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Analysis of Infectious Fractional Disease Model

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Abstract In this paper, we have given the $\mathfrak{SIHR}$ (Susceptible, Infected, Hospitalized and Recovered) model for infectious disease by using the Caputo fractional operator. Fractional order modeling's main advantage is that it examines the model's memory and non-local impacts. In this, we discuss the model's positivity and boundedness in terms of Caputo-fractional order derivatives. The disease-free and endemic equilibria are described in order to make the proposed model more efficient. To examine the sensitivity, global stability and local stability parameters, the fundamental reproduction number (R_0) is also computed. The main advantage of this study is that we describe the fundamental aspects of the new fractional order model in order to evaluate its sensitivity and stability.

1 Introduction

In the past and even in the present, infectious illness outbreaks have consistently been among the major causes of death. The economic and social well-being of many countries, particularly those with lower incomes, is significantly impacted by infectious diseases. Rural residents are subject to an increased chance of death as a result of inadequate medical facilities. Many newly developing and reemerging infectious diseases in the population spread through direct contact between susceptible and infected individuals. Alternative models of epidemics have been proposed all around the world by the authors. Several mathematical models that have been proposed for healthcare facilities do not adequately handle the impact of hospital bed scarcity during an expanding epidemic. Njankou et al. [1] investigated the dynamics of Ebola virus transmission when hospital beds are limited. Therefore, understanding the importance of easily accessible permanent hospital beds with limited medical intervention facilities is the aim of our research. We examine the effects of treatment plans and hospital bed availability within the constraints of restricted resources during the outbreak using a segmented model of a susceptible, infected, hospitalized and recovered population. Mathematical modeling is a useful field that enables us to use mathematical equations to express a variety of natural and scientific processes. Fractional calculus modeling in mathematics has been widely employed in recent years to investigate the dynamic properties of various biological and physical systems. More precise model explanations are often found in fractional calculus.

Fractional modeling is one of the many mathematical models frequently employed in the study of infectious illnesses. Because fractional models can capture more complex dynamics than typical integer-order models, they are becoming increasingly significant in the study of communicable diseases. The following are the primary justifications for why fractional models are necessary in infectious disease modeling: Heterogeneous Population Dynamics, Memory and Hereditary Effects, Non-Local Behavior, Parameter Optimization and the Complexity of Modeling. Therefore, fractional contagious disease models are essential for comprehending and treating illnesses with intricate, memory-dependent and non-linear dynamics. They offer a more precise and adaptable framework for examining how infectious illnesses are managed and affect society across time. Brauer et al. [2] A group of derivative equations of non-integer order makes up the fractional derivatives system. Gottfried Wilhelm Leibniz originally presented these in a letter to Guillaume de L'Hôpital. Leibniz

asked whether the first half-derivative (where $\kappa = \frac{1}{2}$) appears in the letter. Among the different types of fractional derivatives that we can recognize by Ross [3] are the Caputo derivative, the Caputo-Fabrizio (CF) derivative and the Atangana-Baleanu (AB) derivative. The dynamic properties of numerous biological and physical models have been thoroughly studied in recent years using fractional calculus mathematical modeling. Numerous disciplines exhibit it, including the fractional-order Newell-Whitehead-Segel equation Iqbal et al. [4], the brain tumor problem, vector-borne illness model Abboubakar et al. [5] and Hussain et al. [6], oxygen diffusion problem Gour et al. [7], HIV infection of CD4+ cells Aavani et al. [8], and Alshammari et al. [9].
There are various kinds of models in mathematics. employed to investigate infectious illnesses, including fractional modeling, is frequently utilized by Brauer et al. [2]. The mathematical modeling of infectious diseases is crucial in epidemiology for examining disease dynamics and proposing disease control strategies. The concept of fractional calculus has been widely applied in recent decades to investigate the dynamics of real-world situations across various scientific fields, as noted by Beard et al. [10]. The kernels of fractional calculus are non-local and non-singular.In order to determine whether employing control techniques was both highly feasible and economically feasible, they looked into how the basic reproduction number (R_0) affected the control strategy. Therefore, in order to reduce disease at the lowest possible cost, we increase hospital beds, treatment alternatives and knowledge as a dynamic control.
The present article consists of ten sections. The paper's introduction and background are presented in Section 1, while Section 2 provides fundamental definitions, significant lemmas and theorems related to fractional calculus. In Section 3, we defined the infectious disease $\mathfrak{SIHR}$ model using the Caputo fractional operator. The positivity and boundedness of the solution to the mathematical model are covered in Section 4. The DFE and EE points are two important equilibrium points that are presented in Section 5. Section 6 discusses persistence. In which all populations will persist and none will finally go extinct. Section 7 discusses stability analysis, which covers both local and global stability. In Section 8, we discuss sensitivity analysis. The solution's discussion and numerical simulation are included in Section 9. In this, we numerically solve the infectious fractional disease model. Our theoretical findings are validated through numerical simulations using MATLAB software. The solution's conclusion is included in Section 10.

