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On the solutions of certain exponential Diophantine equations

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Abstract We study nonnegative integer solutions of the Diophantine equation $A^x + B^y = z^2$, where $A \equiv 3 \pmod{20}$ with the condition that $(A+1)$ is a perfect square and $B \equiv 5 \pmod{20}$ are positive integers and prove that the equation has a unique solution $(x, y, z) = (1, 0, \sqrt{A+1})$. Then we present a full study of positive integer solutions (p, x, y, z) of the equation $p^x + p^y = 2z^2$, where $p = 2$ and $p \equiv 3, 5 \pmod 8$ are prime numbers. Finally, we present a detailed analysis of positive integer solutions (p, x, y, z) of $p^x - p^y = 2z^2$, when $p \geq 2$ is a prime. At the end, we provide few tables consisting of computational evidences corroborating our results.

1 Introduction

The Diophantine equation of the form $a^x \pm b^y = z^2$, where a and b are positive integers, has been extensively investigated in mathematical literature. A particularly famous result related to such equations is Catalan's Conjecture, which addresses a specific case of this form. The conjecture states that:

Conjucture1.1 [Catalan's Conjecture]. Let a, b, x, and y be integers such that $\min\{a, b, x, y\} > 1$. Then

$$a^x - b^y = 1$$

has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

In a seminal work, Mihăilescu [7] proved this conjecture. Burshtein [1] investigated the equations $p^x + q^y = z^2$ and $p^x - p^y = z^2$, where $p \geq 2$ is a prime, and x, y, and z are positive integers. He showed that these equations have infinitely many solutions. More recently, Srimud and Tadee [9] studied the equation $3^x + b^y = z^2$ under the condition $b \equiv 5 \pmod{20}$, and established that it possesses a unique solution $(x, y, z) = (1, 0, 2)$. The first author, along with Hoque has previously studied related work, and we direct readers to the following references [2, 3, 4, 6, 8, 11] for further explorations in exponential Diophantine equations.

One of the main results of this paper is a generalization of the result in [9]. We consider the equation

$$A^x + B^y = z^2, \tag{1.1}$$

where A, B are positive integers such that $A \equiv 3 \pmod{20}$, and $B \equiv 5 \pmod{20}$. We prove:

Theorem 1.1. *Let A, B be positive integers such that $A \equiv 3 \pmod{20}$ and $B \equiv 5 \pmod{20}$. Then all the solutions of (1.1) are of the form $(x, y, z) = (1, 0, \sqrt{A+1})$, where $(A+1)$ is a perfect square.*

remark 1.1 The restrictions on A and B is necessary for our arguments to go through. It would be of some interest if these restrictions can be slackened. We also examine the positive integer solutions of a variant of the exponential Diophantine equation studied in [1] and prove:

Theorem 1.2. *Let p be an odd prime of the form $p \equiv 3, 5 \pmod 8$ and (x, y, z) are positive integers. Then all the positive integer solutions of*

$$p^x + p^y = 2z^2 \tag{1.2}$$

are of the form $(x, y, z) = (2m, 2m, p^m)$ for some natural number m.

The case $p = 2$ is considered in the following result.

Theorem 1.3. *All the positive integer solutions of*

$$2^x + 2^y = 2z^2 \tag{1.3}$$

are of the form

$$(x, y, z) \in \{(2m, 2m, 2^m), (2n+4, 2n+1, 3.2^n)\},$$

where m, n are certain positive integers.

. Up next we consider the equation

$$p^x - p^y = 2z^2, \tag{1.4}$$

a variant of (1.2).

Theorem 1.4. . *(i) Suppose p be an odd prime such that $p^2 \neq 2t^2 + 1$ for a natural number t. Then for each such prime p, the equation (1.4) has positive integer solutions of the form $(p, x, y, z) = (2n^2 + 1, 2m + 1, 2m, n(2n^2 + 1)^m)$, where m, $n \in \mathbb{N}$.*

(ii) Suppose p is an odd prime such that $p^2 = 2t^2 + 1$, some $t \in \mathbb{N}$ and $p \neq 3$. And x, y and z are positive integers. Then (1.4) has positive integer solutions of the form $(p, x, y, z) = \{(2n^2+1, 2m+1, 2m, n(2n^2+1)^m), (p, 2m+2, 2m, t.p^m)\}$, where $m, n \in \mathbb{N}$.

Corollary 1.2. The integer solutions of

$$3^x - 3^y = 2z^2 \tag{1.5}$$

are of the form $(x, y, z) = \{(2m+1, 2m, 3^m), (2m+2, 2m, 2.3^m), (2m+5, 2m, 11.3^m)\}$, where x, y, z and m are positive integers.

Theorem 1.5. . *The solutions of the equation*

$$2^x - 2^y = 2z^2 \tag{1.6}$$

are of the form $(x, y, z) = (2m+2, 2m+1, 2^m)$*, where* x, y, z *and* m *are positive integers.*

The organization of this paper is as follows: §2 provides a review and proof of several key results that will be instrumental in our study. §3 presents the proof of theorems [Theorem 1.1-1.5] and corollary [1.2]. §4 contains three tables consisting of computational verification of some of our results and §5 nutshells the results obtained.

2 Preliminaries

We prove (or recall) several results which will be used in the sequel.

Theorem 2.1. .*([7] **Mihăilescu's Theorem**).* $(3, 2, 2, 3)$ *is the unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$*, where a, b, x and y are integers such that* $\min\{a, b, x, y\} > 1$.

Theorem 2.2. *([10]). The Diophantine equation* $2x^2 + 1 = y^n$, $n \geq 3$ *has only one positive integer solution, namely*

$$(x, y, n) = (11, 3, 5).$$

Lemma 2.1. [([9] **Lemma 2.3).**] Let B be a positive integer such that $B \equiv 5 \pmod{20}$. Then the equation $1 + B^y = z^2$ has no non-negative integer solutions. The next Lemma considers a simpler case of (1.1).

Lemma 2.2. Let A be a positive integer such that $A \equiv 3 \pmod{20}$ with the additional condition that $A + 1$ is a perfect square. Then the only non-negative integer solution of

$$A^x + 1 = z^2 \tag{2.1}$$

is $(x, z) = (1, \sqrt{A+1})$.

Proof. Let x and z be non-negative integer solution of (2.1). If $x = 0$, then $z^2 = 2$, which is a contradiction. Thus $x \geq 1$.
Case 1: $x = 1$. Then we have $A + 1 = z^2$ and $(x, z) = (1, \sqrt{A+1})$ is a solution of (2.1), where $A + 1$ is a perfect square.
Case 2: $x > 1$. We can see that $A > 1$, as well as $z > 1$. Since $z^2 - A^x = 1$, we get $A = 2$ by Theorem **??**. This implies $2 \equiv 3 \pmod{20}$, which is a contradiction. □

Lemma 2.3. The equation

$$2X^2 - 1 = p^Y \tag{2.2}$$

has no positive integer solutions (X, Y), when $p \equiv 3, 5 \pmod 8$ is a prime number.

Proof. Let X and Y be positive integers such that $2X^2 - 1 = p^Y$. Since $2X^2 \equiv 1 \pmod p$, so we get

$$X^2 \equiv 2^{-1} \pmod p. \tag{2.3}$$

As $p \equiv 3, 5 \pmod 8$, so $\gcd(2, p) = 1$. This implies that $\gcd(2^{-1}, p) = 1$. By using the Legendre symbol property, we get

$$\left(\frac{2^{-1}}{p}\right) = \left(\frac{2^{-2}.2}{p}\right) = \left(\frac{2^{-2}}{p}\right)\left(\frac{2}{p}\right) = 1(-1) = -1.$$

Hence (2.2) has no positive integer solutions (X, Y), when $p \equiv 3, 5 \pmod 8$ is a prime number. □

3 Proof of the theorems

We prove the theorems by using elementary number theory and one of the main tool used is modular arithmetic.

