

Certain Expansion Formulae of Incomplete Yang Y -Function

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Abstract In this paper, we explore the Leibniz rule for derivatives of the Riemann-Liouville type to derive various expansion formulas for the incomplete Yang Y -functions. These functions are instrumental in diverse fields of applied mathematics, including mathematical physics and engineering. We extend and generalize existing expansion formulas by utilizing the Leibniz rule, providing a comprehensive framework that encapsulates several well-known special functions as particular cases. Our significant results encompass the incomplete H -function, Meijer’s G -function, the incomplete Fox-Wright function, and the incomplete generalized hypergeometric function. Each function is elaborated through specific expansion formulas, offering new insights and potential applications in analyzing complex systems. Additionally, this study outlines the development of new expansion formulas for these incomplete special functions, thereby contributing to the broader understanding of fractional calculus and its applications in mathematical modeling. The results presented herein not only generalize but also unify many known results, demonstrating the utility of the Riemann-Liouville derivative in expanding the scope of incomplete special functions.

1 Introduction and Preliminaries

The emergence of new issues in engineering and applied sciences has led to a steady evolution in the large field of special functions. In response to the demands and interests of the public research is still being conducted to examine different facets of these functions and their possible uses. X. J. Yang investigated and defined a new function, known as Yang Y -function. Inspired by the work referenced in [8] certain expansion formulae and of incomplete H -functions and other paper [5] class extension through its incomplete functions, both papers are helpful for solving this research paper.

The Yang Y -function [16] is called a generalized hypergeometric function defined by means of the contour integral:

$$Y_{p,q}^{m,n} \left[v; u; t \left| \begin{matrix} (e_i, E_i)_1^p \\ (f_j, F_j)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\Omega} \vartheta_{p,q}^{m,n}(\eta) v^{-\eta} e^{\eta u} \eta^t d\eta, \tag{1.1}$$

where

$$\vartheta_{p,q}^{m,n}(\eta) = \frac{[\prod_{j=1}^m \Gamma(f_j - F_j \eta)] [\prod_{i=1}^n \Gamma(1 - e_i + E_i \eta)]}{[\prod_{j=p+1}^q \Gamma(1 - f_j + F_j \eta)] [\prod_{i=q+1}^p \Gamma(e_i - E_i \eta)]}, \tag{1.2}$$

here t, u, v are complex number and a contour Ω in the complex plane the orders m, n, p, q are nonnegative integers so that $0 \leq m \leq q, 0 \leq n \leq p$, the parameters $E_m > 0, F_n > 0$ are positive and $e_i, f_j, i = 1, \dots, p; j = 1, \dots, q$ are arbitrary complex which satisfies the following conditions:

$$E_i(f_j + \varrho) \neq F_j(e_i - \varrho' - 1), \tag{1.3}$$

$$(\varrho, \varrho' \in \mathbb{N}_0(0, 1, 2, \dots); i = 1, 2, 3 \dots p; j = 1, 2, 3 \dots q).$$

Our new step here will be to remember that familiar incomplete function $\gamma(\delta, \rho)$ and $\Gamma(\delta, \rho)$ are defined as see ([1, 2])

$$\gamma(\delta, \rho) = \int_0^\rho k^{\delta-1} e^{-k} dk, \quad (\Re(\delta) > 0, \rho \geq 0), \tag{1.4}$$

and

$$\Gamma(\delta, \rho) = \int_\rho^\infty k^{\delta-1} e^{-k} dk, \quad (\Re(\delta) > 0, \rho \geq 0), \tag{1.5}$$

respectively.

Definitions of incomplete gamma function (1.4) and (1.5) easily yield to following decomposition formula:

$$\gamma(\rho, \delta) + \Gamma(\rho, \delta) = \Gamma(\rho), \quad (\Re(\rho) > 0). \tag{1.6}$$

Throughout this article, \mathbb{N} , \mathbb{C} , \mathbb{Z}^- , and \mathbb{R}^+ denote the sets of positive integers, complex numbers, negative integers, and positive real numbers, respectively.

