Palestine Journal of Mathematics

Vol 15(Special Issue I)(2026) , 232–244

SOME THEOREMS ON GENERATING RELATIONS BY GOULD'S IDENTITY AND ITS APPLICATIONS

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MSC 2020 Classifications: Primary:33Cxx, 33C05; Secondary:33C45, 33C65.

Keywords and Phrases: Pathan's generalized hypergeometric polynomials, Srivastava's generalized hypergeometric polynomials, Generalized Rice polynomials of Khandekar, Gould's identity.

"The authors would like to express their sincere gratitude to Editorial Board members for accepting the paper."

Abstract The main aim of the present research paper is to obtain two hypergeometric generating relations with their independent demonstrations using Gould's identity. The methodology used in the demonstrations is series rearrangement technique. Several (presumably new) results are also deduced in the form of applications of our two generating relations. Further generalizations of the main generating relations are also derived.

1 Introduction and Preliminaries

In the present paper, all the sets of number system, the Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ and generalized hypergeometric function ${}_pF_q(.)$ have their usual meanings. The generalizations $[10, pp.348, 349; \text{problems } 212, 216]$ of the familiar binomial expansion ($[20, p.355, eq.(7.1.5)]$; see also $[11, eq.(1.2), eq.(1.5)]$; $[12, 13]$) with the conditions ($[20, p.355, eq.(7.1.6)]$; see also [11, eq.(1.3) and eq.(1.4)]) are mentioned below as Eq. (1.1):

$$\sum_{n=0}^{\infty} \binom{\theta + (\beta+1)n}{n} t^n = \frac{(1+\zeta)^{\theta+1}}{(1-\beta\zeta)}, \tag{1.1}$$

where $\binom{\theta+(\beta+1)n}{n}$ is a binomial coefficient and θ, β are complex numbers independent of n and ζ is a function of 't' defined implicitly by

$$\zeta = t(1+\zeta)^{1+\beta}, \tag{1.2}$$

subject to the condition

$$\zeta(0) = 0. \tag{1.3}$$

Gould's Identity($[15, p.169]$; $[5, p.90]$, **see also [11, eq.(1.6)]; [12, eq.(1.6)]**):

$$\sum_{n=0}^{\infty} \frac{\theta(\sigma+\mu n)}{\{\theta+(\beta+1)n\}} \binom{\theta+(\beta+1)n}{n} t^n = (1+\zeta)^\theta \left(\sigma + \frac{\mu\theta\zeta}{1-\beta\zeta}\right), \tag{1.4}$$

where $\theta, \beta, \sigma, \mu$ are complex parameters independent of n and ζ is given by the equations (1.2) and (1.3).

If we put $\theta = \{\alpha + (\beta+1)mr\}$ and $\sigma = \{\lambda + \mu mr\}$ in Gould's identity (1.4), we get the First modified form of Gould's identity:

$$\sum_{n=0}^{\infty} \frac{\{\alpha+(\beta+1)mr\}(\lambda+\mu n+\mu mr)}{\{\alpha+(\beta+1)mr+(\beta+1)n\}} \binom{\alpha+(\beta+1)mr+(\beta+1)n}{n} t^n$$

$$= (1+\zeta)^{\{\alpha+(\beta+1)mr\}} \left(\lambda + \mu mr + \frac{\mu\{\alpha+(\beta+1)mr\}\zeta}{(1-\beta\zeta)}\right), \tag{1.5}$$

with

$$\zeta = t(1+\zeta)^{\beta+1} \quad ; \zeta(0) = 0.$$

If we put $\theta = \{\alpha + (\beta+1)mr\}$ and $\sigma = \{\lambda + \mu(\beta+1)r\}$ in Gould's identity (1.4), we get the Second modified form of Gould's identity:

$$\sum_{n=0}^{\infty} \frac{\{\alpha+(\beta+1)mr\}\{\lambda+\mu n+\mu(\beta+1)r\}}{\{\alpha+(\beta+1)mr+(\beta+1)n\}} \binom{\alpha+(\beta+1)mr+(\beta+1)n}{n} t^n$$

$$= (1+\zeta)^{\{\alpha+(\beta+1)mr\}} \left(\lambda + \mu(\beta+1)r + \frac{\mu\{\alpha+(\beta+1)mr\}\zeta}{(1-\beta\zeta)} \right), \tag{1.6}$$

with

$$\zeta = t(1+\zeta)^{\beta+1} \quad ; \zeta(0) = 0.$$

Gauss multiplication theorem ([20, $p.23, eq.(1.1.26)$], [11, $eq.(1.9)$]); summation identity ([20, p.101, Lemma(3),eq.(2.1.6)], [11, $eq.(1.10)$]); the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ and $L_n^{(\alpha+n\beta)}(x)$ ([14, p.200, eq.(112.1)] ; see also [11, eq.(1.11), eq.(1.12)]); the Jacobi polynomials of first kind $P_n^{(\alpha,\beta)}(x)$ [14, $p.254,\ eq.(132.1), p.255,\ eq.(132.7)$] and Gauss hypergeometric functions ${}_2F_1$ after particular replacements in Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [11, eq.(1.13) to eq.(1.16)] are also available in the papers [12, 13].
One may refer the paper [11, eq.(1.17) to eq.(1.20)], (may also see [12, 13]) for the generalized Rice polynomials $H_n^{(\alpha,\beta)}[\nu,\sigma,x]$ of Khandekar [8, $p.158, eq.(2.3)$] and for generalized hypergeometric function ${}_3F_2$ after making some replacements in generalized Rice polynomials. Also some Pochhammer relations [11, $eq.(1.21)$ *to* $eq.(1.23)$] are being used in this paper.
Now we shall discuss some special cases of the implicit functions defined by equation (1.2) subject to the condition (1.3).Using Mathematica 9.0, we can find the roots of resulting polynomial equations in ζ for different values of β in equation (1.2).