2 Preliminaries

In this section, we provide an overview of the definitions of the Riemann-Liouville, Caputo and Caputo-Fabrizio (CF) fractional operators, as well as important lemmas and theorems on fractional calculus that are necessary for this study.
Caputo et al. [11] give the Caputo and Fabrizio fractional order derivative, which has an exponential function in the kernel and has no singularity.
This derivative defined on the following set $H^1(\mathfrak{e},\mathfrak{f}) = [g|g \in L^2(\mathfrak{e},\mathfrak{f})$ and $g' \in L^2(\mathfrak{e},\mathfrak{f})$, where $L^2(\mathfrak{e},\mathfrak{f})$ is the space of square-integrable functions on the interval $(\mathfrak{e},\mathfrak{f})$.
Definition 1: Kilbas et al. [12] Let $g \in H^1(\mathfrak{e},\mathfrak{f})$ and $\kappa \in (0,1)$. Then the fractional derivative of Caputo-Fabrizio is defined as follows

$$^{\mathbb{CF}}D^{\kappa}_{\mathfrak{t}}(g(\mathfrak{t})) = \frac{\mathcal{N}(\kappa)}{1-\kappa}\int_{\mathfrak{e}}^{\mathfrak{t}} g'(x)exp\left[-\kappa\frac{\mathfrak{t}-x}{1-\kappa}\right]dx, \tag{2.1}$$

where $\mathcal{N}(\kappa)$ is a function for normalization that ensures $\mathcal{N}(0) = \mathcal{N}(1) = 1$. However, if g $\notin H^1(\mathfrak{e},\mathfrak{f})$, Consequently, the derivative is described as follows

$$^{\mathbb{CF}}D^{\kappa}_{\mathfrak{t}}(g(\mathfrak{t})) = \kappa\frac{\mathcal{N}(\kappa)}{1-\kappa}\int_{\mathfrak{e}}^{\mathfrak{z}} (g(\mathfrak{t})-g(x))exp\left[-\kappa\frac{\mathfrak{t}-x}{1-\kappa}\right]dx. \tag{2.2}$$

Definition 2: Daftardar [13] Let $\kappa \in (0,1]$ and consider a function $\mathfrak{Y} : [\mathfrak{e},\mathfrak{f}] \to \mathbb{R}$. The left- and right-sided

Riemann–Liouville fractional integrals of order $\kappa > 0$ are defined as follows.

$$ {}_{\mathfrak{e}}I_{\mathbb{t}}^{\kappa}(\mathfrak{Y}(\mathbb{t})) = \frac{1}{\Gamma\kappa}\int_{\mathfrak{e}}^{\mathbb{t}}(\mathbb{t}-\mathfrak{s})^{\kappa-1}\mathfrak{Y}(\mathfrak{s})d\mathfrak{s} \tag{2.3} $$

and

$$ {}_{\mathbb{t}}I_{\mathfrak{f}}^{\kappa}(\mathfrak{Y}(\mathbb{t})) = \frac{1}{\Gamma\kappa}\int_{\mathbb{t}}^{\mathfrak{f}}(\mathfrak{s}-\mathbb{t})^{\kappa-1}\mathfrak{Y}(\mathfrak{s})d\mathfrak{s}, \tag{2.4} $$

where $\mathfrak{e} < \mathbb{t} \leq \mathfrak{f} < \infty$ and $\Gamma(.)$ is the gamma function of Euler.