1.1 The Incomplete Y-Functions

Here, in this paper, using the incomplete gamma functions in (1.4) and (1.5), we present the Yang Y-function corresponding to incomplete Y-functions. The incomplete Y-functions are defined as follows [14]:

$$\begin{aligned} \gamma_{p,q}^{m,n}(v; u; t) &= \gamma_{p,q}^{m,n} \left[v; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right] \\ &= \frac{1}{2\pi i} \int_\Omega \gamma_{p,q}^{m,n}(\eta, \rho) v^{-\eta} e^{\eta u} \eta^t d\eta, \end{aligned} \tag{1.7}$$

where

$$\gamma_{p,q}^{m,n}(\eta, \rho) = \frac{\gamma(1 - e_1 + E_1\eta, \rho) [\prod_{j=1}^m \Gamma(f_j - F_j\eta)] [\prod_{i=2}^n \Gamma(1 - e_i + E_i\eta)]}{[\prod_{j=m+1}^q \Gamma(1 - f_j + F_j\eta)] [\prod_{i=n+1}^p \Gamma(e_i - E_i\eta)]}, \tag{1.8}$$

and

$$\begin{aligned} \Gamma_{p,q}^{m,n}(v; u; t) &= \Gamma_{p,q}^{m,n} \left[v; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_n)_1^q \end{matrix} \right] \\ &= \frac{1}{2\pi i} \int_\Omega \Gamma_{p,q}^{m,n}(\eta, \rho) v^{-\eta} e^{\eta u} \eta^t d\eta, \end{aligned} \tag{1.9}$$

where

$$\Gamma_{p,q}^{m,n}(\eta, \rho) = \frac{\Gamma(1 - e_1 + E_1\eta, \rho) [\prod_{j=1}^m \Gamma(f_j - F_j\eta)] [\prod_{i=2}^n \Gamma(1 - e_i + E_i\eta)]}{[\prod_{j=m+1}^q \Gamma(1 - f_j + F_j\eta)] [\prod_{i=n+1}^p \Gamma(e_i - E_i\eta)]}. \tag{1.10}$$

The incomplete Y-functions $\gamma_{p,q}^{m,n}(v; u; t)$ and $\Gamma_{p,q}^{m,n}(v; u; t)$ in (1.7) and (1.9) exist for all $\rho \geq 0$ under the same contour and the same sets of conditions as stated above.

The decomposition formula that follows is easily obtained from these definitions:

$$\gamma_{p,q}^{m,n}(v; u; t) + \Gamma_{p,q}^{m,n}(v; u; t) = Y_{p,q}^{m,n}(v; u; t). \tag{1.11}$$

The result from Equation (1.11) is equal to Equation (1.1); hence, Yang Y-function follows an incomplete function.

Next, we review some basic characteristics of the incomplete Yang Y-functions. The incomplete Yang Y-functions defined in (1.9) and (1.7) exist for all $\rho \geq 0$, under the set of conditions given by Yang [16], with

$$\Delta \geq 0, |arg(v; u; t)| < \Delta \frac{\pi}{2},$$

$$\Delta = \sum_{j=1}^p F_j - \sum_{j=p+1}^s F_j + \sum_{j=1}^q E_j - \sum_{j=q+1}^r E_j.$$

1.2 Riemann-Liouville Fractional Operators

The following is a definition of the Riemann-Liouville fractional operators of order v for the function $f(z)$ (see [4]):

$$I^v f(z) = \frac{1}{\Gamma(v)} \int_0^z (z - s)^{v-1} f(s) ds, \tag{1.12}$$

hence, given that f is a locally integrable function and v is a complex integer in the half plane the integral is well-defined $\Re(v) > 0$,

$$D_z^v f(z) = \frac{1}{\Gamma(-v)} \int_0^z (z - s)^{-v-1} f(s) ds, \quad (\Re(v) < 0), \tag{1.13}$$

if $\Re(v) \geq 0$ and $m \in \mathbb{N}$ is the smallest integer with $m - 1 \leq \Re(v) < m$, then,

$$D_z^v f(z) = \frac{d^m}{dz^m} \cdot D_z^{v-m} f(z) = \frac{d^m}{dz^m} \left[\frac{1}{\Gamma(-v + m)} \int_0^z (z - s)^{-v+m-1} f(s) ds \right]. \tag{1.14}$$