Case I:- When $\beta = 0$ in eq. (1.2),then particular value of ζ(satisfying the condition (1.3)) is denoted by

$$\Theta = \frac{t}{1-t}. \tag{1.7}$$

Case II:- When $\beta = 1$ in eq. (1.2),we get

$$t\zeta^2 + (2t-1)\zeta + t = 0,$$

then one of the values of ζ (satisfying the condition (1.3)) is given by

$$\Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}. \tag{1.8}$$

Case III:- When $\beta = -2$ in eq.(1.2),we get

$$\zeta^2 + \zeta - t = 0,$$

then the particular value of ζ(satisfying the condition (1.3)) is given by

$$\Xi = \frac{-1+\sqrt{(1+4t)}}{2}. \tag{1.9}$$

Case IV:- When $\beta = -3$ in eq. (1.2), we get

$$\zeta^3 + 2\zeta^2 + \zeta - t = 0,$$

then one of the roots(satisfying the condition (1.3)) of above equation is given by

$$\Upsilon = \frac{1}{3}\left[-2 + \frac{2^{\frac{1}{3}}}{\left\{2+27t+3\sqrt{3}\sqrt{(4t+27t^2)}\right\}^{\frac{1}{3}}} + \frac{\left\{2+27t+3\sqrt{3}\sqrt{(4t+27t^2)}\right\}^{\frac{1}{3}}}{2^{\frac{1}{3}}}\right]. \tag{1.10}$$

Case V:- When $\beta = -\frac{1}{2}$ in eq. (1.2), we get

$$\zeta^2 - t^2\zeta - t^2 = 0,$$

then one of the roots(satisfying the condition (1.3)) of above equation is given by

$$U = \frac{t}{2}\{t + \sqrt{(t^2+4)}\}. \tag{1.11}$$

Case VI:- When $\beta = \frac{-3}{2}$ in eq. (1.2),we get

$$\zeta^3 + \zeta^2 - t^2 = 0,$$

then one of the roots(satisfying the condition (1.3)) of above equation is given by

$$\Psi = -\frac{1}{3}\left[1 + \frac{(1+\iota\sqrt{3})}{2^{\frac{2}{3}}\left\{-2+27t^2+3\sqrt{3}\sqrt{(-4t^2+27t^4)}\right\}^{\frac{1}{3}}} + \right.$$
$$\left. + \frac{(1-\iota\sqrt{3})\left\{-2+27t^2+3\sqrt{3}\sqrt{(-4t^2+27t^4)}\right\}^{\frac{1}{3}}}{2^{\frac{4}{3}}}\right], \tag{1.12}$$

where $\iota = \sqrt{(-1)}$.
Case VII:- When $\beta = -\frac{1}{3}$ in eq.(1.2),we obtain

$$\zeta^3 - t^3\zeta^2 - 2t^3\zeta - t^3 = 0,$$

then one of the values of ζ (satisfying the condition (1.3)) is denoted by

$$\Pi = \left[\frac{t^3}{3} - \frac{2^{\frac{1}{3}}(-6t^3 - t^6)}{3\{27t^3+18t^6+2t^9+3\sqrt{3}\sqrt{(27t^6+4t^9)}\}^{\frac{1}{3}}} + \right.$$
$$\left. + \frac{\{27t^3+18t^6+2t^9+3\sqrt{3}\sqrt{(27t^6+4t^9)}\}^{\frac{1}{3}}}{3.2^{\frac{1}{3}}}\right]. \tag{1.13}$$

Case VIII:- When $\beta = -\frac{2}{3}$ in eq.(1.2),we obtain

$$\zeta^3 - t^3\zeta - t^3 = 0,$$

then one of the values of ζ (satisfying the condition (1.3)) is denoted by

$$\Phi = \left[\frac{\left(\frac{2}{3}\right)^{\frac{1}{3}} t^3}{\left\{9t^3+\sqrt{3}\sqrt{(27t^6-4t^9)}\right\}^{\frac{1}{3}}} + \frac{\left\{9t^3+\sqrt{3}\sqrt{(27t^6-4t^9)}\right\}^{\frac{1}{3}}}{2^{\frac{1}{3}}3^{\frac{2}{3}}}\right]. \tag{1.14}$$

Srivastava's generalized hypergeometric polynomials:
Srivastava's generalized hypergeometric polynomials $H_n^{(\alpha,\beta)}(x;m)$ $[20, p.360, eq.(7.3.3)]$ are

given by

$$H_n^{(\alpha,\beta)}(x;m) = \binom{\alpha+(\beta+1)n}{n} {}_{p+m}F_{q+m}\left[\begin{array}{c} \Delta(m;-n),(a_p) \quad ; \\ \Delta(m;1+\alpha+\beta n),(b_q); \end{array} x\right]. \tag{1.15}$$

<u>**Pathan's generalized hypergeometric polynomials**</u>:

Pathan's generalized hypergeometric polynomials $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ [11, $p.140, eq.(2.8)$], are given by

$$\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu) = \binom{\alpha+(\beta+1)n}{n} {}_{p+m+1}F_{q+m+1}\left[\begin{array}{c} \Delta(m;-n),1+\frac{\lambda+\mu n}{\mu(\beta+1-m)},(a_p); \\ \Delta(m;1+\alpha+\beta n),\frac{\lambda+\mu n}{\mu(\beta+1-m)},(b_q); \end{array} x\right]. \tag{1.16}$$

Here in equations (1.15) and (1.16), $\Delta(m;\lambda)$ abbreviates the array of m number of parameters given by

$$\frac{\lambda}{m},\frac{\lambda+1}{m},\ldots,\frac{\lambda+m-1}{m}; m\in\mathbb{N}.$$