Definition 3: Teodoro et al. [14] Let $\kappa \in (0,1]$ and suppose $\mathfrak{Y} : [\mathfrak{e},\mathfrak{f}] \to \mathbb{R}$ is a given function. Then, for $\kappa > 0$, the corresponding left- and right-sided Riemann–Liouville Fractional derivatives are defined as follows

$$ {}_{\mathfrak{e}}^{RL}D_{\mathbb{t}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = \frac{d^n}{d\mathbb{t}^n}({}_{\mathfrak{e}}I_{\mathbb{t}}^{n-\kappa}\mathfrak{Y}(\mathbb{t})) = \frac{1}{\Gamma(n-\kappa)}(\frac{d}{dt})^n\int_{\mathfrak{e}\mathbb{t}}(\mathbb{t}-\mathfrak{s})^{n-\kappa-1}\mathfrak{Y}(\mathfrak{s})d\mathfrak{s} \tag{2.5} $$

and

$$ {}_{\mathbb{t}}^{RL}D_{\mathfrak{f}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = \frac{d^n}{d\mathbb{t}^n}({}_{\mathbb{t}}I_{\mathfrak{f}}^{\kappa}\mathfrak{Y}(\mathbb{t})) = \frac{(-1)^n}{\Gamma(n-\kappa)}(\frac{d}{dt})^n\int_{\mathbb{t}}^{\mathfrak{f}}(\mathfrak{s}-\mathbb{t})^{n-\kappa-1}\mathfrak{Y}(\mathfrak{s})d\mathfrak{s}. \tag{2.6} $$

Definition 4: Teodoro et al. [14] Consider a function $\mathfrak{Y}$: $C^n[\mathfrak{e},\mathfrak{f}] \to$ R where $n-1 < \kappa \leq n, n \in \mathfrak{Y}$. Consequently, the following definitions apply to the left and right Caputo fractional derivatives of order $\kappa > 0$, respectively

$$ {}_{\mathfrak{e}}^{\mathbb{C}}D_{\mathbb{t}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = \frac{1}{\Gamma(n-\kappa)}\int_{\mathfrak{e}}^{\mathbb{t}}(\mathbb{t}-\mathfrak{s})^{n-\kappa-1}\mathfrak{Y}^{(n)}(\mathfrak{s})d\mathfrak{s}, \tag{2.7} $$

$$ {}_{\mathbb{t}}^{\mathbb{C}}D_{\mathfrak{f}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = \frac{1}{\Gamma(n-\kappa)}\int_{\mathbb{t}}^{\mathfrak{f}}(\mathfrak{s}-\mathbb{t})^{n-\kappa-1}\mathfrak{Y}^{(n)}(\mathfrak{s})d\mathfrak{s}. \tag{2.8} $$

Theorem 1: For $\kappa \in (0,1]$ and $\mathfrak{Y} : \mathscr{R}^+ \times \mathscr{R}^n \to \mathscr{R}^n$. consider the form of a fractional order system

$$ {}^{\mathbb{C}}D_{\mathbb{t}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = f(\mathbb{t},\mathfrak{Y}(\mathbb{t})), $$
$$ \mathfrak{Y}(0) = \mathfrak{Y}_{\mathfrak{o}}. \tag{2.9} $$

The equilibrium points of the fractional order system are locally asymptotically stable if all eigenvalues $\lambda_i (i = 1,...,n)$ of the Jacobian matrix $\frac{\partial f}{\partial \mathfrak{Y}}$ computed at the equilibrium points of the fractional order system satisfy $|\arg(\lambda_i)| > \frac{\kappa\pi}{2}$.