1.3 Leibniz Formula for Fractional Derivative

The following is the definition of the classical Leibniz rule for two differentiable functions f and g :

$$D^n [f(s)g(s)] = \sum_{k=0}^n \binom{n}{k} [D^k g(s)][D^{n-k} f(s)]. \tag{1.15}$$

It is possible to extend this Leibniz rule to the derivatives of the Riemann-Liouville type. The fractional generalization of the Leibniz rule is stated as follows if f and g are two functions of class C (see [11]).

$$D^\mu [f(s)g(s)] = \sum_{k=0}^\infty \binom{\mu}{k} [D^k g(s)][D^{\mu-k} f(s)], \quad \mu > 0, k \in \mathbb{N}, \tag{1.16}$$

specifically, if f is a class C function, then

$$D^\mu [s^p f(s)] = \sum_{r=0}^p \binom{\mu}{r} [D^r s^p][D^{\mu-r} f(s)], \quad \mu > 0. \tag{1.17}$$

A mechanism that determines the composition representation of the differential operators can be extracted using the Leibniz rule, which generalizes the differentiation law of the product. Generally speaking, it would be beneficial to deduce some intriguing expansions, summations, generating functions, and transformations pertaining to the various special functions (including q-functions) of one or more variables; refer to works [6, 9, 10, 12, 13] and citations inside. This paper’s main goal is to apply the Leibniz rule for fractional derivatives of the Riemann-Liouville type in order to extract multiple expansion formulae from incomplete Yang Y-functions. Along with the main result, there are a number of intriguing special instances and implementations.

2 Main Findings

Using the Leibniz rule, we have determined a few expansion equations for the incomplete Yang Y-functions.

Theorem 2.1. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} & \Gamma Y_{p+1, q+1}^{m, n+1} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (1 - \lambda, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right] \\ &= \sum_{k=0}^\infty \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma Y_{p+1, q+1}^{m, n+1} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (0, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right]. \end{aligned} \tag{2.1}$$

Proof. To prove the result (2.1), let us consider $f(s) = v^{\lambda-1}$ and

$$g(s) = \Gamma Y_{p,q}^{m,n}(av^\sigma; u; t) = \Gamma Y_{p,q}^{m,n} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right],$$

now, substituting the values of $f(s)$ and $g(s)$ in (1.16), we obtain

$$D^\mu [v^{\lambda-1} \Gamma Y_{p,q}^{m,n}(av^\sigma; u; t)] = \sum_{k=0}^\infty \binom{\mu}{k} [D^k \Gamma Y_{p,q}^{m,n}(av^\sigma; u; t)] [D^{\mu-k} v^{\lambda-1}], \tag{2.2}$$

on taking L.H.S. of Equation (2.2), we obtain

$$\begin{aligned} D^\mu [v^{\lambda-1} \Gamma Y_{p,q}^{m,n}(av^\sigma; u; t)] &= D^\mu \left[v^{\lambda-1} \frac{1}{2\pi i} \int_\Omega \Gamma \vartheta_{p,q}^{m,n}(\eta, \rho)(av^\sigma)^{-\eta} e^{\eta u} \eta^t d\eta \right] \\ &= \frac{1}{2\pi i} \int_\Omega \Gamma \vartheta_{p,q}^{m,n}(\eta, \rho) a^{-\eta} D^\mu [v^{\lambda-1} v^{-\sigma\eta}] e^{\eta u} \eta^t d\eta \\ &= \frac{1}{2\pi i} \int_\Omega \Gamma \vartheta_{p,q}^{m,n}(\eta, \rho) a^{-\eta} \left(\frac{d}{dv}\right)^\mu [v^{\lambda-1-\sigma\eta}] e^{\eta u} \eta^t d\eta \\ &= \frac{1}{2\pi i} \int_\Omega \Gamma \vartheta_{p,q}^{m,n}(\eta, \rho) a^{-\eta} \frac{\Gamma(\lambda - \sigma\eta)}{\Gamma(\lambda - \sigma\eta - \mu)} v^{\lambda-\mu-\sigma\eta-1} e^{\eta u} \eta^t d\eta, \end{aligned}$$

using the Definition (1.9), we obtain

$$D^\mu [v^{\lambda-1} \Gamma Y_{p,q}^{m,n}(av^\sigma; u; t)] = v^{\lambda-\mu-1} \Gamma Y_{p+1,q+1}^{m,n+1} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (1 - \lambda, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right]. \tag{2.3}$$

