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Fractional Hepatitis B Virus Model: An Efficient Approach for Analytical and Numerical Solutions

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Abstract Mathematical models are useful to the understanding and medical management for infectious diseases. They assist researchers and public health personnel in making decisions by obtaining data, evaluating the impact of initiatives, and estimating the spread of diseases. The main objective of the current work is to provide an in-depth analysis of the transmission and control of a hepatitis B model under the Caputo-Fabrizio derivative, including qualitative and semi-analytical investigation. The Picard successive approximation approach and Banach's fixed point theory were used to ensure the model's existence and stability. Numerical computations were performed using the iterative Laplace transform method. The obtained solutions were graphically simulated using MATLAB. These findings hold significant promise for applications in clinical science and medicine, providing valuable insights that could inform healthcare practices and system improvements.

1 Introduction

Hepatitis B is a dangerous disease caused by the hepatitis virus. It is a significant global health problem. This disease leads to serious chronic liver infections, putting people's lives at high risk. Hepatitis B is also a primary cause of liver cancer [1]. The infection occurs when the virus enters the bloodstream and reaches the liver, where it causes damage [2].

Hepatitis B has two stages: an acute stage lasting up to six months, which is often cleared by the immune system, and a chronic stage that lasts longer than six months. Both adults and children can contract this infection. HBV is a serious disease that can be transmitted from an infected individual to a healthy person, making the affected individuals chronic carriers. Nearly 240 million people worldwide have chronic liver infections, and approximately 600,000 people die each year due to this dangerous disease.

Mathematical modeling facilitates precise definition and analysis of real-world events by employing various mathematical approaches, including fractional calculus. Its significance has grown notably within mathematical science, allowing us to transform the real-world issues into particularly equations, mathematical language, and then use appropriate approaches to predict results. Modeling has a wide range of applications, providing as a tool for anticipating or estimating future ramifications, making it an essential approach in many different fields [3, 4]. Over the past decades, several fractional operators, including Riemann-Liouville (RL), Hadamard [5, 6], Caputo [7, 8], Caputo-Fabrizio (CF) [9], and Atangana-Baleanu, have been suggested to enhance the understanding of model dynamics. Each operator has its own advantages and disadvantages. For example, the Caputo fractional operator utilizes initial conditions with integer-order derivatives that have clear physical meaning, but it exhibits singularities at certain points. To address this limitation, Caputo and Fabrizio ([10], [11]) recently proposed a unique fractional derivative operator. This new operator features a nonlocal and nonsingular kernel, making it well-suited for describing and analyzing the dynamics of HBV.

In recent decades, various HBV models have been developed. Mathematical modeling is a powerful tool for understanding infectious diseases and their transmission dynamics. These models

help us comprehend the transmission of diseases like HBV. Numerous articles have investigated the transmission and control of HBV [12, 13, 14, 15, 16, 17]. Kamyad et al. [18] have also explored a mathematical model for HBV as follows:

$$\begin{cases} \frac{dS(\xi)}{d\xi} = \vartheta - [\vartheta p_1 C + \vartheta p_2 R + p'(I+\mu C)S + \vartheta S + \rho_1 S] + \kappa_4 R, \\ \frac{dE(\xi)}{d\xi} = p'(I+\mu C)S - (\vartheta+\kappa_1)E, \\ \frac{dI(\xi)}{d\xi} = \kappa_1 E - (\vartheta+\kappa_2)I, \\ \frac{dC(\xi)}{d\xi} = [\vartheta p_1 - (\vartheta+\kappa_3-\rho_2)]C + p_3\kappa_2 I, \\ \frac{dR(\xi)}{d\xi} = (\vartheta p_2 - \vartheta - \kappa_4)R + (1-p_3)\kappa_2 I + (\kappa_3+\rho_2)C + \rho_1 S. \end{cases} \tag{1.1}$$

In the given model, $S(\xi)$ represents the density of susceptible individuals, $E(\xi)$ denotes exposed individuals, $I(\xi)$ indicates infected individuals, $C(\xi)$ stands for chronic HBV carriers, and $R(\xi)$ refers to recovered individuals. In this context, ϑ is the per capita birth and death rate, while κ_1, κ_2, and κ_3 are the rates at which individuals become exposed, transition to carrier status, and move from carrier to recovered, respectively. Additionally, μ is the infectiousness rate of carriers relative to acute infections, and p_3 is the proportion of acutely infected individuals who become carriers.

According to the law of mass action, the infection transmits horizontally at a rate of $p'(I+\mu C)S$, where p' is the contact rate. The infection also transmits vertically at a rate of p_1 among newborns, represented by the term $\vartheta p_1 C$, where $p_1 < 1$. Furthermore, p_2 of newborns from the recovered class are immune, expressed by $\vartheta p_2 R$, where $p_2 < 1$.