Proof. (Proof of Theorem (1.1)) Let the tuple (x, y, z) be a non-negative integer solution of (1.1). Note that, as $A \equiv 3 \pmod{20}$ and $B \equiv 5 \pmod{20}$, both A and B are odd. This implies that $A^x + B^y$ is even and so z^2 is even. Thus, z is even and $z^2 \equiv 0 \pmod 4$. This implies $A^x + B^y \equiv 0 \pmod 4$ since $B \equiv 5 \pmod{20}$, hence $B \equiv 1 \pmod 4$. Thus, $B^y \equiv 1 \pmod 4$. As $A \equiv 3 \pmod{20}$, which implies $A \equiv 3 \pmod 4$.

Furthermore, $A^x \equiv 1, 3 \pmod 4$ depending on whether x is even or odd, respectively. Now, if x is even, $A^x + 1 \equiv 0 \pmod 4$, then $2 \equiv 0 \pmod 4$. So x is odd. The following cases need to be considered.
Case 1: In case if $y = 0$, then by Lemma (2.2), the solution is $(x, y, z) = (1, 0, \sqrt{A+1})$.
Case 2: Let $y \geq 1$. Since $B \equiv 5 \pmod{20}$, this implies that $B \equiv 0 \pmod 5$ and so $B^y \equiv 0 \pmod 5$.

Since $A \equiv 3 \pmod{20}$, we get $A \equiv 3 \pmod 5$. That entails $A^x \equiv 3^x \pmod 5$ and $3^x \equiv 2, 3 \pmod 5$. Thus (1.1) reduces to

$$z^2 \equiv 3^x \equiv 2, 3 \pmod 5.$$

On the other hand, $z^2 \equiv 0, 1, 4 \pmod 5$. This fact and the above relation contradict each other. This completes the proof. □

We note a few interesting corollaries.

Corollary 3.1. The equations

$$43^x + 5^y = z^2$$
$$103^x + 25^y = z^2$$

have no non-negative integer solutions.

Corollary 3.2. The equation

$$3^x + 85^y = z^2$$

has the unique non-negative integer solution $(x, y, z) = (1, 0, 2)$.

Corollary 3.3. The equation

$$3^x + k^y = z^2$$

has the unique non-negative integer solution $(x, y, z) = (1, 0, 2)$, where $k \equiv 5 \pmod{20}$.
The proof of Theorem 1.2 follows from the result of Mihăilescu's Theorem 2.1 and Lemma 2.3.

Proof. *Case (1):* If $x = y$, then $2p^x = 2z^2$, yields $x = 2m$, where $m \in \mathbb{N}$ and $z = p^m$. So the equation $p^x + p^y = 2z^2$ has infinitely many solutions of the form

$$(x, y, z) = (2m, 2m, p^m). \tag{3.1}$$

Let $x > y$. The following cases need to be considered.
Case (2): $x > y$, where $x = 2k_1$ and $y = 2k_2 + 1$, $k_1, k_2 \in \mathbb{N}$. Further let $2k_1 = (2k_2 + 1) + t_1$, where t_1 is some odd natural number. Let (1.2) has a solution, then

$$p^{2k_1} + p^{2k_2+1} = p^{(2k_2+1)+t_1} + p^{2k_2+1} = p^{2k_2+1}(p^{t_1} + 1) = 2z^2.$$

The factor $p^{t_1} + 1$ is not a multiple of p and p^{2k_2+1} is also not a perfect square. This implies that $p^{2k_2+1}(p^{t_1} + 1) \neq 2z^2$. So case (2) has no solutions.
Case (3): $x > y$, where $x = 2k_1 + 1$ and $y = 2k_2 + 1$, $k_1, k_2 \in \mathbb{N}$.
We write $2k_1 + 1 = (2k_2 + 1) + t_2$, where t_2 is some even natural number. Suppose (1.2) has a solution, then

$$p^{2k_1+1} + p^{2k_2+1} = p^{2k_2+1}(p^{t_2} + 1) = 2z^2$$

Arguing as in the previous case, we can conclude that case (3) also has no solutions.
Case (4): $x > y$, where $x = 2k_1$ and $y = 2k_2$, $k_1, k_2 \in \mathbb{N}$.
We can write $2k_1 = 2k_2 + t_3$, where t_3 is some even natural number. Let x, y and z are positive integers that satisfy (1.2),

$$p^{2k_1} + p^{2k_2} = p^{2k_2}(p^{t_3} + 1) = (p^{k_2})^2(p^{t_3} + 1) = 2z^2.$$