Similarly, the R.H.S. of Equation (2.3) is the immediate consequences of the Definitions (1.9) and (1.16), we obtain

$$\begin{aligned} &\sum_{k=0}^\infty \binom{\mu}{k} [D^k \Gamma Y_{p,q}^{m,n}(av^\sigma; u; t)] [D^{\mu-k} v^{\lambda-1}] \\ &= v^{\lambda-\mu-1} \sum_{k=0}^\infty \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma Y_{p+1,q+1}^{m,n+1} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (0, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right], \end{aligned} \tag{2.4}$$

substituting the Equation (2.3) and (2.4) into (2.2), we get the required result (2.1). The following theorem is provided without proof because it is the direct result of Definitions (1.7), (1.9), and (1.16). □

Theorem 2.2. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} &\Gamma Y_{p+1,q+1}^{m+1,n} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, F_j)_1^q \end{matrix} \right] \\ &= \sum_{k=0}^\infty \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma Y_{p+1,q+1}^{m+1,n} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (1 - k, \sigma) \\ (1, \sigma), (f_j, F_j)_1^q \end{matrix} \right]. \end{aligned} \tag{2.5}$$

Proof. To prove the result (2.5), let us consider $f(s) = v^{\lambda-1}$ and

$$g(s) = \Gamma Y_{p,q}^{m,n}(av^{-\sigma}; u; t) = \Gamma Y_{p,q}^{m,n} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right],$$

now, substituting the values of $f(s)$ and $g(s)$ in (1.16), we obtain

$$D^\mu [v^{\lambda-1} \Gamma_{Y_{p,q}^{m,n}}(av^{-\sigma}; u; t)] = \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k \Gamma_{Y_{p,q}^{m,n}}(av^{-\sigma}; u; t)] [D^{\mu-k} v^{\lambda-1}], \tag{2.6}$$

on taking L.H.S. of Equation (2.6), we obtain

$$\begin{aligned} D^\mu [v^{\lambda-1} \Gamma_{Y_{p,q}^{m,n}}(av^{-\sigma}; u; t)] &= D^\mu \left[v^{\lambda-1} \frac{1}{2\pi i} \int_{\Omega} \Gamma_{\vartheta_{p,q}^{m,n}}(\eta, \rho)(av^{-\sigma})^{-\eta} e^{\eta u} \eta^t d\eta \right] \\ &= \frac{1}{2\pi i} \int_{\Omega} \Gamma_{\vartheta_{p,q}^{m,n}}(\eta, \rho) a^{-\eta} D^\mu [v^{\lambda-1} v^{\sigma\eta}] e^{\eta u} \eta^t d\eta \\ &= \frac{1}{2\pi i} \int_{\Omega} \Gamma_{\vartheta_{p,q}^{m,n}}(\eta, \rho) a^{-\eta} \left(\frac{d}{dv}\right)^\mu [v^{\lambda-1+\sigma\eta}] e^{\eta u} \eta^t d\eta \\ &= \frac{1}{2\pi i} \int_{\Omega} \Gamma_{\vartheta_{p,q}^{m,n}}(\eta, \rho) a^{-\eta} \frac{\Gamma(\lambda + \sigma\eta)}{\Gamma(\lambda + \sigma\eta - \mu)} v^{\lambda-\mu+\sigma\eta-1} e^{\eta u} \eta^t d\eta, \end{aligned}$$

using the Definition (1.9), we obtain

$$D^\mu [v^{\lambda-1} \Gamma_{Y_{p,q}^{m,n}}(av^{-\sigma}; u; t)] = v^{\lambda-\mu-1} \Gamma_{Y_{p+1,q+1}^{m,n+1}} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, F_j)_1^q \end{matrix} \right]. \tag{2.7}$$