<u>Some useful Pochhammer's relations</u>

$$\frac{(\lambda+\mu n)\left(\frac{\lambda+\mu n}{\mu(\beta+1-m)}+1\right)_r}{\left(\frac{\lambda+\mu n}{\mu(\beta+1-m)}\right)_r} = \lambda+\mu n+\mu(\beta+1)r-\mu mr \tag{1.17}$$

$$\lambda+\mu(\beta+1)r+\frac{\mu\zeta m(\beta+1)r}{(1-\beta\zeta)} = \lambda\frac{\left(\frac{\lambda(1-\beta\zeta)}{\mu(\beta+1)(1-\beta\zeta+m\zeta)}+1\right)_r}{\left(\frac{\lambda(1-\beta\zeta)}{\mu(\beta+1)(1-\beta\zeta+m\zeta)}\right)_r} \tag{1.18}$$

$$\alpha+m(\beta+1)r = \alpha\frac{\left(\frac{\alpha}{m(\beta+1)}+1\right)_r}{\left(\frac{\alpha}{m(\beta+1)}\right)_r} \tag{1.19}$$

where

$$r = 0,1,2,3,\ldots$$

Srivastava's generalized hypergeometric polynomials and Pathan's generalized hypergeometric polynomials are the generalisations of generalised Laguerre polynomials, generalised Rice polynomials, Khandekar polynomials, Jacobi polynomials and other such polynomials.
Our present investigation is motivated by the work on generating relations by Brown [1, 2], Calvez and Génin [3],Carlitz [4], Joshi and Prajapat [6],Karande and Thakare [7], Srivastava [16, 17, 18, 19], Srivastava and Singhal [21], Whittaker and Watson [22] and work on Jacobi polynomials by Milch [9].
The rest of the paper is structured as follows:The two main generating relations have been derived in section 2. Variety of applications of first generating relation have been deduced in section 3, also applications of second main result have been deduced in section 4. Further generalizations of our main generating relations (2.1) and (2.4) have been given in section 5. Concluding remarks of the research paper are reflected in section 6 followed by references.

2 Theorems on Generating Relations

First Theorem on Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}H_n^{(\alpha,\beta)}(x;m)\,t^n = (1+\zeta)^\alpha\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c} \frac{\alpha}{m\,(\beta+1)},1+\frac{\lambda\,(1-\beta\,\zeta)}{m\,\mu\,(1+\zeta)},(a_p); \\ 1+\frac{\alpha}{m\,(\beta+1)},\frac{\lambda\,(1-\beta\,\zeta)}{m\,\mu\,(1+\zeta)},(b_q); \end{array} x(-\zeta)^m\right]+\right.$$

$$+\frac{\mu\zeta}{(1-\beta\zeta)}\,{}_{p+1}F_{q+1}\left[\begin{array}{l}\frac{\alpha}{m\,(\beta+1)},(a_p)\,;\\ \frac{\alpha}{m\,(\beta+1)}+1,(b_q)\,;\end{array}\;x(-\zeta)^m\right]\Bigg\},\tag{2.1}$$

where

$$\zeta=t(1+\zeta)^{1+\beta};\zeta(0)=0.$$

provided that involved series on both sides are absolutely convergent.

Here Srivastava's generalized hypergeometric polynomials $H_n^{(\alpha,\beta)}(x;m)$ are given by eq. (1.15).

Independent Demonstration:
Using the definition (1.15) of $H_n^{(\alpha,\beta)}(x;m)$ and then the power series form of ${}_{p+m}F_{q+m}[x]$ in left hand side of equation (2.1), using Gauss multiplication theorem [11, (1.9)] in the equation so obtained, then, applying summation identity [11, (1.10)] and then simplifying further, and then using first modified Gould's identity (1.5),we get (2.2)

$$\Omega=(1+\zeta)^\alpha\sum_{r=0}^{\infty}\frac{(\alpha)_{m(\beta+1)r}\;[(a_p)]_r\;x^r\;(-t)^{mr}}{\alpha\;(\alpha+1)_{m(\beta+1)r}\;[(b_q)]_r\;r!}(1+\zeta)^{m(\beta+1)r}\times$$

$$\times\left[\frac{\mu\zeta\alpha}{(1-\beta\zeta)}+\left(\lambda+\mu\,mr+\frac{\mu\zeta(\beta+1)mr}{(1-\beta\zeta)}\right)\right],\tag{2.2}$$

Simplifying it further, we get Eq.(2.3)

$$\Omega=(1+\zeta)^\alpha\sum_{r=0}^{\infty}\frac{(\frac{\alpha}{m(\beta+1)})_r\;[(a_p)]_r\;x^r\;(-\zeta)^{mr}}{\alpha\;(\frac{\alpha}{m(\beta+1)}+1)_r\;[(b_q)]_r\;r!}\times$$

$$\times\left[\lambda\frac{\left(\frac{\lambda(1-\beta\zeta)}{\mu\,m(1+\zeta)}+1\right)_r}{\left(\frac{\lambda(1-\beta\zeta)}{\mu\,m(1+\zeta)}\right)_r}+\frac{\mu\zeta\alpha}{(1-\beta\zeta)}\right],\tag{2.3}$$

After solving it further,we get the result (2.1) in the form of sum of two generalized hypergeometric functions of one variable.

Second Theorem on Generating Relation:
If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)\;t^n=(1+\zeta)^\alpha\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{l}\frac{\alpha}{m(\beta+1)},1+\frac{\lambda(1-\beta\zeta)}{\mu(\beta+1)(1-\beta\zeta+m\,\zeta)},(a_p);\\ 1+\frac{\alpha}{m(\beta+1)},\frac{\lambda(1-\beta\zeta)}{\mu(\beta+1)(1-\beta\zeta+m\,\zeta)},(b_q);\end{array}\;x(-\zeta)^m\right]+\right.$$

$$+\frac{\mu\zeta}{(1-\beta\zeta)}\,{}_{p+1}F_{q+1}\left[\begin{array}{l}\frac{\alpha}{m(\beta+1)},(a_p);\\ \frac{\alpha}{m(\beta+1)}+1,(b_q);\end{array}\;x(-\zeta)^m\right]\Bigg\},\tag{2.4}$$

where

$$\zeta=t(1+\zeta)^{1+\beta};\zeta(0)=0.$$

provided that involved series on both sides are absolutely convergent.
Here Pathan's generalized Hypergeometric polynomials, $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$,are given by eq.(1.16).