So the factor $p^{t_3} + 1 = 2A^2$, where A is a factor of z. We can rewrite it as $2A^2 - 1 = p^{t_3}$, and by Lemma 2.3, it is not solvable. Hence case (4) has no solutions.
Case(5): $x > y$, where $x = 2k_1 + 1$ and $y = 2k_2$, $k_1, k_2 \in \mathbb{N}$. Let $2k_1 + 1 = 2k_2 + t_4$, where t_4 is odd. Suppose (1.2) has a solution, then

$$p^{2k_1+1} + p^{2k_2} = p^{2k_2}(p^{t_4} + 1) = (p^{k_2})^2(p^{t_4} + 1) = 2z^2.$$

Thus $p^{t_4} + 1 = 2B^2$, where B is a factor of z. By the same argument as in case (4); the case (5) also has no solutions. Hence (3.1) represents all the positive integer solutions of (1.2). □

Now, we present all the positive integer solutions of $p^x + p^y = 2z^2$, when $p = 2$.

Proof. *Case (1):* If $x = y$, then we have $2.2^x = 2z^2$, this implies $x = 2m$, where $m \in \mathbb{N}$. Thus $z = 2^m$. So $2^x + y^2 = 2z^2$ has infinitely many solutions of the form

$$(x, y, z) = (2m, 2m, 2^m). \tag{3.2}$$

To proceed further, without loss of generality, let $x > y$. We discuss more cases related to this condition.
Case (2): $x > y$, where $x = 2k_1$ and $y = 2k_2 + 1$, $k_1, k_2 \in \mathbb{N}$.
We write $2k_1 = (2k_2 + 1) + t_1$, where t_1 is an odd natural number. Let (1.3) has a solution, then

$$2^{2k_2+1}(2^{t_1} + 1) = 2.(2^{k_2})^2(2^{t_1} + 1) = 2z^2.$$

So the factor $2^{t_1} + 1 = A^2$, where A is a factor of z. We can see that $\min\{2, t_1, A\} > 1$, So by appealing to Theorem 2.1, we get $t_1 = A = 3$. Let us rename $k_2 = n$. Hence the required solutions are

$$(x, y, z) = (2n + 4, 2n + 1, 3.2^n). \tag{3.3}$$

Case (3): $x > y$, where $x = 2k_1 + 1$ and $y = 2k_2$, $k_1, k_2 \in \mathbb{N}$.
We can write $2k_1 + 1 = 2k_2 + t_2$, where t_2 is an odd natural number. Suppose (1.3) has a solution, then

$$2^{2k_2}(2^{t_2} + 1) = (2^{k_2})^2(2^{t_2} + 1) = 2z^2.$$

The factor $2^{t_1} + 1$ is not divisible by 2, so $(2^{k_2})^2(2^{t_2} + 1) = 2z^2$ is not solvable. Case (3) has no solutions.
Case(4): $x > y$, where $x = 2k_1$ and $y = 2k_2$, $k_1, k_2 \in \mathbb{N}$.
We can write $2k_1 = 2k_2 + t_3$, where t_3 is some even natural number. Assume that (1.3) has a solution, then

$$2^{2k_2}(2^{t_3} + 1) = (2^{k_2})^2(2^{t_3} + 1) = 2z^2$$

This equation is not solvable, it follows the same argument as case (3).
Case (5): $x > y$, where $x = 2k_1 + 1$ and $y = 2k_2 + 1$, $k_1, k_2 \in \mathbb{N}$.
We arrange $2k_1 + 1 = (2k_2 + 1) + t_4$, where t_4 is some even natural number. Assume that (1.3) has a solution, then

$$2^{2k_2+1}(2^{t_4} + 1) = 2.2^{2k_2}(2^{t_4} + 1) = 2z^2$$

and we get $2^{t_4} + 1 = B^2$, where B is a factor of z. Analogously as in case (2), we arrive on to $t_4 = 3$ but t_4 is even. This is a contradiction. So, in this case (1.3) does not have any solution. After summing up all the cases, we get (3.2) and (3.3) represent all the solutions of (1.3). □