Similarly, the R.H.S. of Equation (2.7) is the immediate consequences of the Definitions (1.9) and (1.16), we obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} \binom{\mu}{k} [D^k \Gamma_{Y_{p,q}^{m,n}}(av^{-\sigma}; u; t)] [D^{\mu-k} v^{\lambda-1}] \\ &= v^{\lambda-\mu-1} \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma_{Y_{p+1,q+1}^{m,n+1}} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (1 - k, \sigma) \\ (1, \sigma), (f_j, F_j)_1^q \end{matrix} \right], \end{aligned} \tag{2.8}$$

substituting the Equation (2.7) and (2.8) into (2.6), we get the required result (2.5). The following theorem is provided without proof because it is the direct result of Definitions (1.7), (1.9), and (1.16). □

Theorem 2.3. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} &\gamma_{Y_{p+1,q+1}^{m,n+1}} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (1 - \lambda, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \gamma_{Y_{p+1,q+1}^{m,n+1}} \left[av^\sigma; u; t \mid \begin{matrix} (e_1, E_1, \rho), (0, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right]. \end{aligned} \tag{2.9}$$

Theorem 2.4. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} &\gamma_{Y_{p+1,q+1}^{m+1,n}} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, F_j)_1^q \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \gamma_{Y_{p+1,q+1}^{m+1,n}} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (1 - k, \sigma) \\ (1, \sigma), (f_j, F_j)_1^q \end{matrix} \right]. \end{aligned} \tag{2.10}$$

3 Special Cases

The following results are obtained in this section as particular instances of the theorems pertaining to the incomplete Y -function, the Y -function, the incomplete H -function, the H -function the incomplete Meijer’s G -function. We can acquire special situations that demonstrate the application of the fundamental results by adjusting specific factors.

(i) Yang Y -Function

If we replace $\rho = 0$ in the (1.7) and (1.9) then it change into Yang Y -function (1.1),as follow:

$$\Gamma_{Y_{p,q}^{m,n}} \left[v; u; t \mid \begin{matrix} (e_1, E_1, 0), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right] = Y_{p,q}^{m,n} \left[v; u; t \mid \begin{matrix} (e_i, E_i)_1^p \\ (f_j, F_j)_1^q \end{matrix} \right]. \tag{3.1}$$

Applying the relationships above in Theorem (2.1) to Theorem (2.2), we get the following corollaries:

Corollary 3.1. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$ and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} & Y_{p+1,q+1}^{m,n+1} \left[av^\sigma; u; t \mid \begin{matrix} (1 - \lambda, \sigma), (e_i, E_i)_1^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} Y_{p+1,q+1}^{m,n+1} \left[av^\sigma; u; t \mid \begin{matrix} (0, \sigma), (e_i, E_i)_1^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right]. \end{aligned} \tag{3.2}$$

Corollary 3.2. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} & Y_{p+1,q+1}^{m+1,n} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_i, E_i)_1^p, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, F_j)_1^q \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} Y_{p+1,q+1}^{m+1,n} \left[av^{-\sigma}; u; t \mid \begin{matrix} (e_i, E_i)_1^p, (1-k, \sigma) \\ (1, \sigma), (f_j, F_j)_1^q \end{matrix} \right]. \end{aligned} \tag{3.3}$$

Similarly we will obtain results for all the theorem given above.

(ii) Incomplete H -Function

If we replace $t = u = 0$ and $\frac{1}{v}$ in place of v in (1.7) and (1.9), it reduces to the incomplete H -function [8], as follows:

$$\Gamma_{Y_{p,q}^{m,n}} \left[\frac{1}{v}; 0; 0 \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right] = \Gamma_{p,q}^{m,n} \left[v \mid \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right]. \tag{3.4}$$

We get the following corollaries by applying the relationships above in Theorem (2.1) and Theorem (2.2).

Similarly, for the upper Yang Y -function, applying the relationships outlined in Theorem (2.3) and Theorem (2.4), we obtain the following corollaries.