Independent Demonstration:
Using the definition (1.16) of $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ and then the power series form of ${}_{p+m+1}F_{q+m+1}[x]$ in left hand side of equation (2.4),using Gauss multiplication theorem [11, (1.9)] and result [11, (1.21)], now applying summation identity [11, (1.10)] in the equation so obtained, then sim-

plifying further, then using second modified Gould's identity (1.6),we get

$$\Omega^* = (1+\zeta)^{\alpha} \sum_{r=0}^{\infty} \frac{[(a_p)]_r\, x^r\, (-t)^{mr}\, (1+\zeta)^{m(\beta+1)r}}{[(b_q)]_r \{\alpha + m\,(\beta+1)r\}\, r!} \times$$
$$\times \left[\frac{\mu\zeta\alpha}{(1-\beta\zeta)} + \left\{ \lambda + \mu(\beta+1)r + \frac{\mu\,\zeta(\beta+1)mr}{(1-\beta\zeta)} \right\} \right]. \tag{2.5}$$

Now using equations [11, (1.22)and(1.23)] in above equation and summing it up into hypergeometric form further, we get the desired result (2.4).

3 Applications of First Theorem (2.1)

Using the definition (1.15) of Srivastava polynomials in hypergeometric form and
1). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (2.1) and after simplifying, we get

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} {}_{p+m}F_{q+m} \left[\begin{matrix} \Delta(m;-n), (a_p) \quad ; \\ \Delta(m; 1+\alpha+n\beta), (b_q); \end{matrix}\; x \right] t^n$$
$$= (1+\zeta)^{\alpha} \left\{ {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m\,(\beta+1)}, 1 + \frac{\alpha\,(1-\beta\,\zeta)}{m\,(\beta+1)\,(1+\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m\,(\beta+1)}, \frac{\alpha\,(1-\beta\,\zeta)}{m\,(\beta+1)\,(1+\zeta)}, (b_q); \end{matrix}\; x(-\zeta)^m \right] + \right.$$
$$\left. + \frac{(\beta+1)\zeta}{(1-\beta\zeta)} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m\,(\beta+1)}, (a_p)\,; \\ \frac{\alpha}{m\,(\beta+1)} + 1, (b_q)\,; \end{matrix}\; x(-\zeta)^m \right] \right\}. \tag{3.1}$$

2). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ and $m = 1$ in equation (2.1) and using the definition of $H_n^{(\alpha,\beta)}(x;1)$ and after simplification, we get

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n\,, (a_p) \quad ; \\ 1+\alpha+n\beta, (b_q); \end{matrix}\; x \right] t^n$$
$$= (1+\zeta)^{\alpha} \left\{ {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, 1 + \frac{\alpha\,(1-\beta\,\zeta)}{(\beta+1)\,(1+\zeta)}, (a_p); \\ 1 + \frac{\alpha}{(\beta+1)}, \frac{\alpha\,(1-\beta\,\zeta)}{(\beta+1)\,(1+\zeta)}, (b_q); \end{matrix}\; x(-\zeta) \right] + \right.$$
$$\left. + \frac{(\beta+1)\zeta}{(1-\beta\zeta)} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, (a_p) \quad ; \\ \frac{\alpha}{(\beta+1)} + 1, (b_q)\,; \end{matrix}\; x(-\zeta) \right] \right\}. \tag{3.2}$$

3). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}$ and $m = 1$ in equation (2.1),using the definition(1.15) of $H_n^{(\alpha,-\frac{1}{2})}(x;1) = f_n^{(\alpha)}(x)$ and replacing ζ by U, we get

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \binom{\alpha+\frac{n}{2}}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, (a_p) \quad ; \\ 1+\alpha-\frac{n}{2}, (b_q) \quad ; \end{matrix}\; x \right] t^n$$
$$= [1+U]^{\alpha} \left\{ {}_{p+2}F_{q+2} \left[\begin{matrix} 2\alpha, 1 + \frac{\alpha\,(2+U)}{(U+1)}, (a_p); \\ 1+2\alpha, \frac{\alpha\,(2+U)}{(1+U)}, (b_q); \end{matrix}\; x(-U) \right] + \right.$$
$$\left. + \frac{U}{(2+U)} {}_{p+1}F_{q+1} \left[\begin{matrix} 2\alpha, (a_p) \quad ; \\ 2\alpha+1, (b_q)\,; \end{matrix}\; x(-U) \right] \right\}, \tag{3.3}$$

where

$$U = \frac{t}{2}\{t + \sqrt{(t^2+4)}\}.$$

4). Putting $\beta=0$ and $\zeta=\Theta=\frac{t}{1-t}$ from equation (1.7) in equation (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+n)}\binom{\alpha+n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad;\\ \Delta(m;1+\alpha),(b_q);\end{array}x\right]t^n=(1-t)^{-\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m},1+\frac{\lambda(1-t)}{m\mu},(a_p);\\ 1+\frac{\alpha}{m},\frac{\lambda(1-t)}{m\mu},(b_q);\end{array}x\left(\frac{t}{t-1}\right)^m\right]+\right.$$

$$\left.+\frac{\mu t}{(1-t)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{\alpha}{m},(a_p)\ ;\\ 1+\frac{\alpha}{m},(b_q)\,;\end{array}x\left(\frac{t}{t-1}\right)^m\right]\right\}. \tag{3.4}$$

5). Putting $\beta=1$ and $\zeta=\Lambda=\frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (1.8) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+2n)}\binom{\alpha+2n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad;\\ \Delta(m;1+\alpha+n),(b_q);\end{array}x\right]t^n=(1+\Lambda)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{2m},1+\frac{\lambda(1-\Lambda)}{m\mu(1+\Lambda)},(a_p);\\ 1+\frac{\alpha}{2m},\frac{\lambda(1-\Lambda)}{m\mu(1+\Lambda)},(b_q);\end{array}x(-\Lambda)^m\right]+\right.$$

$$\left.+\frac{\mu\Lambda}{(1-\Lambda)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{\alpha}{2m},(a_p)\,;\\ 1+\frac{\alpha}{2m},(b_q)\,;\end{array}x(-\Lambda)^m\right]\right\}. \tag{3.5}$$

6). Putting $\beta=-2$ and $\zeta=\Xi=\frac{-1+\sqrt{(1+4t)}}{2}$ from equation (1.9) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-n)}\binom{\alpha-n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad;\\ \Delta(m;1+\alpha-2n),(b_q);\end{array}x\right]t^n=(1+\Xi)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-\alpha}{m},1+\frac{\lambda(1+2\Xi)}{m\mu(1+\Xi)},(a_p);\\ 1-\frac{\alpha}{m},\frac{\lambda(1+2\Xi)}{m\mu(1+\Xi)},(b_q);\end{array}x(-\Xi)^m\right]+\frac{\mu\,\Xi}{(1+2\,\Xi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}-\frac{\alpha}{m},(a_p)\,;\\ 1-\frac{\alpha}{m},(b_q)\,;\end{array}x(-\Xi)^m\right]\right\}. \tag{3.6}$$