Proof. (Proof of Theorem 1.4) (i) It is clear that $x > y$.
Case (1): $x = 2k_1$ and $y = 2k_2 + 1,\ \ k_1, k_2 \in \mathbb{N}$.
After rearranging, $2k_1 = (2k_2 + 1) + t_1$. If (1.4) has a solution, then

$$p^{2k_1} - p^{2k_2+1} = p^{2k_2+1}(p^{t_1} - 1) = 2z^2.$$

The value p^{2k_2+1} is not a perfect square and also $p^{t_1} - 1$ is not a multiple of p. Therefore, $p^{2k_2+1}(p^{t_1} - 1) = 2z^2$ is not solvable.
Case (2): $x = 2k_1 + 1$ and $y = 2k_2 + 1,\ \ k_1, k_2 \in \mathbb{N}$.
We can write $2k_1 + 1 = (2k_2 + 1) + t_2$, where t_2 is even. If (1.4) has a solution, then

$$p^{2k_2+1}(p^{t_2} - 1) = 2z^2.$$

Analogously as in case (1), we conclude that $p^{2k_2+1}(p^{t_2} - 1) = 2z^2$ is not solvable.
Case(3): $x = 2k_1 + 1$ and $y = 2k_2,\ \ k_1, k_2 \in \mathbb{N}$.
We can write $2k_1 + 1 = 2k_2 + t_3$, where t_3 is odd. If (1.4) has a solution, then

$$p^{2k_2}(p^{t_3} - 1) = (p^{k_2})^2(p^{t_3} - 1) = 2z^2.$$

This implies that $p^{t_3} - 1 = 2n^2$, where n is factor of z. Upon rearranging $2n^2 + 1 = p^{t_3}$ and appealing to Theorem 2.2, we conclude that for $t_3 \geq 3$, it has only one solution

$$(n, p, t_3) = (11, 3, 5), \tag{3.4}$$

but $3^2 = 2.2^2 + 1$ and it violates the condition $p^2 \neq 2t^2 + 1$.
Now the case $t_3 = 1$ leads to $p = 2n^2 + 1$ and after renaming $k_2 = m$, we obtain

$$(p, x, y, z) = (2n^2 + 1, 2m + 1, 2m, n(2n^2 + 1)^m). \tag{3.5}$$

Case (4): $x = 2k_1$ and $y = 2k_2,\ \ k_1, k_2 \in \mathbb{N}$.
We can rearrange $2k_1 = 2k_2 + t_4$, where t_4 is an even number. Let (1.4) has a solution, then

$$p^{2k_2}(p^{t_4} - 1) = (p^{k_2})^2(p^{t_4} - 1) = 2z^2.$$

We obtain, $p^{t_4} - 1 = 2t^2$, where t is a factor of z. Hence by Theorem **??**, we get for $t_4 \geq 3$ has no solutions, because t_4 is even.
For $t_4 = 2$, we get $p^2 = 2t^2 + 1$, which is not possible. Therefore in this case, (1.4) has no solutions. After summing all the cases, we get (3.5) represents all the solutions of (1.4).
(ii) Here p is an odd prime $(p \neq 3)$ and $p^2 = 2t^2 + 1$, for some $t \in \mathbb{N}$.
The cases (1),(2) and (3) follows analogously as in (i). In case (4), for $t_4 = 2$, we get $p^2 = 2t^2 + 1$. Therefore after renaming $k_2 = m$, we have

$$(p, x, y, z) = (p, 2m + 2, 2m, t.p^m). \tag{3.6}$$

Thus we get infinitely many solutions in case (4) of the form (3.6).
To sum it up, we obtain (3.5) and (3.6) represent all the positive integer solutions of $p^x - p^y = 2z^2$, where $p^2 = 2t^2 + 1$ and $p \neq 3$. □

Proof. (Proof of Corollary 1.1) Cases (1) and (2) follow verbatim as in (i) of Theorem 1.4. In case (3), for $p = 3$, (3.4) gives $(x, y, z) = (2m + 5, 2m, 11.3^m)$. From (3.5), we get another class of solutions $(x, y, z) = (2m + 1, 2m, 3^m)$. Furthermore, 3^2 is of the form $2.2^2 + 1$, so from (3.6), we get a class of solution $(x, y, z) = (2m + 2, 2m, 2.3^m)$. □

Proof. (Proof of Theorem 1.5) Here too $x > y$. As before we need to divide the proof into several cases.
Case(1): $x = 2k_1$ and $y = 2k_2 + 1,\ \ k_1, k_2 \in \mathbb{N}$.
Let $2k_1 = (2k_2 + 1) + t_1$, where t_1 is odd. If (1.6) has a solution, then

$$2^{2k_2+1}(2^{t_1} - 1) = 2z^2.$$

The factor $2^{t_1} - 1$ is not a multiple of 2, so it must be equal to A^2, where A is a factor of z. This implies $2^{t_1} - 1 = A^2$. If $t_1 > 1$, then $A > 1$. After rearranging, we get $2^{t_1} - A^2 = 1$, and the $\min\{2, A, t_1\} > 1$. Then by Theorem 1.2 this has the unique solution $(2, t_1, A, 2) = (3, 2, 2, 3)$, which is a contradiction.
For $t_1 = 1$, we get $A = 1$, which implies $z = 2^{k_2}$. Let $k_2 = m$, and in this case we obtain solutions

$$(x, y, z) = (2m + 2, 2m + 1, 2^m) \tag{3.7}$$

where m is a positive integer.
Case (2): $x = 2k_1 + 1$ and $y = 2k_2 + 1,\ \ k_1, k_2 \in \mathbb{N}$.
After rearranging, we get $2k_1 + 1 = (2k_2 + 1) + t_2$, where t_2 is some even number. If (1.6) has a solution, then

$$2^{2k_2+1}(2^{t_2} - 1) = 2z^2.$$

By the same argument of case (1), if $t_2 > 1$, it is not solvable. $t_2 \neq 1$, because t_2 is an even natural number. So case (2) has no solutions.
Case(3): $x = 2k_1 + 1$ and $y = 2k_2 \qquad k_1, k_2 \in \mathbb{N}$.
We can write $2k_1 + 1 = 2k_2 + t_3$, where t_3 is some odd natural number. If (1.6) has a solution, then

$$2^{2k_2}(2^{t_3} - 1) = (2^{k_2})^2(2^{t_3} - 1) = 2z^2$$

and we obtain $(2^{t_3} - 1) = 2A^2$, where A is factor of z. But this equation is not solvable as left hand side is odd and right hand side is even. So case (3) has no solutions.
Case (4): $x = 2k_1$ and $y = 2k_2,\ \ k_1, k_2 \in \mathbb{N}$.
Let $2k_1 = 2k_2 + t_4$, where t_4 is some even natural number. If (1.6) has a solution, then

$$2^{2k_2}(2^{t_4} - 1) = 2z^2.$$

By analogous argumentas in case (3), this equation is also not solvable. Case (4) anyways has no solutions.
Hence, after summing up all the cases, we find that (3.7) represents all the positive integer solutions of (1.6). □

4 Numerical Examples

In this section, we provide some numerical evidences corroborating our results in Theorem 1.2 and Theorem 1.4.