Corollary 3.3. *Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:*

$$\begin{aligned} & \Gamma_{p+1,q+1}^{m,n+1} \left[av^\sigma \mid \begin{matrix} (e_1, E_1, \rho), (1 - \lambda, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma_{p+1,q+1}^{m,n+1} \left[av^\sigma \mid \begin{matrix} (e_1, E_1, \rho), (0, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right]. \end{aligned} \tag{3.5}$$

Corollary 3.4. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & \Gamma_{p+1,q+1}^{m+1,n} \left[av^{-\sigma} \left| \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, F_j)_1^q \end{matrix} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma_{p+1,q+1}^{m+1,n} \left[av^{-\sigma} \left| \begin{matrix} (e_1, E_1, \rho), (e_i, E_i)_2^p, (1 - k, \sigma) \\ (1, \sigma), (f_j, F_j)_1^q \end{matrix} \right. \right]. \end{aligned} \tag{3.6}$$

Similarly we will obtain results for all the Theorem given above.

(iii) H-Function

If we replace $t = u = \rho = 0$ and $\frac{1}{v}$ in place of v in (1.7) and (1.9), it reduces to the H -function [8], as follows:

$$\Gamma_{p,q}^{m,n} \left[\frac{1}{v}; 0; 0 \left| \begin{matrix} (e_1, E_1, 0), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right. \right] = H_{p,q}^{m,n} \left[v \left| \begin{matrix} (e_i, E_i)_1^p \\ (f_j, F_j)_1^q \end{matrix} \right. \right]. \tag{3.7}$$

Applying the relationships outlined in Theorem (2.3) to Theorem (2.4), we obtain the following corollaries.

Corollary 3.5. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & H_{p+1,q+1}^{m,n+1} \left[av^\sigma \left| \begin{matrix} (1 - \lambda, \sigma), (e_i, E_i)_1^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} H_{p+1,q+1}^{m,n+1} \left[av^\sigma \left| \begin{matrix} (0, \sigma), (e_i, E_i)_1^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right. \right]. \end{aligned} \tag{3.8}$$

Corollary 3.6. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & H_{p+1,q+1}^{m+1,n} \left[av^{-\sigma} \left| \begin{matrix} (e_i, E_i)_1^p, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, F_j)_1^q \end{matrix} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} H_{p+1,q+1}^{m+1,n} \left[av^{-\sigma} \left| \begin{matrix} (e_i, E_i)_1^p, (1 - k, \sigma) \\ (1, \sigma), (f_j, F_j)_1^q \end{matrix} \right. \right]. \end{aligned} \tag{3.9}$$

Similarly we will obtain results for all the Theorem given above.

(iv) Incomplete Meijar’s G-Function

If we replace $t = u = 0, E_i = 1(i = 1, \dots, p), F_j = 1(j = 1, \dots, q), \sigma = 1$ and $\frac{1}{v}$ in place of v in (1.7) and (1.9), it reduces to the incomplete Meijar’s G -function [7], as follows:

$$(\Gamma)Y_{p,q}^{m,n} \left[\frac{1}{v}; 0; 0 \left| \begin{matrix} (e_1, 1, \rho), (e_i, 1)_2^p \\ (f_j, 1)_1^q \end{matrix} \right. \right] = (\Gamma)G_{p,q}^{m,n} \left[v \left| \begin{matrix} (e_1, \rho), (e_i)_2^p \\ (f_j)_1^q \end{matrix} \right. \right]. \tag{3.10}$$

Applying the relationships above in Theorem (2.1) to Theorem (2.2), we get the following corollaries:

Corollary 3.7. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & (\Gamma)G_{p+1,q+1}^{m,n+1} \left[av \left| \begin{matrix} (e_1, \rho), (1 - \lambda), (e_i)_2^p \\ (f_j)_1^q, (1 - \lambda + \mu) \end{matrix} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} (\Gamma)G_{p+1,q+1}^{m,n+1} \left[av \left| \begin{matrix} (e_1, \rho), 0, (e_i)_2^p \\ (f_j)_1^q, k \end{matrix} \right. \right]. \end{aligned} \tag{3.11}$$