Theorem 2: Assume that $\forall\ \mathbb{t} > 0, \mathfrak{Y} : [0,\mathfrak{f}] \to R$ is a function such that $n-1 < \kappa \leq n$. The relationship between Caputo fractional derivatives and Riemann-Liouville fractional derivatives is always as follows

$$ {}_{0}^{RL}D_{\mathbb{t}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = {}_{\mathfrak{e}}^{\mathbb{C}}D_{\mathbb{t}}^{\kappa}\mathfrak{Y}(\mathbb{t})) + \sum_{\kappa=0}^{n-1}\frac{\mathfrak{Y}^{(\kappa)}(0)}{\Gamma(\alpha-\kappa+1)}\mathbb{t}^{(\alpha-\kappa)} \tag{2.10} $$

and

$$ {}_{\mathbb{t}}^{RL}D_{\mathfrak{f}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = {}_{\mathbb{t}}^{\mathbb{C}}D_{\mathfrak{f}}^{\kappa}(\mathfrak{Y}(\mathbb{t})) + \sum_{\kappa=0}^{n-1}\frac{\mathfrak{Y}^{(\alpha)}(\mathfrak{f})}{\Gamma(\alpha-\kappa+1}(\mathfrak{f}-\mathbb{t})^{(\alpha-\kappa)}. \tag{2.11} $$

Thus, if $\mathfrak{Y}(0) = \mathfrak{Y}^{'}(0) = ... = \mathfrak{Y}^{(n-1)}(0) = 0$, then ${}_{0}^{RL}D_{\mathbb{t}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = {}_{\mathfrak{e}}^{\mathbb{C}}D_{\mathbb{t}}^{\kappa}(\mathfrak{Y}(\mathbb{t}))$
and if $\mathfrak{Y}(\mathfrak{f}) = \mathfrak{Y}^{'}((\mathfrak{f})) = ... = \mathfrak{Y}^{(n-1)}((\mathfrak{f})) = 0$, then ${}_{\mathbb{t}}^{RL}D_{\mathfrak{f}}^{\kappa}\mathfrak{Y}(\mathbb{t}) = {}_{\mathbb{t}}^{\mathbb{C}}D_{\mathfrak{f}}^{\kappa}(\mathfrak{Y}(\mathbb{t}))$.

Lemma 1 : Ameen et al. [15] Suppose that $\mathfrak{Y} : [0,\mathbb{T}] \to R$ and $0 < \kappa \leq 1$, thus

$$ {}_{\mathbb{t}}^{\mathbb{C}}D_{\mathbb{T}}^{\kappa}(\mathfrak{Y}(\mathbb{t})) = {}_{0}^{\mathbb{C}}D_{\mathbb{T}}^{\kappa}(\mathfrak{Y}(\mathbb{T}-\mathbb{t})). \tag{2.12} $$

Lemma 2: Li et al. [16] If $\mathfrak{V}(\mathbb{t}) \in \mathscr{R}^+$ is a differentiable function then for any $\mathbb{t} < 0$

$$ {}^{\mathbb{C}}D_{\mathbb{t}}^{\kappa}[\mathfrak{V}(\mathbb{t}) - \mathfrak{V}^* - \mathfrak{V}^* ln\frac{\mathfrak{V}(\mathbb{t})}{\mathfrak{V}^*}] \leq (1-\frac{\mathfrak{V}^*}{\mathfrak{V}(\mathbb{t})})^{\mathbb{C}}D_{\mathbb{t}}^{\kappa}\mathfrak{V}(\mathbb{t}), \mathfrak{V}^* \in \mathscr{R}^+, \forall\kappa \in (0,1). \tag{2.13} $$

In the next section, we present the fractional infectious disease compartment model $\mathfrak{SIHR}$ along with its flow chart. Also, we discuss the parameters, biological description and values of the parameters and sources in Table 1.