Prime p	Value of m	Solution (x, y, z)	Verification (Equation)
$p = 3$	$m = 1$	$(2, 2, 3)$	$3^2 + 3^2 = 2 \cdot 3^2$
	$m = 2$	$(4, 4, 9)$	$3^4 + 3^4 = 2 \cdot 9^2$
	$m = 3$	$(6, 6, 27)$	$3^6 + 3^6 = 2 \cdot 27^2$
$p = 5$	$m = 1$	$(2, 2, 5)$	$5^2 + 5^2 = 2 \cdot 5^2$
	$m = 2$	$(4, 4, 25)$	$5^4 + 5^4 = 2 \cdot 25^2$
	$m = 3$	$(6, 6, 125)$	$5^6 + 5^6 = 2 \cdot 125^2$
$p = 11$	$m = 1$	$(2, 2, 11)$	$11^2 + 11^2 = 2 \cdot 11^2$
	$m = 2$	$(4, 4, 121)$	$11^4 + 11^4 = 2 \cdot 121^2$
	$m = 3$	$(6, 6, 1331)$	$11^6 + 11^6 = 2 \cdot 1331^2$
$p = 19$	$m = 1$	$(2, 2, 19)$	$19^2 + 19^2 = 2 \cdot 19^2$
	$m = 2$	$(4, 4, 361)$	$19^4 + 19^4 = 2 \cdot 361^2$
	$m = 3$	$(6, 6, 6859)$	$19^6 + 19^6 = 2 \cdot 6859^2$
$p = 29$	$m = 1$	$(2, 2, 29)$	$29^2 + 29^2 = 2 \cdot 29^2$
	$m = 2$	$(4, 4, 841)$	$29^4 + 29^4 = 2 \cdot 841^2$
	$m = 3$	$(6, 6, 24389)$	$29^6 + 29^6 = 2 \cdot 24389^2$

Table 1. Solutions for the equation $p^x + p^y = 2z^2$ for primes $p \equiv 3, 5 \pmod 8$

Prime p	Condition $p^2 \neq 2t^2 + 1$	Value of n	Value of m	Solution (x, y, z)
$p = 19$	$19^2 = 361 \neq 2t^2 + 1$	$n = 3$	$m = 1$	$(3, 2, 57)$
		$n = 3$	$m = 2$	$(5, 4, 1083)$
		$n = 3$	$m = 3$	$(7, 6, 20577)$
$p = 73$	$73^2 = 5329 \neq 2t^2 + 1$	$n = 6$	$m = 1$	$(3, 2, 438)$
		$n = 6$	$m = 2$	$(5, 4, 31974)$
		$n = 6$	$m = 3$	$(7, 6, 2334102)$
$p = 163$	$163^2 = 26569 \neq 2t^2 + 1$	$n = 9$	$m = 1$	$(3, 2, 1467)$
		$n = 9$	$m = 2$	$(5, 4, 239121)$
		$n = 9$	$m = 3$	$(7, 6, 38976723)$
$p = 397$	$397^2 = 157609 \neq 2t^2 + 1$	$n = 14$	$m = 1$	$(3, 2, 5558)$
		$n = 14$	$m = 2$	$(5, 4, 2206526)$
		$n = 14$	$m = 3$	$(7, 6, 875990822)$

Table 2. Solutions to the equation $p^x - p^y = 2z^2$ for primes p satisfying $p^2 \neq 2t^2 + 1$.

5 Conclusions

In this study, we have identified all non-negative integer solutions (x, y, z) for the Diophantine equation $A^x + B^y = z^2$, where A, B are positive integers such that $A \equiv 3 \pmod{20}$ and $B \equiv 5 \pmod{20}$, extending the work of Srimud and Tadee [9].

Furthermore, we have considered more equations, $p^x + p^y = 2z^2$, where $p \equiv 3, 5 \pmod 8$. We found that this equation has infinitely many positive integer solutions. Additionally, we showed $2^x + 2^y = 2z^2$, has infinitely many positive integer solutions and with this we present all the class of positive integer solutions for $p^x - p^y = 2z^2$ for all $p \geq 2$. We have also derived a number of interesting corollaries from our main results.

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Prime p	**Condition** $p^2 = 2t^2 + 1$	**Value of** m	**Solution** (x, y, z)
$p = 17$	$17^2 = 2(12)^2 + 1$	$m = 1$	$(4, 2, 204)$
		$m = 2$	$(6, 4, 3468)$
		$m = 3$	$(8, 6, 58956)$
$p = 577$	$577^2 = 2(408)^2 + 1$	$m = 1$	$(4, 2, 235416)$
		$m = 2$	$(6, 4, 135835032)$
		$m = 3$	$(8, 6, 78376813464)$
$p = 665857$	$665857^2 = 2(470832)^2 + 1$	$m = 1$	$(4, 2, 313506783024)$

Table 3. Solutions for the equation $p^x - p^y = 2z^2$ for primes p satisfying $p^2 = 2t^2 + 1$.

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