7). Putting $\beta=-3$ and $\zeta=\Upsilon$ from equation (1.10) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-2n)}\binom{\alpha-2n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad;\\ \Delta(m;1+\alpha-3n),(b_q);\end{array}x\right]t^n=(1+\Upsilon)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-\alpha}{2m},1+\frac{\lambda(1+3\Upsilon)}{m\mu(1+\Upsilon)},(a_p);\\ 1-\frac{\alpha}{2m},\frac{\lambda(1+3\Upsilon)}{m\mu(1+\Upsilon)},(b_q);\end{array}x(-\Upsilon)^m\right]+\frac{\mu\,\Upsilon}{(1+3\,\Upsilon)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}-\frac{\alpha}{2m},(a_p)\,;\\ 1-\frac{\alpha}{2m},(b_q)\,;\end{array}x(-\Upsilon)^m\right]\right\}. \tag{3.7}$$

8). Putting $\beta=\frac{-1}{2}$ and $\zeta=U$ from equation (1.11) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{2}n)}\binom{\alpha+\frac{1}{2}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad;\\ \Delta(m;1+\alpha-\frac{1}{2}n),(b_q);\end{array}x\right]t^n=(1+U)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{2\alpha}{m},1+\frac{\lambda(1+\frac{U}{2})}{m\mu(1+U)},(a_p);\\ 1+\frac{2\alpha}{m},\frac{\lambda(1+\frac{U}{2})}{m\mu(1+U)},(b_q);\end{array}x(-U)^m\right]+\frac{\mu\,U}{(1+\frac{1}{2}\,U)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{2\alpha}{m},(a_p)\quad;\\ \frac{2\alpha}{m}+1,(b_q)\,;\end{array}x(-U)^m\right]\right\}. \tag{3.8}$$

9). Putting $\beta=\frac{-3}{2}$ and $\zeta=\Psi$ from equation (1.12) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-\frac{1}{2}n)}\binom{\alpha-\frac{1}{2}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad;\\ \Delta(m;1+\alpha-\frac{3}{2}n),(b_q);\end{array}x\right]t^n=(1+\Psi)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-2\alpha}{m},1+\frac{\lambda\,(1+\frac{3}{2}\Psi)}{m\,\mu\,(1+\Psi)},(a_p);\\ 1-\frac{2\alpha}{m},\frac{\lambda\,(1+\frac{3}{2}\Psi)}{m\,\mu\,(1+\Psi)},(b_q);\end{array}x(-\Psi)^m\right]+\frac{\mu\,\Psi}{(1+\frac{3}{2}\,\Psi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}-\frac{2\alpha}{m},(a_p)\,;\\ 1-\frac{2\alpha}{m},(b_q)\,;\end{array}x(-\Psi)^m\right]\right\}. \tag{3.9}$$

10). Putting $\beta=\frac{-1}{3}$ and $\zeta=\Pi$ from equation (1.13) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{2}{3}n)}\binom{\alpha+\frac{2}{3}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\qquad;\\ \Delta(m;1+\alpha-\frac{1}{3}n),(b_q);\end{array}x\right]t^n=(1+\Pi)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{3\alpha}{2m},1+\frac{\lambda\,(1+\frac{\Pi}{3})}{m\,\mu\,(1+\Pi)},(a_p);\\ 1+\frac{3\alpha}{2\,m},\frac{\lambda\,(1+\frac{\Pi}{3})}{m\,\mu\,(1+\Pi)},(b_q);\end{array}x(-\Pi)^m\right]+\frac{\mu\,\Pi}{(1+\frac{1}{3}\,\Pi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{3\alpha}{2\,m},(a_p)\,;\\ 1+\frac{3\alpha}{2\,m},(b_q)\,;\end{array}x(-\Pi)^m\right]\right\} \tag{3.10}$$

. 11). Putting $\beta=\frac{-2}{3}$ and $\zeta=\Phi$ from equation (1.14) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{3}n)}\binom{\alpha+\frac{1}{3}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\qquad;\\ \Delta(m;1+\alpha-\frac{2}{3}n),(b_q);\end{array}x\right]t^n=(1+\Phi)^{\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{3\alpha}{m},1+\frac{\lambda\,(1+\frac{2\Phi}{3})}{m\,\mu\,(1+\Phi)},(a_p);\\ 1+\frac{3\alpha}{m},\frac{\lambda\,(1+\frac{2\Phi}{3})}{m\,\mu\,(1+\Phi)},(b_q);\end{array}x(-\Phi)^m\right]+\frac{\mu\,\Phi}{(1+\frac{2}{3}\,\Phi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{3\alpha}{m},(a_p)\quad;\\ 1+\frac{3\alpha}{m},(b_q)\,;\end{array}x(-\Phi)^m\right]\right\}. \tag{3.11}$$

4 Applications of Second Theorem (2.4)

Using the definition (1.16) of Pathan's polynomials in hypergeometric polynomial form and

1). Putting $\lambda=1,\mu=\frac{\beta+1}{\alpha}$ in equation (2.4) and after simplifying, we get

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),\;1+\frac{\{\alpha+(\beta+1)\,n\}}{(\beta+1)(\beta+1-m)},\;(a_p);\\ \Delta(m;1+\alpha+\beta\,n),\;\frac{\{\alpha+(\beta+1)\,n\}}{(\beta+1)(\beta+1-m)},\;(b_q);\end{array}x\right]t^n=(1+\zeta)^{\alpha}\times$$

$$\times\left\{{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m(\beta+1)},1+\frac{\alpha(1-\beta\,\zeta)}{(\beta+1)^2\,(1-\beta\,\zeta+m\zeta)},(a_p);\\ 1+\frac{\alpha}{m(\beta+1)},\frac{\alpha(1-\beta\,\zeta)}{(\beta+1)^2(1-\beta\,\zeta+m\zeta)},(b_q);\end{array}x(-\zeta)^m\right]+\right.$$

$$\left.+\frac{(\beta+1)\zeta}{(1-\beta\zeta)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{\alpha}{m(\beta+1)},(a_p);\\ 1+\frac{\alpha}{m(\beta+1)},(b_q);\end{array}x(-\zeta)^m\right]\right\}, \tag{4.1}$$

where ζ is given by equations (1.2) and (1.3).