3 Mathematical Model

This section utilizes the Caputo fractional order derivative to develop a non-linear, four-dimensional infectious disease model. The symbol for the overall population, represented as $\mathfrak{N}(\mathbb{t})$, is separated into four compartments s.t. the susceptible class $\mathfrak{S}(\mathbb{t})$, the infected class $\mathfrak{I}(\mathbb{t})$, the hospitalized class, $\mathfrak{H}(\mathbb{t})$ and the recovered class $\mathfrak{R}(\mathbb{t})$. Susceptible people $\mathfrak{S}(\mathbb{t})$, who are at risk of contracting the virus but have not yet contracted it. $\mathfrak{I}(\mathbb{t})$ stands for infected individuals, who are presently infected and able to spread the illness. People in hospitals $\mathfrak{H}(\mathbb{t})$ stands for the number of hospitalized and self-isolated infected patients receiving treatment. Those receiving treatment, living alone and ceasing to socialize are included in this group. People in the recovery class $\mathfrak{R}(\mathbb{t})$ are not susceptible or contagious. Consequently, $\mathfrak{N}(\mathbb{t}) = \mathfrak{S}(\mathbb{t}) + \mathfrak{I}(\mathbb{t}) + \mathfrak{H}(\mathbb{t}) + \mathfrak{R}(\mathbb{t})$ gives the total population of the compartment model.

We proposed the fractional order $\mathfrak{SIHR}$ model of the infectious disease. The main advantage of the proposed fractional order model is that we get the knowledge of the disease dynamic memory effect and the local effect of the proposed model.

$$
\begin{aligned}
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{S} &= \aleph^{\kappa} - \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \gamma^{\kappa})\mathfrak{S} + \varrho^{\kappa}\mathfrak{I},\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{I} &= \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{a}} - \mathfrak{H})\mathfrak{I},\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{H} &= \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{a}} - \mathfrak{H})\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{a}^{\kappa}\mathfrak{H}}{1 + \mathfrak{b}^{\kappa}\mathfrak{H}},\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{R} &= \zeta^{\kappa}\mathfrak{I} + \frac{\mathfrak{a}^{\kappa}\mathfrak{H}}{1 + \mathfrak{b}^{\kappa}\mathfrak{H}} + \gamma^{\kappa}\mathfrak{S} - \varsigma^{\kappa}\mathfrak{R}.
\end{aligned}
\tag{3.1}
$$

along with initial conditions

$$
\mathfrak{S}(0) = \mathfrak{S}_0 \geq 0, \mathfrak{I}(0) = \mathfrak{I}_0 \geq 0, \mathfrak{H}(0) = \mathfrak{H}_0 \geq 0, \mathfrak{R}(0) = \mathfrak{R}_0 \geq 0. \tag{3.2}
$$

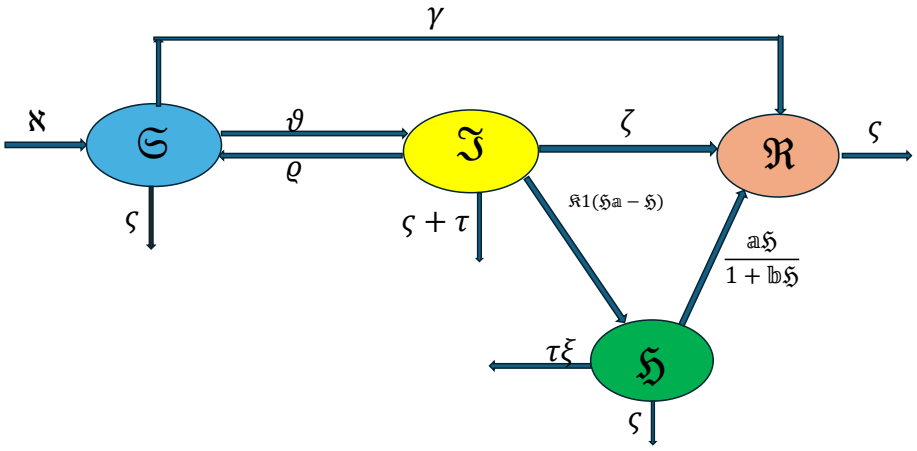

Figure 1: $\mathfrak{SIHR}$-Model

In the next section, we will discuss the positivity and boundedness of the fractional order model (FOM) (3.1). