$$\tilde{\Gamma}_q(n)=\int_0^{\infty}t^{n-1}e_q(-t)\,d_qt,(n>0). \tag{1.15}$$

The Jackson q-derivative is defined as [5]

$$D_qf(t)=\frac{f(t)-f(qt)}{(1-q)t}. \tag{1.16}$$

The definite Jackson q-integral is defined by (see [1] and [5])

$$\int_0^x f(t)\,d_qt=(1-q)\sum_{n=0}^{\infty}f(q^nx)\,xq^n. \tag{1.17}$$

Further q-analogue of the integration by parts is given by [5]

$$\int_a^b f(t)\,D_qg(t)\,d_qt=\{f(t)\,g(t)\}_a^b-\int_a^b g(qt)\,D_qf(t)\,d_qt. \tag{1.18}$$

The first and second kind q-Laplace transforms, respectively, are defined by [1] as:

$$F_q(s)=L_q\{f(t);s\}=\int_0^{\infty}E_q(-qst)\,f(t)\,d_qt,\qquad(s>0), \tag{1.19}$$

$$\tilde{F}_q(s)=\tilde{L}_q\{f(t);s\}=\int_0^{\infty}e_q(-st)\,f(t)\,d_qt,\qquad(s>0). \tag{1.20}$$

This paper is organized as follows. We first introduce the q-Tarig transform of the first kind, establish its fundamental properties, and obtain the transform of certain functions. We then employ this transform, together with its inverse, to solve certain q-differential equations. Thereafter, we present the q-Tarig transform of the second kind, derive its properties, and apply it to specific functions.

2 *q*-Tarig transform of First Kind

We define P_q as the set of functions $f(t)$, that are of q-exponential order, given by

$$P_q = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < M E_q\left(\frac{|t|}{k_j}\right), \quad t \in (-1)^j \times [0, \infty) \right\}, \tag{2.1}$$

Here, M denotes a finite number, while k_1 and k_2 may take finite or infinite values.
The q-analogue of the Tarig transform of the first kind is introduced in the function space P_q as

Definition 2.1. The **q-Tarig transform** of the first kind denoted by T_q, is defined as

$$T_q\left[f(t); v\right] = F_q(v) = \frac{1}{v}\int_0^\infty E_q\left(-\frac{qt}{v^2}\right) f(t)\, d_q t. \tag{2.2}$$

It can also be expressed in the form

$$T_q\left[f(t); v\right] = F_q(v) = \int_0^\infty E_q\left(-\frac{qt}{v}\right) f(vt)\, d_q t. \tag{2.3}$$

From (2.2) and (2.3), it is clear that $\lim_{q\to 1} T_q[f(t); v] = T[f(t); v]$.
Substituting $s = \frac{1}{v^2}$ in (1.19) together with $t = \frac{w}{v}$ in (2.2), yields the following duality relation between the q-Tarig and q-Laplace transforms of the first kind for a function $f(t)$.

Theorem 2.2. ***(Duality Relation)*** *If q-Tarig transform of $f(t)$ is $F_q(v)$ and its q-Laplace transform is $G_q(s)$. Then*

$$F_q(v) = \frac{G_q\left(\frac{1}{v^2}\right)}{v}. \tag{2.4}$$

Also, we can observe that $F_q(1) = G_q(1)$. Therefore, both q-Tarig and q-Laplace transform coincide at $v = 1$ and $s = 1$.

Using the duality relation (2.4) together with the convolution theorem for the q-Laplace transform [1], given by

$$L_q\left\{(f * g)_q(t)\right\} = L_q\{f(t)\}.L_q\{g(t)\}, \tag{2.5}$$

For piecewise continuous functions f and g defined on $(0, \infty)$, and with convolution $(f * g)\, q(t) = \int_0^t f(\tau)g(t-\tau), d\tau$, we derive the corresponding convolution property for the q-Tarig transform.

Theorem 2.3. ***(Convolution Property)*** *If $f(t)$, $g(t) \in P_q$ with q-Laplace transform of the first kind $F_q(s)$ and $G_q(s)$ respectively and q-Tarig transform of $f(t)$ and $g(t) \in P_q$ are $M_q(v)$ and $N_q(v)$ respectively, then*

$$T_q\left[(f * g)(t); v\right] = v M_q(v) N_q(v).$$

From Definition 2.1, we get the following linearity and change of scale properties.

Theorem 2.4. ***(Linearty Property)*** *If $f(t)$, $g(t) \in P_q$ and a, $b \in \mathbb{R}$, then the following identity holds:*

$$T_q\left[af(t) + bg(t); v\right] = aT_q\left[f(t); v\right] + bT_q\left[g(t); v\right]. \tag{2.6}$$

Theorem 2.5. ***(Change of Scale Property)*** *If $f(t) \in P_q$ and $T_q\left[f(t); v\right] = F_q(v)$, then*

$$T_q\left[f(at); v\right] = \frac{1}{\sqrt{a}} F_q\left(\sqrt{a} v\right) = \frac{1}{\sqrt{a}} T_q\left[f(t); \sqrt{a} v\right]. \tag{2.7}$$

Theorem 2.6. ***(Transform of q-Derivative)*** *If $f(t), D_q f(t), D_q^2 f(t) \ \ldots\ D_q^n f(t) \in P_q$, then we have*

$$T_q\left[D_q^n f(t); v\right] = F_q(v) - \sum_{i=1}^{n} v^{2(i-n)-1} f^{(i-1)}(0), \tag{2.8}$$

where $f^{(n)}(t) = D_q^n f(t)$.

Proof. Here, we will use mathematical induction. By using the definition given by (2.2) and q-integration by parts (1.18), we have

$$T_q\left[D_q f(t);\nu\right] = -\frac{1}{v}f(0) + \frac{1}{v^2}T_q\left[f(t);\nu\right], \tag{2.9}$$

Now, suppose this relation is true for $n = k > 0$, that is

$$T_q\left[D_q^k f(t);\nu\right] = \frac{F_q(v)}{v^{2k}} - \sum_{i=1}^{k} v^{2(i-k)-1} f^{(i-1)}(0). \tag{2.10}$$

Let $g(t) = D_q^k f(t)$, so that $D_q^{k+1} f(t) = D_q g(t)$, thus from (2.9), we have

$$\begin{aligned} T_q\left[D_q^{k+1} f(t);\nu\right] &= T_q\left[D_q g(t);\nu\right] = -\tfrac{1}{v}g(0) + \tfrac{1}{v^2}T_q\left[g(t);\nu\right] \\ &= -\tfrac{1}{v}\left[D_q^k f(0)\right] + \tfrac{1}{v^2}T_q\left[D_q^k f(t);\nu\right] \end{aligned}. \tag{2.11}$$

Now, on using (2.10), we have

$$T_q\left[D_q^{k+1} f(t);\nu\right] = \frac{T_q\left[f(t);\nu\right]}{v^{2(k+1)}} - \sum_{i=1}^{k+1} v^{2(i-(k+1))-1} f^{(i-1)}(0).$$

This establishes that (2.8) holds for $n = k+1$, and consequently for all $n \in \mathbb{N}$. Thus, the proof of the theorem is complete. □

Theorem 2.7. ***(q-Derivative of Transform)*** *If $f(t) \in P_q$ and $T_q\left[f(t);v\right] = F_q(v)$, then for all $n \in \mathbb{N}$*

$$T_q\left[t^n f(t);\nu\right] = \frac{q^{n(n+2)}}{v}\left(\frac{v^3}{1+q}\frac{\partial_q}{\partial_q v}\right)^n\left[vF_q\left(vq^{n/2}\right)\right]. \tag{2.12}$$

Proof. From [1], the q-derivative of the q-Laplace transform can be expressed as

$$L_q\left\{t^n f(t)\right\} = (-1)^n q^{\begin{pmatrix} n \\ 2 \end{pmatrix}}\left(\frac{\partial_q}{\partial_q s}\right)^n G_q\left(q^{-n}s\right), \tag{2.13}$$

where $G_q\left(q^{-n}s\right) = \int_0^\infty E_q\left(\frac{-qts}{q^n}\right) f(t)\, d_q t$.
Here, by replacing s with $\frac{1}{v^2}$ and applying (2.4), we have

$$\begin{aligned} T_q\left[t^n f(t);\nu\right] &= \frac{q^{\begin{pmatrix} n \\ 2 \end{pmatrix}}}{v}\left(\frac{q^2v^3}{1+q}\frac{\partial_q}{\partial_q v}\right)^n \int_0^\infty E_q\left(-\frac{qt}{v^2q^n}\right) f(t)\, d_q t \\ &= \frac{q^{n(n+2)}}{v}\left(\frac{v^3}{1+q}\frac{\partial_q}{\partial_q v}\right)^n\left[vF_q\left(vq^{n/2}\right)\right]. \end{aligned} \tag{2.14}$$

□

Theorem 2.8. ***(Integral property)*** *If $f(t) \in P_q$ and $T_q\left[f(t);v\right] = F_q(v)$, then*

$$T_q\left[\int_0^t f(u)\, d_q u; v\right] = v^2 T_q\left[f(t);v\right] = v^2 F_q(v). \tag{2.15}$$

Proof. If we set $g(t) = E_q\left(\frac{-t}{v^2}\right)$ and $F(t) = \int_0^t f(u)\, d_q u$ in q-analogue of integration by parts formula (1.18), we get

$$\begin{aligned} &\int_0^\infty \left\{\int_0^t f(u)\, d_q u\right\}.D_q\left[E_q\left(\tfrac{-t}{v^2}\right)\right] d_q t \\ &= \left\{\left\{\int_0^t f(u)\, d_q u\right\} E_q\left(\tfrac{-t}{v^2}\right)\right\}_0^\infty - \int_0^\infty E_q\left(\tfrac{-qt}{v^2}\right) D_q\left\{\int_0^t f(u)\, d_q u\right\} d_q t \end{aligned} \tag{2.16}$$

or

$$\frac{-1}{v^2}\int_0^\infty E_q\left(\frac{-qt}{v^2}\right)\left\{\int_0^t f(u)\, d_q u\right\} d_q t = -\int_0^\infty E_q\left(\frac{-qt}{v^2}\right) f(t)\, d_q t.$$

Hence, we arrive at the required result. □

Next, applying the q-gamma function of the first kind (1.14) to (2.3) leads to the q-Tarig transform of the first kind, tabulated for certain functions in following table.

Table 1. q-Tarig transform of First Kind of Certain Functions

S. No.	$f(t)$	$\tilde{T}_q[f(t);v]=\tilde{F}_q(v)$
1	t^n	$[n]_q!v^{2n+1}$
2	$e_q(at)$	$\left(\frac{v}{1-av^2}\right)$
3	$E_q(at)$	$v\sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}\left(av^2\right)$
4	$Cos_q(at)$	$v\sum_{n=0}^{\infty}(-1)^n q^{n(2n-1)}\left(av^2\right)^{2n}$
5	$Sin_q(at)$	$v\sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)}\left(av^2\right)^{2n+1}$
6	$\cos_q(at)$	$\left(\frac{v}{1+a^2v^4}\right)$
7	$\sin_q(at)$	$\left(\frac{av^3}{1+a^2v^4}\right)$
8	$Cosh_q(at)$	$v\sum_{n=0}^{\infty}q^{n(2n-1)}\left(av^2\right)^{2n}$
9	$Sinh_q(at)$	$v\sum_{n=0}^{\infty}q^{n(2n+1)}\left(av^2\right)^{2n+1}$
10	$\cosh_q(at)$	$\left(\frac{v}{1-a^2v^4}\right)$
11	$\sinh_q(at)$	$\left(\frac{av^3}{1-a^2v^4}\right)$
12	$H(t-\alpha)$	$vE_q\left(\frac{-q\alpha}{v^2}\right)$

3 Applications of q-Tarig transform of First Kind

In this section, solutions of certain q-differential equations are obtained through the q-Tarig transform of the first kind. For this purpose, we introduce the inverse of the q-Tarig transform.

Definition 3.1. For $f(t)\in P_q$ and $T_q[f(t);v]=F_q(v)$, the function $f(t)$ is termed the inverse q-Tarig transform of the first kind of $F_q(v)$, expressed as

$$f(t)=T_q^{-1}[F_q(v);t]. \tag{3.1}$$

Linearity for q-Tarig transform also holds

$$T_q^{-1}[\alpha F_q(v)+\beta G_q(v);t]=\alpha T_q^{-1}[F_q(v);t]+\beta T_q^{-1}[G_q(v);t]. \tag{3.2}$$

where α and β are constant.
The following examples illustrate that the q-Tarig transform of the first kind is a powerful method for solving q-initial value problems.
Example 3.1 Solve the following q-differential equation

$$D_qy(t)+cy(t)=0,\ y(t)\in P_q, \tag{3.3}$$

with $y(0)=1$.
Solution: Applying the q-Tarig transform of the first kind to both sides of equation (3.3), and

using (2.9), we obtain

$$\frac{1}{v^2}F_q\left(v\right)-\frac{1}{v}F_q(0)+cF_q\left(v\right)=0,$$

$$\left(\frac{1}{v^2}+c\right)F_q\left(v\right)=\frac{1}{v},$$

$$F_q(v)=v\frac{1}{(1+cv^2)}. \tag{3.4}$$

Using the inverse q-Tarig transform in (3.4), and in view of Result 2 in Table 1, we get the solution $y\left(t\right)=e_q(-ct)$.

Example 3.2 Solve the following second order q-differential equation

$$D_q^2y\left(t\right)+4y\left(t\right)=0,\qquad y\left(t\right)\in P_q, \tag{3.5}$$

with $y\left(0\right)=1$ and $D_qy\left(0\right)=3$.
Solution: On taking the q-Tarig transform both sides of the equation (3.5), in view of (2.8), we have

$$\frac{1}{v^4}F_q(v)-\left\{\frac{y\left(0\right)}{v^3}+\frac{D_qy\left(0\right)}{v}\right\}+4F_q(v)=0.$$

Using initial conditions, we get

$$F_q(v)=\frac{v}{(1+4v^4)}+\frac{3v^3}{(1+4v^4)}. \tag{3.6}$$

Taking the inverse q-Tarig transform of (3.6), in view of Results 6 and 7 in Table 1, we get the solution

$$y\left(t\right)=\cos_q\left(2t\right)+\frac{3}{2}\sin_q\left(2t\right). \tag{3.7}$$

Example 3.3 Solve the following q-initial value problem

$$D_q^2y\left(t\right)+16y\left(t\right)=9t,\qquad y\left(t\right)\in P_q \tag{3.8}$$

with $y\left(0\right)=0$ and $D_qy\left(0\right)=7$.
Solution: Applying the q-Tarig transform of the first kind to both sides of equation (3.8), and in view of (2.8) with the given conditions, we obtain

$$\frac{F_q\left(v\right)}{v^4}-\frac{7}{v}+16F_q\left(v\right)=9v^3,$$

$$F_q(v)=\frac{9v^7}{(1+16v^4)}+\frac{7v^3}{(1+16v^4)},$$

$$F_q(v)=\frac{9v^3}{16}+\frac{103}{16}\frac{v^3}{(1+16v^4)}. \tag{3.9}$$

Taking the inverse q-Tarig transform of (3.9), in view of Results 1 and 7 in Table 1, we get the result

$$y\left(t\right)=\frac{9t}{16}+\frac{103}{64}\sin_q\left(4t\right).$$

4 q-Tarig transform of Second Kind

We define $\tilde{P}_q$ as the set of functions $f\left(t\right)$, that are of q-exponential order, given by

$$\tilde{P}_q=\left\{f(t):\exists M,k_1,k_2>0,|f(t)|<Me_q\left(\frac{|t|}{k_j}\right),\ t\in(-1)^j\times[0,\infty)\right\}, \tag{4.1}$$

Here, M denotes a finite number, while k_1 and k_2 may take finite or infinite values.
The q-analogue of the Tarig transform of the second kind is introduced in the function space $\tilde{P}_q$ as

Definition 4.1. The q**-Tarig transform** of the second kind denoted by $\tilde{T}_q$, is defined as

$$\tilde{T}_q\left[f(t);v\right]=\tilde{F}_q(v)=\frac{1}{v}\int_0^{\infty}e_q\left(-\frac{t}{v^2}\right)f(t)\,d_qt. \tag{4.2}$$

It can also be expressed in the form

$$\tilde{T}_q\left[f(t);v\right]=\tilde{F}_q(v)=\int_0^{\infty}e_q\left(-\frac{t}{v}\right)f(vt)\,d_qt. \tag{4.3}$$

From (4.2), it is clear that $\lim\limits_{q\to 1}\tilde{T}_q[f(t);v]=T[f(t);v]$.

The following theorem establishes the close relationship between the q-Tarig (4.2) and q-Laplace (1.20) transforms of the second kind.

Theorem 4.2. ***(Duality Relation)*** *If q-Tarig transform of second kind of function $f(t)$ is $\tilde{F}_q(v)$ and its q-Laplace transform of second kind is $\tilde{G}_q(s)$. Then*

$$\tilde{F}_q(v)=\frac{\tilde{G}_q\left(\frac{1}{v^2}\right)}{v}. \tag{4.4}$$

Theorem 4.3. ***(Linearty Property)*** *If $f(t),\ g(t)\in\tilde{P}_q$ and $a,\ b\in\mathbb{R}$, then the following identity holds:*

$$\tilde{T}_q\left[af(t)+bg(t);v\right]=a\tilde{T}_qf(t)+b\tilde{T}_qg(t), \tag{4.5}$$

Theorem 4.4. ***(Change of Scale Property)*** *If $f(t)\in\tilde{P}_q$ and $\tilde{T}_q\left[f(t);v\right]=\tilde{F}_q(v)$, then*

$$\tilde{T}_q\left[f(at);v\right]=\frac{1}{\sqrt{a}}\tilde{F}_q\left(\sqrt{a}v\right)=\frac{1}{\sqrt{a}}\tilde{T}_q\left[f(t);\sqrt{a}v\right]. \tag{4.6}$$

Theorem 4.5. ***(Transform of q-Derivative)*** *If $f(t), D_qf(t), D_q^2f(t),\ldots\ldots D_q^nf(t)\in\tilde{P}_q$, then we have*

$$\tilde{T}_q\left[D_q^nf(t);v\right]=\frac{1}{v^{2n}}q^{-\left(\frac{n^2}{2}\right)}\tilde{F}_q\left(q^{\frac{n}{2}}v\right)-\sum_{j=0}^{n}v^{-2(n-j)+1}q^{-\begin{pmatrix}n-j\\2\end{pmatrix}}f^{(j-1)}(0), \tag{4.7}$$

where $f^{(n)}(t)=D_q^nf(t)$.

Proof. Here, we will use mathematical induction. By using the Definition 4.1 and q-integration by parts (1.18), we have

$$\tilde{T}_q\left[D_qf(t);v\right]=-\frac{1}{v}f(0)+\frac{q^{\left(-\frac{1}{2}\right)}}{v^2}\tilde{F}_q\left[q^{\frac{1}{2}}v\right]. \tag{4.8}$$

Now, suppose this relation is true for $n=k>0$, that is

$$\tilde{T}_q\left[D_q^kf(t);v\right]=\frac{1}{v^{2k}}q^{-\left(\frac{k^2}{2}\right)}\tilde{F}_q\left(q^{\frac{k}{2}}v\right)-\sum_{j=0}^{k}v^{-2(k-j)-1}q^{-\begin{pmatrix}k-j\\2\end{pmatrix}}f^{(j-1)}(0). \tag{4.9}$$

Let $g(t)=D_q^kf(t)$, so that $D_q^{k+1}f(t)=D_qg(t)$, thus

$$\begin{aligned}\tilde{T}_q\left[D_q^{k+1}f(t);v\right]=\tilde{T}_q\left[D_q^kg(t);v\right]&=-\tfrac{1}{v}g(0)+\tfrac{q^{-\left(\frac{1}{2}\right)}}{v^2}\tilde{T}_q\left[g(t);q^{\left(\frac{1}{2}\right)}v\right]\\&=-\tfrac{1}{v}D_q^kf(0)+\tfrac{q^{\left(\frac{-1}{2}\right)}}{v^2}\tilde{T}_q\left[D_q^kf(t);q^{\left(\frac{1}{2}\right)}v\right].\end{aligned} \tag{4.10}$$

Now, on using (4.9), we have

$$\begin{aligned}&\tilde{T}_q\left[D_q^{k+1}f(t);v\right]=-\tfrac{1}{v}D_q^kf(0)\\&+\tfrac{q^{\left(\frac{-1}{2}\right)}}{v^2}\left[\left\{\tfrac{1}{v^{2k}}q^{\left(\frac{-k^2}{2}-k\right)}\tilde{F}_q\left(q^{\left(\frac{k+1}{2}\right)}v\right)\right\}-\sum_{j=0}^{k}v^{-2(k-j)-3}q^{-\left(\begin{array}{c}k+1-j\\2\end{array}\right)}f^{(j-1)}(0)\right]\\&=\tfrac{1}{v^{2(k+1)}}q^{-\left(\frac{(k+1)^2}{2}\right)}\tilde{F}_q\left(q^{\frac{k+1}{2}}v\right)-\sum_{j=0}^{k}v^{-2(k+1-j)+1}q^{-\left(\begin{array}{c}k+1-j\\2\end{array}\right)}f^{(j-1)}(0).\end{aligned} \tag{4.11}$$

This establishes that (4.7) holds for $n=k+1$, and consequently for all $n\in\mathbb{N}$. Thus, the proof of the theorem is complete. □

Theorem 4.6. *(q-**Derivative of Transform**). If $f(t)\in\tilde{P}_q$ and $\tilde{T}_q[f(t);v]=\tilde{F}_q(v)$, then for all $n\in\mathbb{N}$*

$$\tilde{T}_q[t^nf(t);v]=\left(\frac{q^2v^3}{1+q}\frac{\partial_q}{\partial_qv}\right)^n\tilde{F}_q(v). \tag{4.12}$$

Proof. From [1], the q-derivative of the q-Laplace transform of the second kind can be expressed as

$$\tilde{L}_q\{t^nf(t)\}=(-1)^n\left(\frac{\partial_q}{\partial_qs}\right)^n\tilde{G}_q(s), \tag{4.13}$$

where $\tilde{G}_q(s)=\int_0^\infty e_q(-st)f(t)d_qt$. Here, by replacing s with $\frac{1}{v^2}$ and applying (4.4), we have

$$\tilde{T}_q[t^nf(t);v]=(-1)^n\left(\frac{-q^2v^3}{1+q}\frac{\partial_q}{\partial_qv}\right)^n\frac{1}{v}\int_0^\infty e_q\left(-\frac{t}{v^2}\right)f(t)d_qt=\left(\frac{q^2v^3}{1+q}\frac{\partial_q}{\partial_qv}\right)^n\tilde{F}_q(v).$$

□

Theorem 4.7. ***(Integral Property)*** *If $f(t)\in\tilde{P}_q$ and $\tilde{T}_q[f(t);v]=\tilde{F}_q(v)$, then*

$$\tilde{T}_q\left[\int_0^tf(u)d_qu;v\right]=v^2q^{\left(\frac{-1}{2}\right)}\tilde{T}_q\left[f(t);vq^{\left(\frac{-1}{2}\right)}\right]=v^2q^{\left(\frac{-1}{2}\right)}\tilde{F}_q\left(vq^{\left(\frac{-1}{2}\right)}\right). \tag{4.14}$$