Corollary 3.8. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned}
 & (\Gamma)G_{p+1,q+1}^{m+1,n} \left[av^{-1} \mid \begin{matrix} (e_1, \rho), (e_i)_2^p, (\lambda - \mu) \\ \lambda, (f_j)_1^q \end{matrix} \right] \\
 &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} (\Gamma)G_{p+1,q+1}^{m+1,n} \left[av^{-1} \mid \begin{matrix} (e_1, \rho), (e_i)_2^p, (1 - k) \\ 1, (f_j)_1^q \end{matrix} \right]. \tag{3.12}
 \end{aligned}$$

Additionally, using the following functional connection [15], if we substitute $a = -a, m = 1, n = p, q = q + 1, e_i \rightarrow (1 - e_i) (i = 1, \dots, p)$ and $f_j \rightarrow (1 - f_j) (j = 1, \dots, q)$ in (2.1) and (2.5),

$$(\Gamma)Y_{p,q+1}^{1,p} \left[\frac{1}{v}; 0; 0 \mid \begin{matrix} (1 - e_1, E_1, \rho), (1 - e_i, E_i)_2^p \\ (0, 1), (1 - f_j, F_j)_1^q \end{matrix} \right] = {}_p\psi_q^{(\Gamma)} \left[v \mid \begin{matrix} (1 - e_1, E_1, \rho), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q \end{matrix} \right], \tag{3.13}$$

we obtain the following corollaries:

Corollary 3.9. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned}
 & {}_{p+1}\psi_{q+1}^{(\Gamma)} \left[v \mid \begin{matrix} (e_1, E_1, \rho), (1 - \lambda, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (1 - \lambda + \mu, \sigma) \end{matrix} \right] \\
 &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} {}_{p+1}\psi_{q+1}^{(\Gamma)} \left[v \mid \begin{matrix} (e_1, E_1, \rho), (0, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (k, \sigma) \end{matrix} \right]. \tag{3.14}
 \end{aligned}$$

Corollary 3.10. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned}
 & {}_{p+1}\psi_{q+1}^{(\Gamma)} \left[v \mid \begin{matrix} (e_1, E_1, \rho), (\lambda - \mu, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (\lambda, \sigma) \end{matrix} \right] \\
 &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} {}_{p+1}\psi_{q+1}^{(\Gamma)} \left[v \mid \begin{matrix} (e_1, E_1, \rho), (1 - k, \sigma), (e_i, E_i)_2^p \\ (f_j, F_j)_1^q, (1, \sigma) \end{matrix} \right]. \tag{3.15}
 \end{aligned}$$

Once more, using the relation [15], if we substitute $\sigma = 1$ in (3.14) and (3.15) with $E_i = 1, (i = 1, \dots, p)$ and $F_j = 1, (j = 1, \dots, q)$,

$$(\Gamma)Y_{p,q+1}^{1,p} \left[\frac{1}{v}; 0; 0 \mid \begin{matrix} (1 - e_1, 1, \rho), (1 - e_i, 1)_2^p \\ (0, 1), (1 - f_j, 1)_1^q \end{matrix} \right] = \mathbb{C}_q^p {}_p\Gamma_q \left[\begin{matrix} (e_1, \rho), (e_i)_2^p; \\ (f_j)_1^q; \end{matrix} v \right] \tag{3.16}$$

where, $\mathbb{C}_q^p = \frac{\prod_{i=1}^p \Gamma(e_i)}{\prod_{j=1}^q \Gamma(f_j)}$, consequently, the following corollaries are obtained:

Corollary 3.11. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned}
 & \frac{\Gamma(1 - \lambda)}{\Gamma(1 - \lambda + \mu)} {}_{p+1}\Gamma_{q+1} \left[\begin{matrix} (e_1, \rho), (1 - \lambda), (e_i)_2^p; \\ (f_j)_1^q, (1 - \lambda + \mu); \end{matrix} av \right] \\
 &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)\Gamma(k)} {}_{p+1}\Gamma_{q+1} \left[\begin{matrix} (e_1, \rho), 0, (e_i)_2^p; \\ (f_j)_1^q, k; \end{matrix} av \right]. \tag{3.17}
 \end{aligned}$$

Corollary 3.12. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned}
 & \frac{\Gamma(\lambda - \mu)}{\Gamma(\lambda)} {}_{p+1}\Gamma_{q+1} \left[\begin{matrix} (e_1, \rho), \lambda - \mu, (e_i)_2^p; \\ (f_j)_1^q, \lambda; \end{matrix} av \right] \\
 &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)\Gamma(1 - k)}{\Gamma(\lambda - \mu + k)} {}_{p+1}\Gamma_{q+1} \left[\begin{matrix} (e_1, \rho), 1 - k, (e_i)_2^p; \\ (f_j)_1^q, 1; \end{matrix} av \right]. \tag{3.18}
 \end{aligned}$$

For example, by substituting $p = 2, q = 1$, and $p = q = 1$ into (3.17) and (3.18), we obtain the following results involving incomplete Gauss's and Kummer's hypergeometric functions:

Example 3.13. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & \frac{\Gamma(1 - \lambda)}{\Gamma(1 - \lambda + \mu)} {}_3\Gamma_2 \left[\begin{matrix} (e_1, \rho), (1 - \lambda), e_2; \\ f_1, 1 - \lambda + \mu; \end{matrix} \begin{matrix} av \\ \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)\Gamma(k)} {}_3\Gamma_2 \left[\begin{matrix} (e_1, \rho), 0, e_2; \\ f_1, k; \end{matrix} \begin{matrix} av \\ \end{matrix} \right]. \end{aligned} \tag{3.19}$$

Example 3.14. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & \frac{\Gamma(\lambda - \mu)}{\Gamma(\lambda)} {}_3\Gamma_2 \left[\begin{matrix} (e_1, \rho), \lambda - \mu, e_2; \\ f_1, \lambda; \end{matrix} \begin{matrix} av \\ \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)\Gamma(1 - k)}{\Gamma(\lambda - \mu + k)} {}_3\Gamma_2 \left[\begin{matrix} (e_1, \rho), 1 - k, e_2; \\ f_1, 1; \end{matrix} \begin{matrix} av \\ \end{matrix} \right]. \end{aligned} \tag{3.20}$$

Example 3.15. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$ and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & \frac{\Gamma(1 - \lambda)}{\Gamma(1 - \lambda + \mu)} {}_2\Gamma_2 \left[\begin{matrix} (e_1, \rho), (1 - \lambda); \\ f_1, 1 - \lambda + \mu; \end{matrix} \begin{matrix} av \\ \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)\Gamma(k)} {}_2\Gamma_2 \left[\begin{matrix} (e_1, \rho), 0; \\ f_1, k; \end{matrix} \begin{matrix} av \\ \end{matrix} \right]. \end{aligned} \tag{3.21}$$

Example 3.16. Let $\lambda \geq 1, \sigma > 0, k \in \mathbb{N}, m \in \mathbb{N}$, and $m - 1 \leq \Re(\mu) \leq m$, then the following results holds:

$$\begin{aligned} & \frac{\Gamma(\lambda - \mu)}{\Gamma(\lambda)} {}_2\Gamma_2 \left[\begin{matrix} (e_1, \rho), \lambda - \mu; \\ f_1, \lambda; \end{matrix} \begin{matrix} av \\ \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)\Gamma(1 - k)}{\Gamma(\lambda - \mu + k)} {}_2\Gamma_2 \left[\begin{matrix} (e_1, \rho), 1 - k; \\ f_1, 1; \end{matrix} \begin{matrix} av \\ \end{matrix} \right]. \end{aligned} \tag{3.22}$$

Theorems (2.3) and Theorem (2.4) may also have special instances that can be deduced.

4 Conclusions

In this research paper, we have applied the Leibniz rule for the Riemann-Liouville type fractional derivatives to develop certain expansion equations for incomplete Yang Y -functions. Special cases of our main results, presented in Section 3, demonstrate their broader applicability to various special functions such as the H -function, incomplete Meijer's G -function hypergeometric functions and the Fox-Wright function under appropriate parameter conditions. This demonstrates how the findings are flexible and widely applicable in the field of special functions and their extensions.

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