2). Putting $\lambda=1,\mu=\frac{\beta+1}{\alpha}$, $m=1$ in equation (2.4) and using the definition (1.16) of $\mathscr{B}_n^{(\alpha,\beta)}(x;1,1,\frac{\beta+1}{\alpha})$, we get

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}{}_{p+2}F_{q+2}\left[\begin{array}{c}-n,\;1+\frac{\{\alpha+(\beta+1)\,n\}}{(\beta+1)\beta},\;(a_p);\\ 1+\alpha+\beta\,n,\;\frac{\{\alpha+(\beta+1)\,n\}}{(\beta+1)\beta},\;(b_q);\end{array}x\right]t^n=(1+\zeta)^{\alpha}\times$$

$$\times\left\{{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{(\beta+1)},1+\frac{\alpha(1-\beta\,\zeta)}{(\beta+1)^2\,(1-\beta\,\zeta+\zeta)},(a_p);\\ 1+\frac{\alpha}{(\beta+1)},\frac{\alpha(1-\beta\,\zeta)}{(\beta+1)^2(1-\beta\,\zeta+\zeta)},(b_q);\end{array}x(-\zeta)\right]+\right.$$

$$\left.+\frac{(\beta+1)\zeta}{(1-\beta\zeta)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{\alpha}{(\beta+1)},(a_p);\\ 1+\frac{\alpha}{(\beta+1)},(b_q);\end{array}x(-\zeta)\right]\right\}, \tag{4.2}$$

where ζ is given by equations (1.2) and (1.3).
3). Putting $\lambda=1,\mu=\frac{1}{2\alpha},\beta=\frac{-1}{2}$, $m=1$ in equation (2.4),using the definition (1.16) of

$\mathscr{B}_n^{(\alpha,-\frac{1}{2})}(x;1,1,\frac{1}{2\alpha})$ and replacing ζ by U from equation (1.11), we get

$$\sum_{n=0}^{\infty}\binom{\alpha+\frac{n}{2}}{n}{}_{p+2}F_{q+2}\left[\begin{array}{c}-n,1-4\alpha-2n,\ \ (a_p);\\ 1+\alpha-\frac{n}{2},-4\alpha-2n,\ (b_q)\ \ ;\end{array} x\right]t^n=(1+U)^{\alpha}\times$$

$$\times\left\{{}_{p+2}F_{q+2}\left[\begin{array}{c}2\alpha,1+\frac{4\alpha(2+U)}{(2+3U)},(a_p);\\ 1+2\alpha,\frac{4\alpha(2+U)}{(2+3U)},(b_q);\end{array} x(-U)\right]+\frac{U}{(2+U)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}2\alpha,(a_p);\\ 1+2\alpha,(b_q);\end{array} x(-U)\right]\right\}. \tag{4.3}$$

4). Putting $\beta=0$ and $\zeta=\Theta=\frac{t}{1-t}$ from equation (1.7) in equation (2.4), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+n)}\binom{\alpha+n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{\lambda+\mu n}{\mu(1-m)},(a_p);\\ \Delta(m;1+\alpha),\ \frac{\lambda+\mu n}{\mu(1-m)},\ \ (b_q)\ ;\end{array} x\right]t^n=(1-t)^{-\alpha}\times$$

$$\times\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m},1+\frac{\lambda\,(1-t)}{\mu\,(1-t+mt)},(a_p);\\ 1+\frac{\alpha}{m},\frac{\lambda\,(1-t)}{\mu\,(1-t+mt)},(b_q);\end{array} x\left(\frac{t}{t-1}\right)^m\right]+\right.$$

$$\left.+\mu\left(\frac{t}{1-t}\right){}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{\alpha}{m},(a_p)\,;\\ \frac{\alpha}{m}+1(b_q)\,;\end{array} x\left(\frac{t}{t-1}\right)^m\right]\right\}. \tag{4.4}$$

5). Putting $\beta=1$ and $\zeta=\Lambda=\frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (1.8) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+2n)}\binom{\alpha+2n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{\lambda+\mu n}{\mu(2-m)},(a_p);\\ \Delta(m;1+\alpha+\ n),\frac{\lambda+\mu n}{\mu(2-m)},(b_q);\end{array} x\right]t^n$$

$$=(1+\Lambda)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{2m},1+\frac{\lambda\,(1-\Lambda)}{2\,\mu\,(1-\Lambda+m\Lambda)},(a_p);\\ 1+\frac{\alpha}{2m},\frac{\lambda\,(1-\Lambda)}{2\,\mu\,(1-\Lambda+m\Lambda)},(b_q);\end{array} x(-\Lambda)^m\right]+\right.$$

$$\left.+\frac{\mu\Lambda}{(1-\Lambda)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{\alpha}{2m},(a_p)\,;\\ 1+\frac{\alpha}{2m},(b_q)\,;\end{array} x(-\Lambda)^m\right]\right\}. \tag{4.5}$$

6). Putting $\beta=-2$ and $\zeta=\Xi=\frac{-1+\sqrt{(1+4t)}}{2}$ from equation (1.9) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-n)}\binom{\alpha-n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1-\frac{(\lambda+\mu\,n)}{\mu(1+m)},(a_p)\ \ ;\\ \Delta(m;1+\alpha-2\,n),-\frac{(\lambda+\mu\,n)}{\mu(1+m)},(b_q);\end{array} x\right]t^n$$

$$=(1+\Xi)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-\alpha}{m},1-\frac{\lambda\,(1+2\Xi)}{\mu\,(1+2\Xi+m\Xi)},(a_p);\\ 1-\frac{\alpha}{m},\frac{-\lambda\,(1+2\Xi)}{\mu\,(1+2\Xi+m\Xi)},(b_q);\end{array} x(-\Xi)^m\right]+\right.$$

$$\left.+\frac{\mu\,\Xi}{(1+2\,\Xi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}-\frac{\alpha}{m},(a_p)\,;\\ 1-\frac{\alpha}{m},(b_q)\,;\end{array} x(-\Xi)^m\right]\right\}. \tag{4.6}$$