Proof. If we set $g(t)=e_q\left(\frac{-t}{v^2}\right)$ and $F(t)=\int_0^tf(u)d_qu$ in q-analogue of integration by parts formula (1.18), we get

$$\begin{aligned}&\int_0^\infty\left\{\int_0^tf(u)d_qu\right\}D_q\left[e_q\left(\tfrac{-t}{v^2}\right)\right]d_qt\\&=\left\{\left\{\int_0^tf(u)d_qu\right\}e_q\left(\tfrac{-t}{v^2}\right)\right\}_0^\infty-\int_0^\infty e_q\left(\tfrac{-qt}{v^2}\right)D_q\left\{\int_0^tf(u)d_qu\right\}d_qt,\end{aligned} \tag{4.15}$$

which gives

$$\frac{-1}{v}.\frac{1}{v}\int_0^\infty e_q\left(\frac{-t}{v^2}\right)\left\{\int_0^tf(u)d_qu\right\}d_qt=-v.\frac{q^{\frac{-1}{2}}}{vq^{\frac{-1}{2}}}\int_0^\infty e_q\left(\frac{-t}{v^2q^{-1}}\right)f(t)d_qt.$$

Hence

$$\tilde{T}_q[F(t);v]=v^2q^{\left(\frac{-1}{2}\right)}\tilde{T}_q\left[f(t);vq^{\left(\frac{-1}{2}\right)}\right].$$

□

Next, applying the q-gamma function of the second kind (1.15) to (4.3) leads to the q-Tarig transform of the second kind, tabulated for certain functions in following table.

Table 2. q-Tarig transform of Second Kind of Certain Functions

S. No.	$f(t)$	$\tilde{T}_q[f(t);v] = \tilde{F}_q(v)$
1	t^n	$q^{-\binom{n+1}{2}} [n]_q! v^{2n+1}$
2	$e_q(at)$	$\sum_{n=0}^{\infty} a^n q^{-\binom{n+1}{2}} v^{2n+1}$
3	$E_q(at)$	$\frac{vq}{q-av^2}$
4	$Cos_q(at)$	$\left(\frac{q^2 v}{q^2+a^2v^4}\right)$
5	$Sin_q(at)$	$\left(\frac{aqv^3}{q^2+a^2v^4}\right)$
6	$\cos_q(at)$	$v\sum_{n=0}^{\infty}(-1)^n q^{-n(2n+1)}\left(av^2\right)^{2n}$
7	$\sin_q(at)$	$v\sum_{n=0}^{\infty}(-1)^n q^{-(n+1)(2n+1)}\left(av^2\right)^{2n+1}$
8	$Cosh_q(at)$	$\left(\frac{vq^2}{q^2-a^2v^4}\right)$
9	$Sinh_q(at)$	$\left(\frac{aqv^3}{q^2-a^2v^4}\right)$
10	$\cosh_q(at)$	$v\sum_{n=0}^{\infty} q^{-n(2n+1)}\left(av^2\right)^{2n}$
11	$\sinh_q(at)$	$v\sum_{n=0}^{\infty} q^{-(n+1)(2n+1)}\left(av^2\right)^{2n+1}$
12	$H(t-\alpha)$	$ve_q\left(\frac{-\alpha}{v^2}\right)$

5 Conclusion Remarks

In this paper, we have defined two q-analogues of the Tarig transforms and established their key properties, including linearity, scaling, and convolution. Also, the q-Tarig transform of derivatives and integrals of a function, and some special functions are obtained. As applications, we have solved some q-differential equations by using the q-Tarig transform of the first kind. The findings highlight the effectiveness of the q-Tarig transform as a valuable analytical tool in quantum calculus and its applications in various scientific domains.

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