We believe that a suitable mathematical model will assist health officials in taking effective measures to contain the spread of the HBV. Motivated by the aforementioned literature, we studied the graphical nature of model (1.1) using the CF fractional derivative and is given as:

$$^{CF}D_\xi^\varsigma S(\xi) = \vartheta - [\vartheta p_1 C + \vartheta p_2 R + p'(I+\mu C)S + \vartheta S + \rho_1 S] + \kappa_4 R, \tag{1.2}$$

$$^{CF}D_\xi^\varsigma E(\xi) = p'(I+\mu C)S - (\vartheta+\kappa_1)E, \tag{1.3}$$

$$^{CF}D_\xi^\varsigma I(\xi) = \kappa_1 E - (\vartheta+\kappa_2)I, \tag{1.4}$$

$$^{CF}D_\xi^\varsigma C(\xi) = [\vartheta p_1 - (\vartheta+\kappa_3-\rho_2)]C + p_3\kappa_2 I, \tag{1.5}$$

$$^{CF}D_\xi^\varsigma R(\xi) = (\vartheta p_2 - \vartheta - \kappa_4)R + (1-p_3)\kappa_2 I + (\kappa_3+\rho_2)C + \rho_1 S, \quad 0 < \varsigma \leq 1. \tag{1.6}$$

with initial conditions

$$S(0) = S_0, \tag{1.7}$$

$$E(0) = E_0, \tag{1.8}$$

$$I(0) = I_0, \tag{1.9}$$

$$C(0) = C_0, \tag{1.10}$$

and

$$R(0) = R_0. \tag{1.11}$$

In this article, the Laplace transform method (LTM) is combined with new iterative technique (NIM) called the iterative Laplace transform method (ILTM) to achieve the approximate solution and graphical results. Moreover, the Picard successive approximation approach and Banach's fixed point theory are employed to demonstrate the stability conditions of the suggested model.

2 Fractional calculus

Fractional calculus stands as a prized asset within mathematics, offering exceptional significance in the realm of mathematical modeling. Its capacity to characterize natural phenomena surpasses that of classical calculus, providing more precise descriptions. Moreover, it proves highly advantageous for delineating nonlinear phenomena across various scientific and technological domains. Currently, numerous scholars harness this potent tool, enhancing the quality and depth of their research endeavors. In medical science, fractional calculus finds extensive application

in the study of fractional-order disease models, like coronavirus dynamics [19, 20, 21, 22, 23], HIV [24], HBV [25, 26, 27, 28, 29, 30, 35] and serval others [31, 32, 33, 34]. Here, we present some useful definitions of fractional calculus that aid in our current research.

Definition 2.1. The Caputo derivative for $0 < \varsigma < 1$, is defined as:

$$D_{\xi}^{\varsigma}\varkappa(\xi) = \frac{1}{\Gamma(p-\varsigma)}\int_{0}^{\xi}(\xi-\Psi)^{p-\varsigma-1}\varkappa(\Psi)\,d\Psi,\ \xi>0,\ p-1<\varsigma\leq p,\ p\in Z^{+}. \tag{2.1}$$

Definition 2.2. Let $\varkappa \in G^{1}(a_1, b_1)$ $b_1 > a_1$ and $0 < \varsigma < 1$, then CF derivative ([10], [11]) is defined as:

$${}^{CF}D_{\xi}^{\varsigma}\varkappa(\xi) = \frac{1}{(1-\varsigma)}\int_{a_1}^{\xi}\varkappa'(\Psi)\exp\left[-\varsigma\frac{\xi-\Psi}{1-\varsigma}\right]d\Psi, \tag{2.2}$$

here, $G^{1}(a_1, b_1) = \left\{\varkappa |\ \varkappa \in L^{2}(a_1, b_1)\, and\, \varkappa' \in L^{2}(a_1, b_1)\right\}$ and $L^{2}(a_1, b_1)$ is the space of square integrable functions on the interval (a_1, b_1).

Definition 2.3. The CF fractional integral operator of order $0 < \varsigma < 1$, is defined as:

$${}^{CF}J_{\xi}^{\varsigma}\varkappa(\xi) = \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\varkappa(\xi) + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_{0}^{\xi}\varkappa(\Psi)\,d\Psi,\ \ \xi\geq 0, \tag{2.3}$$

where ${}^{CF}D_{\xi}^{\varsigma}\varkappa(\xi) = 0$, if $\varkappa$ is a constant function.

Remark 2.4. From the previous definitions, it has been observed that the fractional integral of a function with order $0 < \varsigma \leq 1$ represents an average between the respective functions and their integrals of order one. This further implies that

$$\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\varkappa(\xi) + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} = 1.$$

The previous equation provides a clear formula for

$$M(\varsigma) = \frac{2\varsigma}{(2-\varsigma)},\ \ 0\leq\varsigma\leq 1.$$

Definition 2.5. The Laplace transform (LT) of the CF derivative of order $0 < \varsigma \leq 1$ for $m \in \mathbb{N}$ is given by:

$$\begin{aligned} L\left({}^{CF}D_{\xi}^{m+\varsigma}\varkappa(\xi)\right)(q) &= \frac{1}{1-\varsigma}L\left(\varkappa^{(m+1)}(\xi)\right)\ L\left(exp\left(-\frac{\varsigma}{1-\varsigma}\xi\right)\right) \\ &= \frac{q^{m+1}L(\varkappa(\xi)) - q^{m}\varkappa(0) - q^{m-1}\varkappa'(0) - .... - \varkappa^{(m)}(0)}{q+\varsigma(1-q)} \end{aligned} \tag{2.4}$$

In particular, we have

$$L\left({}^{CF}D_{\xi}^{\varsigma}\varkappa(\xi)\right)(q) = \frac{qL(\varkappa(\xi)) - \varkappa(0)}{q+\varsigma(1-q)},\ \ m=0,$$

$$L\left({}^{CF}D_{\xi}^{\varsigma+1}\varkappa(\xi)\right)(q) = \frac{q^{2}L(\varkappa(\xi)) - q\varkappa(0) - \varkappa'(0)}{q+\varsigma(1-q)},\ \ m=1.$$

3 Proposed Methodology

By applying the Laplace transform to both sides of the system given by the HBV model (1.2)-(1.6) along with the initial conditions (1.7)-(1.11), we can observe that the terms (CS) and *IS* in this model exhibit nonlinearity.

$$\frac{qL\left(S(\xi)\right)-S(0)}{q+\varsigma(1-q)}=L\left(\vartheta-\left[\vartheta p_1C+\vartheta p_2R+p'\left(I+\mu C\right)S+\vartheta S+\rho_1S\right]+\kappa_4R\right),$$
$$\frac{qL\left(E(\xi)\right)-E(0)}{q+\varsigma(1-q)}=L\left(p'\left(I+\mu C\right)S-\left(\vartheta+\kappa_1\right)E\right),$$
$$\frac{qL\left(I(\xi)\right)-I(0)}{q+\varsigma(1-q)}=L\left(\kappa_1E-\left(\vartheta+\kappa_2\right)I\right),$$
$$\frac{qL\left(C(\xi)\right)-C(0)}{q+\varsigma(1-q)}=L\left(\left[\vartheta p_1-\left(\vartheta+\kappa_3-\rho_2\right)\right]C+p_3\kappa_2I\right),$$
$$\frac{qL\left(R(\xi)\right)-R(0)}{q+\varsigma(1-q)}=L\left(\left(\vartheta p_2-\vartheta-\kappa_4\right)R+\left(1-p_3\right)\kappa_2I+\left(\kappa_3+\rho_2\right)C+\rho_1S\right). \tag{3.1}$$

Rearranging, we get

$$L\left(S(\xi)\right)=\frac{S(0)}{q}+\left(\frac{q+\varsigma(1-q)}{q}\right)L\big(\vartheta-\left[\vartheta p_1C+\vartheta p_2R+p'\left(I+\mu C\right)S+\vartheta S+\rho_1S\right]+\kappa_4R\big),$$
$$L\left(E(\xi)\right)=\frac{E(0)}{q}+\left(\frac{q+\varsigma(1-q)}{q}\right)L\left(p'\left(I+\mu C\right)S-\left(\vartheta+\kappa_1\right)E\right),$$
$$L\left(I(\xi)\right)=\frac{I(0)}{q}+\left(\frac{q+\varsigma(1-q)}{q}\right)L\left(\kappa_1E-\left(\vartheta+\kappa_2\right)I\right),$$
$$L\left(C(\xi)\right)=\frac{C(0)}{q}+\left(\frac{q+\varsigma(1-q)}{q}\right)L\left(\left[\vartheta p_1-\left(\vartheta+\kappa_3-\rho_2\right)\right]C+p_3\kappa_2I\right),$$
$$L\left(R(\xi)\right)=\frac{R(0)}{q}+\left(\frac{q+\varsigma(1-q)}{q}\right)L\big(\left(\vartheta p_2-\vartheta-\kappa_4\right)R+\left(1-p_3\right)\kappa_2I+\left(\kappa_3+\rho_2\right)C+\rho_1S\big). \tag{3.2}$$

Further, the inverse LT of equations (3.2) yields

$$S(\xi)=S(0)+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)L(\vartheta-\left[\vartheta p_1C+\vartheta p_2R+p'\left(I+\mu C\right)S+\vartheta S+\rho_1S\right]+\kappa_4R)\right],$$
$$E(\xi)=E(0)+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)L\left(p'\left(I+\mu C\right)S-\left(\vartheta+\kappa_1\right)E\right)\right],$$
$$I(\xi)=I(0)+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)L\left(\kappa_1E-\left(\vartheta+\kappa_2\right)I\right)\right],$$
$$C(\xi)=C(0)+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)L\left(\left[\vartheta p_1-\left(\vartheta+\kappa_3-\rho_2\right)\right]C+p_3\kappa_2I\right)\right],$$
$$R(\xi)=R(0)+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)L(\left(\vartheta p_2-\vartheta-\kappa_4\right)R+\left(1-p_3\right)\kappa_2I+\left(\kappa_3+\rho_2\right)C+\rho_1S)\right]. \tag{3.3}$$

The series solutions achieved by the method are given by

$$S=\sum_{\tau=0}^{\infty}S_\tau \qquad E=\sum_{\tau=0}^{\infty}E_\tau \qquad I=\sum_{\tau=0}^{\infty}I_\tau,$$
$$C=\sum_{\tau=0}^{\infty}C_\tau \qquad R=\sum_{\tau=0}^{\infty}R_\tau. \tag{3.4}$$

The nonlinear term CS can be written as

$$CS = \sum_{\tau=0}^{\infty} G_{\tau},$$

where G_τ is further decomposed as follows:

$$G_\tau = \sum_{r=0}^{\tau} C_r \sum_{r=0}^{\tau} S_r - \sum_{r=0}^{\tau-1} C_r \sum_{r=0}^{\tau-1} S_r.$$

We get the following recursive relation by using initial conditions

$$\begin{aligned}
S_{n+1}(\xi) =& S_n(0) + L^{-1}\Bigg[\left(\frac{q+\varsigma(1-q)}{q}\right) L(\vartheta - [\vartheta p_1 C_n + \vartheta p_2 R_n + p'\left(I_n + \mu C_n\right) S_n] \\
&+ \vartheta S_n + \rho_1 S_n + \kappa_4 R_n)\Bigg], \\
E_{n+1}(\xi) =& E_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(p'\left(I_n + \mu C_n\right) S_n - \left(\vartheta + \kappa_1\right) E_n\right)\right], \\
I_{n+1}(\xi) =& I_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\kappa_1 E_n - \left(\vartheta + \kappa_2\right) I_n\right)\right], \\
C_{n+1}(\xi) =& C_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left([\vartheta p_1 - \left(\vartheta + \kappa_3 - \rho_2\right)] C_n + p_3 \kappa_2 I_n\right)\right], \\
R_{n+1}(\xi) =& R_n(0) + L^{-1}\Bigg[\left(\frac{q+\varsigma(1-q)}{q}\right) L((\vartheta p_2 - \vartheta - \kappa_4) R_n + (1-p_3)\kappa_2 I_n \\
&+ \left(\kappa_3 + \rho_2\right) C_n + \rho_1 S_n)\Bigg]. \qquad (3.5)
\end{aligned}$$

4 Stability Analysis

Let Banach space$(B, ||.||)$ is a endomorphism Λ on B. The recurrence formula $\zeta_{n+1} = p(\Lambda, \zeta_n)$ denotes an exact recurrence. The set of fixed points of Λ is represented by $U(\Lambda)$. Moreover, Λ has minimum one element ζ_n that converges to a point $x \in U(\Lambda)$. Let $\{\beta_n\} \in B$ and consider $\sigma_n = ||\beta_{n+1} - p(\Lambda, \beta_n)||$. If $\lim_{n\to\infty} \sigma_n = 0$ implies $\lim_{n\to\infty} \beta_n = x$, then the given iteration method $\zeta_{n+1} = p(\Lambda, \zeta_n)$ is known as Λ-stable.Thus, the sequence β_n is bounded from above, and the process is recognized as Picard's iteration. Additionally, it achieves Λ-stability when the specified conditions hold true for $\zeta\_n + 1 = \Lambda, \zeta\_n$.

Theorem 4.1. *Let Λ be a self-map on Banach space $(B, ||.||)$ that satisfies*

$$\|\Lambda a - \Lambda b\| \leq \Gamma\|a - \Lambda a\| + \epsilon\|a - b\|$$

$\forall\ a, b \in B$, where $0 \leq \Gamma$ and $0 \leq \epsilon < 1$. Suppose Λ is Picard Λ-stable. Consider equations (3.5) related to (1.1)-(1.6).

Theorem 4.2. *Consider a self-map* Λ *defined as*

$$\begin{aligned}
\Lambda\left(S_n(\xi)\right) =& S_{n+1}(\xi)\\
=& S_n(0) + L^{-1}\Bigg[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\vartheta - \left[\vartheta p_1 C_n + \vartheta p_2 R_n + p'\left(I_n + \mu C_n\right) S_n\right.\\
&\left. + \vartheta S_n + \rho_1 S_n\right] + \kappa_4 R_n\Big)\Bigg],\\
\Lambda\left(E_n(\xi)\right) =& E_{n+1}(\xi) = E_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(p'\left(I_n + \mu C_n\right) S_n - \left(\vartheta + \kappa_1\right) E_n\right)\right],\\
\Lambda\left(I_n(\xi)\right) =& I_{n+1}(\xi) = I_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\kappa_1 E_n - \left(\vartheta + \kappa_2\right) I_n\right)\right],\\
\Lambda\left(C_n(\xi)\right) =& C_{n+1}(\xi) = C_n(0) + L^{-1}\Bigg[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\left[\vartheta p_1 - \left(\vartheta + \kappa_3 - \rho_2\right)\right] C_n\\
&+ p_3\kappa_2 I_n\Big)\Bigg],\\
\Lambda\left(R_n(\xi)\right) =& R_{n+1}(\xi) = R_n(0) + L^{-1}\Bigg[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\left(\vartheta p_2 - \vartheta - \kappa_4\right) R_n + \left(1 - p_3\right)\right.\\
&\left.\kappa_2 I_n + \left(\kappa_3 + \rho_2\right) C_n + \rho_1 S_n\right)\Bigg],
\end{aligned} \tag{4.1}$$

where Lagrange's multiplier in fractional form is defined as $\frac{q+\tau(1-q)}{q}$. *It is* Λ*-stable in* $L^1(a_1, b_1)$ *if*

$$\begin{aligned}
&\Big(1 - (\vartheta p_1 + \vartheta p_2 + \vartheta + \rho_1 - \kappa_4)F(\varkappa) - (p' + p'\mu)K_1 G(\varkappa) - p' K_2 H(\varkappa)\\
&- p'\mu K_3 J(\varkappa)\Big) < 1,\\
&\Big(1 + (p' + p'\mu)F_1(\varkappa) + p' K_2 G_1(\varkappa) + p'\mu K_3 H_1(\varkappa) - (\vartheta + \kappa_1)J_1(\varkappa)\Big) < 1,\\
&\Big(1 + \kappa_1 F_2(\varkappa) - (\vartheta + \kappa_2)G_2(\varkappa)\Big) < 1,\\
&\Big(1 - (\vartheta + \kappa_3 - \rho_2)F_3(\varkappa) + p_3\kappa 2 G_3(\varkappa)\Big) < 1,\\
&\Big(1 + (\vartheta p_2 - \kappa_4 - \vartheta)F_4(\varkappa) + (1 - p_3)\kappa 2 G_4(\varkappa) + (\kappa_3 + \rho_2)H_4(\varkappa) + \rho_1 J_4(\varkappa)\Big) < 1.
\end{aligned}$$

Proof. The proof starts by showing that Λ possesses a fixed point. Hence, for all $((m, n) \in$

$\mathbb{N} \times \mathbb{N}$), we compute the following variances:

$$
\begin{aligned}
&\Lambda\left(S_m(\xi)\right) - \Lambda\left(S_n(\xi)\right) = S_m(0) - S_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\vartheta - \big[\vartheta p_1 C_m\right.\\
&\left. + \vartheta p_2 R_m + p'\left(I_m + \mu C_m\right) S_m + \vartheta S_m + \rho_1 S_m\big] + \kappa_4 R_m\Big)\right]\\
&- L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\vartheta - \big[\vartheta p_1 C_n + \vartheta p_2 R_n + p'\left(I_n + \mu C_n\right) S_n\right.\\
&\left. + \vartheta S_n + \rho_1 S_n\big] + \kappa_4 R_n\Big)\right],\\
&\Lambda\left(E_m(\xi)\right) - \Lambda\left(E_n(\xi)\right) = E_m(0) - E_n(0)\\
&+ L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(p'\left(I_m + \mu C_m\right) S_m - \left(\vartheta + \kappa_1\right) E_m\right)\right]\\
&- L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(p'\left(I_n + \mu C_n\right) S_n - \left(\vartheta + \kappa_1\right) E_n\right)\right],\\
&\Lambda\left(I_m(\xi)\right) - \Lambda\left(I_n(\xi)\right) = I_m(0) - I_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\kappa_1 E_m - \left(\vartheta + \kappa_2\right) I_m\right)\right]\\
&- L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\kappa_1 E_n - \left(\vartheta + \kappa_2\right) I_n\right)\right]\\
&\Lambda\left(C_m(\xi)\right) - \Lambda\left(C_n(\xi)\right) = C_m(0) - C_n(0)\\
&+ L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\left[\vartheta p_1 - \left(\vartheta + \kappa_3 - \rho_2\right)\right] C_m + p_3 \kappa_2 I_m\Big)\right],\\
&- L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\left[\vartheta p_1 - \left(\vartheta + \kappa_3 - \rho_2\right)\right] C_n + p_3 \kappa_2 I_n\Big)\right]\\
&\Lambda\left(R_m(\xi)\right) - \Lambda\left(R_n(\xi)\right) = R_m(0) - R_n(0)\\
&+ L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\left(\vartheta p_2 - \vartheta - \kappa_4\right) R_m + \left(1 - p_3\right) \kappa_2 I_m + \left(\kappa_3 + \rho_2\right) C_m + \rho_1 S_m\right)\right]\\
&- L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\left(\left(\vartheta p_2 - \vartheta - \kappa_4\right) R_n + \left(1 - p_3\right) \kappa_2 I_n + \left(\kappa_3 + \rho_2\right) C_n + \rho_1 S_n\right)\right]
\end{aligned}
\tag{4.2}
$$

Considering the first equation of (4.2) and applying the norm, we obtain the following expression, without loss of generality:

$$
\begin{aligned}
&\left|\left|\Lambda\left(S_m(\xi)\right) - \Lambda\left(S_n(\xi)\right)\right|\right| = \Big|\Big| S_m(0) - S_n(0) + L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\vartheta - \big[\vartheta p_1 C_m\right.\\
&\left. + \vartheta p_2 R_m + p'\left(I_m + \mu C_m\right) S_m + \vartheta S_m + \rho_1 S_m\big] + \kappa_4 R_m\Big)\right]\\
&- L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right) L\Big(\vartheta - \big[\vartheta p_1 C_n + \vartheta p_2 R_n + p'\left(I_n + \mu C_n\right) S_n\right.\\
&\left. + \vartheta S_n + \rho_1 S_n\big] + \kappa_4 R_n\Big)\right]\Big|\Big|,
\end{aligned}
\tag{4.3}
$$

next, utilizing triangle inequality and simplifying (4.3), we get

$$\begin{aligned}||\Lambda\left(S_m(\xi)\right)-\Lambda\left(S_n(\xi)\right)||\leq&||S_m(0)-S_n(0)||+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)\right.\\&L\Big[||-\vartheta p_1(C_m-C_n)||+||-\vartheta p_2(R_m-R_n)||\\&+||-p'S_n(I_m-I_n)||+||-p'I_m(S_m-S_n)||\\&+||-p'\mu S_n(C_m-C_n)||+||-p'\mu C_m(S_m-S_n)||\\&\left.+||-(\vartheta+\rho_1)(S_m-S_n)||+||\kappa_4(R_m-R_n)||\Big]\right].\end{aligned}\tag{4.4}$$

Since both solutions have similar impacts, it is presumed that

$$\begin{aligned}||S_m(\xi)-S_n(\xi)||&=||E_m(\xi)-E_n(\xi)||\\||S_m(\xi)-S_n(\xi)||&=||I_m(\xi)-E_n(\xi)||\\||S_m(\xi)-S_n(\xi)||&=||C_m(\xi)-E_n(\xi)||\\||S_m(\xi)-S_n(\xi)||&=|RE_m(\xi)-E_n(\xi)||\end{aligned}$$

Substituting this into (4.4), we obtain the following equation:

$$\begin{aligned}||\Lambda\left(S_m(\xi)\right)-\Lambda\left(S_n(\xi)\right)||\leq&||S_m-S_n(0)||+L^{-1}\left[\left(\frac{q+\varsigma(1-q)}{q}\right)\right.\\&L\Big[||-\vartheta p_1(S_m-S_n)||+||-\vartheta p_2(S_m-S_n)||\\&+||-p'S_n(S_m-S_n)||+||-p'I_m(S_m-S_n)||\\&+||-p'\mu S_n(S_m-S_n)||+||-p'\mu C_m(S_m-S_n)||\\&\left.+||-(\vartheta+\rho_1)(S_m-S_n)||+||\kappa_4(S_m-S_n)||\Big]\right].\end{aligned}\tag{4.5}$$

Since $S_n, I_m, andC_m$ are bounded sequences, we have three distinct positive constants, K_1, K_2, and K_3 for every ξ, such that

$$||S_n||\leq K_1,\qquad ||I_m||\leq K_2,\qquad ||C_m||\leq K_3,\quad (m,n)\in\mathbb{N}\times\mathbb{N}.\tag{4.6}$$

Furthermore, taking into account equations (4.5) and (4.6), we obtain

$$\begin{aligned}||\Lambda\left(S_m(\xi)\right)-\Lambda\left(S_n(\xi)\right)||\leq&\Big(1-(\vartheta p_1+\vartheta p_2+\vartheta+\rho_1-\kappa_4)F(\varkappa)-(p'+p'\mu)K_1G(\varkappa)\\&-p'K_2H(\varkappa)-p'\mu K_3J(\varkappa)\Big)||S_m-S_n||.\end{aligned}\tag{4.7}$$

where F, G, H, and J are functions of $L^{-1}\left\{L\left(\frac{q+\alpha(1-q)}{q}\right)\right\}$. Similarly, we can obtain

$$
\begin{aligned}
||\Lambda(E_m(\xi)) - \Lambda(E_n(\xi))|| \leq & \Big(1 + (p' + p'\mu)F_1(\varkappa) + p'K_2G_1(\varkappa) + p'\mu K_3H_1(\varkappa) \\
& - (\vartheta + \kappa_1)J_1(\varkappa)\Big)||E_m - E_n|| \\
||\Lambda(I_m(\xi)) - \Lambda(I_n(\xi))|| \leq & \Big(1 + \kappa_1F_2(\varkappa) - (\vartheta + \kappa_2)G_2(\varkappa)\Big)||I_m - I_n|| \\
||\Lambda(C_m(\xi)) - \Lambda(C_n(\xi))|| \leq & \Big(1 - (\vartheta + \kappa_3 - \rho_2)F_3(\varkappa) + p_3\kappa 2G_3(\varkappa)\Big)||C_m - C_n|| \\
||\Lambda(R_m(\xi)) - \Lambda(R_n(\xi))|| \leq & \Big(1 + (\vartheta p_2 - \kappa_4 - \vartheta)F_4(\varkappa) + (1 - p_3)\kappa 2G_4(\varkappa) \\
& + (\kappa_3 + \rho_2)H_4(\varkappa) + \rho_1J_4(\varkappa)\Big)||R_m - R_n||, \qquad (4.8)
\end{aligned}
$$

where

$$
\begin{aligned}
& \Big(1 - (\vartheta p_1 + \vartheta p_2 + \vartheta + \rho_1 - \kappa_4)F(\varkappa) - (p' + p'\mu)K_1G(\varkappa) - p'K_2H(\varkappa) \\
& - p'\mu K_3J(\varkappa)\Big) < 1, \\
& \Big(1 + (p' + p'\mu)F_1(\varkappa) + p'K_2G_1(\varkappa) + p'\mu K_3H_1(\varkappa) - (\vartheta + \kappa_1)J_1(\varkappa)\Big) < 1, \\
& \Big(1 + \kappa_1F_2(\varkappa) - (\vartheta + \kappa_2)G_2(\varkappa)\Big) < 1, \\
& \Big(1 - (\vartheta + \kappa_3 - \rho_2)F_3(\varkappa) + p_3\kappa 2G_3(\varkappa)\Big) < 1, \\
& \Big(1 + (\vartheta p_2 - \kappa_4 - \vartheta)F_4(\varkappa) + (1 - p_3)\kappa 2G_4(\varkappa) + (\kappa_3 + \rho_2)H_4(\varkappa) + \rho_1J_4(\varkappa)\Big) < 1.
\end{aligned}
$$

Consequently, the non-linear self-mapping (Λ) possesses a fixed point. Subsequently, we will demonstrate that (Λ) fulfills all the conditions outlined in Theorem 4.1. Given the validity of equations (4.7) and (4.8), we will proceed to show, we use $\varepsilon = (0, 0, 0, 0, 0)$ and

$$
\Gamma = \begin{cases}
1 - (\vartheta p_1 + \vartheta p_2 + \vartheta + \rho_1 - \kappa_4)F(\varkappa) - (p' + p'\mu)K_1G(\varkappa) - p'K_2H(\varkappa) - p'\mu K_3J(\varkappa), \\
1 + (p' + p'\mu)F_1(\varkappa) + p'K_2G_1(\varkappa) + p'\mu K_3H_1(\varkappa) - (\vartheta + \kappa_1)J_1(\varkappa), \\
1 + \kappa_1F_2(\varkappa) - (\vartheta + \kappa_2)G_2(\varkappa), \\
1 - (\vartheta + \kappa_3 - \rho_2)F_3(\varkappa) + p_3\kappa 2G_3(\varkappa), \\
1 + (\vartheta p_2 - \kappa_4 - \vartheta)F_4(\varkappa) + (1 - p_3)\kappa 2G_4(\varkappa) + (\kappa_3 + \rho_2)H_4(\varkappa) + \rho_1J_4(\varkappa).
\end{cases}
$$

□

5 numerical results

Here, we provide the analytical and graphical solutions for the considered model. Considering the values mentioned in [18] as $p_1 = 0.11$, $p_2 = 0.1$, $p_3 = 0.059$, $\vartheta = 0.0121$, $p^{'} = 0.820$, $\mu = 0.1$, $\mu_1 = 0.45$, $\mu = 0.9$, $\kappa_1 = 6$ per year, $\kappa_2 = 4$ per year, $\kappa_3 = 0.025$ per year, and $\kappa_4 = 0.06$, we plot the graphical solutions using Matlab in figures 1-5.

From figure 1, it can be observed that the density of the susceptible population decreases sharply at different rates corresponding to various fractional orders. With the implementation of vaccination, the densities of the exposed, and infected of HBV decrease, with a faster decline observed at

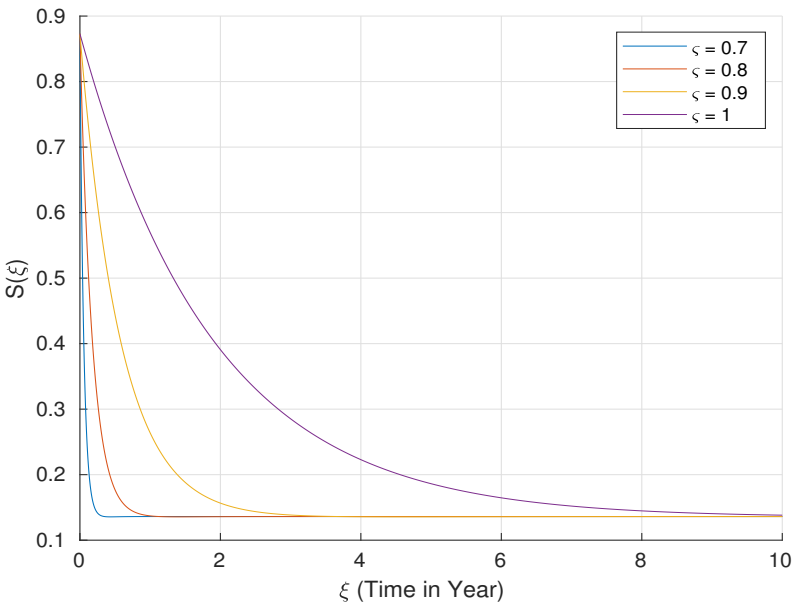

Figure 1. Graphical nature of susceptible individuals $S(\xi)$with respect to ξ at different order of ς.

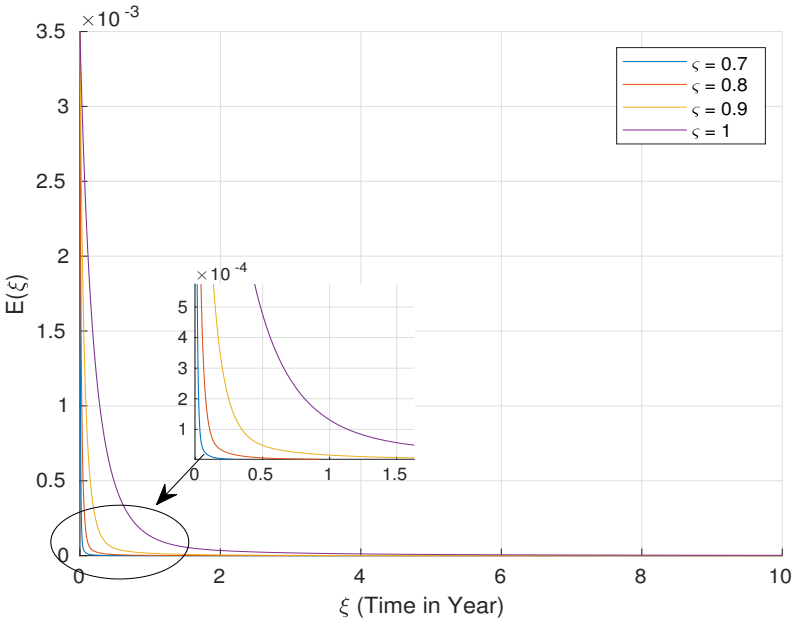

Figure 2. Graphical nature of infected exposed $E(\xi)$ with respect to ξ at different order of ς.

lower fractional orders, as shown in figures 2-3. The density of the chronic carriers, corresponding to different fractional orders in the proposed model (1.2), decreases, as depicted in figure 4. Consequently, the density of the recovered population, corresponding to different fractional orders in the proposed model (1.2), increases, as depicted in figure 5.

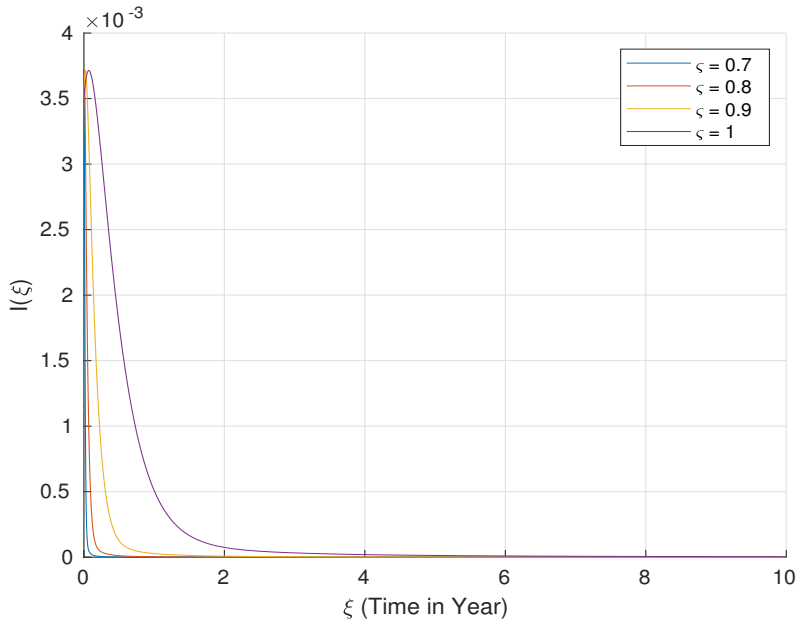

Figure 3. Graphical nature of infected individuals $I(\xi)$ with respect to ξ at different order of ς.

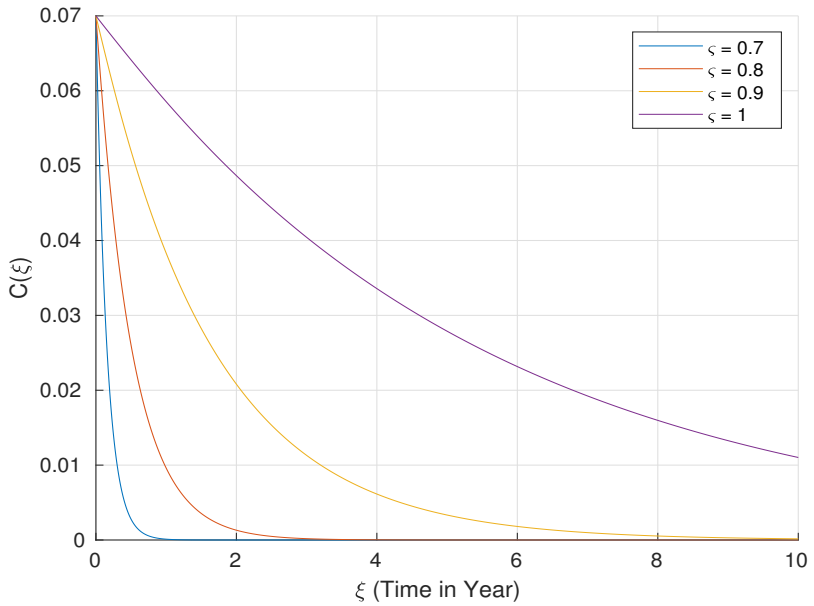

Figure 4. Graphical nature of chronic individuals $C(\xi)$ with respect to ξ at different order of ς.

6 Conclusions

In this paper, we developed criteria for analyzing the HBV model from both qualitative and analytical viewpoints. We demonstrated the existence and solution of the proposed model using Banach's theorem. To explore the transmission and vaccination processes of HBV more thoroughly, we created an algorithm to find approximate solutions for the model. Fractional model reveals that vaccination leads to a reduction in the density of these carriers, with a more rapid decline observed at lower fractional orders. This allows us to accurately estimate the vaccination rate required to control the epidemic. The graphs show that varying the fractional order influences the dynamics of susceptible and infected populations. In conclusion, the proposed

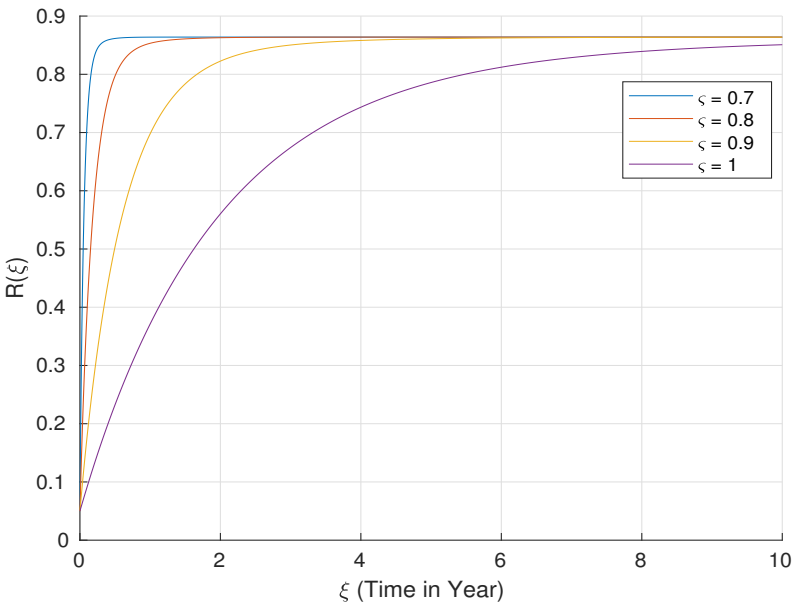

Figure 5. Graphical nature of recovered individuals $R(\xi)$ with respect to ξ at different order of ς.

technique proves effective for a detailed study of biological models. Additionally, the relevance of the S-E-I-C-R model to real-life problems underscores the importance of our study. Despite several limitations, our approach aligns with international experiences in hepatitis B control and offers insights into the long-term benefits of preventive health measures for high-prevalence populations.

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