7). Putting $\beta=-3$ and $\zeta=\Upsilon$ from equation (1.10) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-2n)}\binom{\alpha-2n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),\ 1-\frac{(\lambda+\mu\,n)}{\mu(2+m)},\ \ (a_p);\\ \Delta(m;1+\alpha-3\,n),-\frac{(\lambda+\mu\,n)}{\mu(2+m)},(b_q);\end{array} x\right]t^n$$

$$=(1+\Upsilon)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-\alpha}{2m},1-\frac{\lambda\,(1+3\Upsilon)}{2\,\mu\,(1+3\Upsilon+m\Upsilon)},(a_p);\\ 1-\frac{\alpha}{2\,m},\frac{-\lambda\,(1+3\Upsilon)}{2\,\mu\,(1+3\Upsilon+m\Upsilon)},(b_q);\end{array} x(-\Upsilon)^m\right]+\right.$$

$$\left.+\frac{\mu\,\Upsilon}{(1+3\,\Upsilon)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}-\frac{\alpha}{2\,m},(a_p)\,;\\ 1-\frac{\alpha}{2\,m},(b_q)\,;\end{array} x(-\Upsilon)^m\right]\right\}. \tag{4.7}$$

8). Putting $\beta=\frac{-1}{2}$ and $\zeta=U$ from equation (1.11) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{2}n)}\binom{\alpha+\frac{1}{2}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}-m)},(a_p);\\ \Delta(m;1+\alpha-\frac{1}{2}n),\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}-m)},(b_q);\end{array}x\right]t^n$$

$$=(1+U)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{2\alpha}{m},1+\frac{2\,\lambda\,(2+U)}{\mu\,(2+U+2mU)},(a_p);\\ 1+\frac{2\alpha}{m},\frac{2\lambda\,(2+U)}{\mu\,(2+U+2mU)},(b_q);\end{array}x(-U)^m\right]+\right.$$

$$\left.+\frac{\mu\,U}{(1+\frac{1}{2}\,U)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{2\alpha}{m},(a_p);\\ 1+\frac{2\alpha}{m},(b_q);\end{array}x(-U)^m\right]\right\}. \tag{4.8}$$

9). Putting $\beta=\frac{-3}{2}$ and $\zeta=\Psi$ from equation (1.12) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-\frac{1}{2}n)}\binom{\alpha-\frac{1}{2}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1-\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}+m)},(a_p);\\ \Delta(m;1+\alpha-\frac{3}{2}\,n),-\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}+m)},(b_q);\end{array}x\right]t^n$$

$$=(1+\Psi)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{cc}\frac{-2\alpha}{m},1-\frac{2\lambda\,(1+\frac{3}{2}\Psi)}{\mu\,(1+\frac{3}{2}\Psi+m\Psi)}, & (a_p);\\ 1-\frac{2\alpha}{m},\frac{-2\lambda\,(1+\frac{3}{2}\Psi)}{\mu\,(1+\frac{3}{2}\Psi+m\Psi)}, & (b_q);\end{array}x(-\Psi)^m\right]+\right.$$

$$\left.+\frac{\mu\,\Psi}{(1+\frac{3}{2}\,\Psi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{-2\alpha}{m},(a_p);\\ 1-\frac{2\alpha}{m},(b_q);\end{array}x(-\Psi)^m\right]\right\}. \tag{4.9}$$

10). Putting $\beta=\frac{-1}{3}$ and $\zeta=\Pi$ from equation (1.13) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{2}{3}n)}\binom{\alpha+\frac{2}{3}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{(\lambda+\mu\,n)}{\mu(\frac{2}{3}-m)},(a_p);\\ \Delta(m;1+\alpha-\frac{1}{3}\,n),\frac{(\lambda+\mu\,n)}{\mu(\frac{2}{3}-m)},(b_q);\end{array}x\right]t^n$$

$$=(1+\Pi)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{3\alpha}{2m},1+\frac{3\,\lambda\,(1+\frac{\Pi}{3})}{2\,\mu\,(1+\frac{\Pi}{3}+m\Pi)},(a_p);\\ 1+\frac{3\alpha}{2\,m},\frac{3\lambda\,(1+\frac{\Pi}{3})}{2\,\mu\,(1+\frac{\Pi}{3}+m\Pi)},(b_q);\end{array}x(-\Pi)^m\right]+\right.$$

$$\left.+\frac{\mu\,\Pi}{(1+\frac{1}{3}\,\Pi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{3\alpha}{2\,m},(a_p);\\ 1+\frac{3\alpha}{2\,m},(b_q);\end{array}x(-\Pi)^m\right]\right\}. \tag{4.10}$$

11). Putting $\beta=\frac{-2}{3}$ and $\zeta=\Phi$ from equation (1.14) in equation (2.4),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{3}n)}\binom{\alpha+\frac{1}{3}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{3}-m)},(a_p);\\ \Delta(m;1+\alpha-\frac{2}{3}\,n),\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{3}-m)},(b_q);\end{array}x\right]t^n$$

$$=(1+\Phi)^{\alpha}\left\{\frac{\lambda}{\alpha}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{3\alpha}{m},1+\frac{3\,\lambda\,(1+\frac{2}{3}\Phi)}{\mu\,(1+\frac{2}{3}\Phi+m\Phi)},(a_p);\\ 1+\frac{3\alpha}{m},\frac{3\lambda\,(1+\frac{2}{3}\Phi)}{\mu\,(1+\frac{2}{3}\Phi+m\Phi)},(b_q);\end{array}x(-\Phi)^m\right]+\right.$$

$$\left.+\frac{\mu\,\Phi}{(1+\frac{2}{3}\,\Phi)}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\frac{3\alpha}{m},(a_p);\\ 1+\frac{3\alpha}{m},(b_q);\end{array}x(-\Phi)^m\right]\right\}. \tag{4.11}$$

5 Further Generalizations of Generating Relations (2.1) and (2.4)

Generalization of (2.1):

Let

$$S_n^{(\alpha,\beta)}(x;m) = \sum_{r=0}^{\left[\frac{n}{m}\right]} \binom{\alpha+(\beta+1)n}{n-mr} \gamma_r\, x^r, \tag{5.1}$$

where α,β are complex parameters independent of 'n'; m is an arbitrary positive integer and $\{\gamma_r\}$ is a bounded sequence of arbitrary real and complex numbers such that $\gamma_r \neq 0$.Then

$$\sum_{n=0}^{\infty} \frac{\{\lambda+\mu\, n\}}{\{\alpha+(\beta+1)n\}} S_n^{(\alpha,\beta)}(x;m)t^n$$

$$= (1+\zeta)^{\alpha} \left\{ \frac{\mu\,\zeta\,\alpha}{(1-\beta\,\zeta)} \sum_{n=0}^{\infty} \frac{1}{\{\alpha+(\beta+1)m\,n\}} \gamma_n\, x^n\, \zeta^{m\,n} + \sum_{n=0}^{\infty} \left(\lambda+\mu\, m\, n + \frac{\mu\,\zeta\,(\beta+1)mn}{(1-\beta\,\zeta)} \right) \times \right.$$

$$\left. \times \frac{1}{\{\alpha+(\beta+1)m\,n\}} \gamma_n\, x^n\, \zeta^{m\,n} \right\}, \tag{5.2}$$

where ζ is given by

$$\zeta = t(1+\zeta)^{(\beta+1)};\ \zeta(0) = 0. \tag{5.3}$$

provided that each of the series involved is absolutely convergent.

Independent Demonstration:

Using the definition (5.1) of $S_n^{(\alpha,\beta)}(x;m)$ in left hand side of equation (5.2),we get

$$\Omega^{**} = \sum_{n=0}^{\infty} \frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}} S_n^{(\alpha,\beta)}(x;m)\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{(\lambda+\mu n)\, \Gamma\{\alpha+(\beta+1)n+1\}}{\{\alpha+(\beta+1)n\}\, \Gamma(n-mr+1)\, \Gamma\{\alpha+\beta\, n+mr+1\}} \gamma_r\, x^r\, t^n, \tag{5.4}$$

Applying summation identity [11, (1.10)] and then simplifying further, now using first modified Gould's identity (1.5) with condition (5.3), and then changing the summation index from r to n and after solving it further,we get the general result (5.2) corresponding to our first generating relation (2.1) subject to the conditions (5.3).

Generalization of (2.4):

Let

$$T_n^{(\alpha,\beta)}(x;m,\lambda,\mu) = \frac{1}{(\lambda+\mu\, n)} \sum_{r=0}^{\left[\frac{n}{m}\right]} \left\{ \binom{\alpha+(\beta+1)n}{n-mr} [\lambda+\mu\, n+\mu(\beta+1-m)r] \right\} \gamma_r\, x^r, \tag{5.5}$$

where α, β,λ,μ are complex parameters independent of 'n'; m is an arbitrary positive integer and $\{\gamma_r\}$ is a bounded sequence of arbitrary real and complex numbers such that $\gamma_r \neq 0$.Then

$$\sum_{n=0}^{\infty} \frac{(\lambda+\mu\, n)}{\{\alpha+(\beta+1)n\}} T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)t^n = (1+\zeta)^{\alpha} \times$$

$$\times \left\{ \frac{\mu\,\zeta\,\alpha}{(1-\beta\,\zeta)} \sum_{n=0}^{\infty} \frac{1}{\{\alpha+(\beta+1)m\,n\}} \gamma_n\, x^n\, \zeta^{m\,n} + \right.$$

$$\left. + \sum_{n=0}^{\infty} \left(\lambda+\mu\,(\beta+1)\,n + \frac{\mu\,\zeta\,(\beta+1)m\,n}{(1-\beta\,\zeta)} \right) \frac{1}{\{\alpha+(\beta+1)m\,n\}} \gamma_n\, x^n\, \zeta^{m\,n} \right\}, \tag{5.6}$$

where ζ is given by

$$\zeta = t(1+\zeta)^{(\beta+1)}; \zeta(0) = 0.$$

provided that each of the series involved is absolutely convergent.

Independent Demonstration:

Using the definition (5.5) of $T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ in left hand side of equation (5.6),we get

$$\Omega^{***} = \sum_{n=0}^{\infty} \frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}} T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{m}]} \left\{ \binom{\alpha+(\beta+1)n}{n-mr} \frac{\{\lambda+\mu n+\mu(\beta+1)r-\mu\, mr\}}{\{\alpha+(\beta+1)n\}} \right\} \gamma_r\, x^r\, t^n, \tag{5.7}$$

Applying summation identity [11, (1.10)] and then simplifying further, now using second modified Gould's identity (1.6) with conditions (5.3), then changing the summation index from r to n and after solving it further,we get the general result (5.6) corresponding to our second generating relation (2.4) subject to the conditions (5.3).
In the definitions of generalized polynomials given by $S_n^{(\alpha,\beta)}(x;m)$ and $T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$, putting

$$\gamma_r = \frac{(-1)^{m\,r}(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_q)_r\, r!},$$

we obtain Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha,\beta)}(x;m)$ and Pathan's generalized hypergeometric polynomials of one variable $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ respectively.

6 Conclusion

On the basis of the present paper,we observe that similar types of generating relations can also be obtained in an analogous manner by using another new identity, different from Gould's identity, related hypergeometric polynomials and series arrangement techniques. Our results are stated in preceding sections as generating relations (2.1), (2.4); generalizations (5.2) and (5.4); their applications from (3.1) to (3.11) and (4.1) to (4.11).
We remark further that many of the results, which have been derived in this paper, are quite significant and are expected to be beneficial for the researchers in the fields of mathematical physics, applied mathematics, statistics and other branches of science and engineering.
We also hope for the future applications of these two generating relations and their further generalizations.

Conflicts of Interest

The authors declare no conflict of interest.

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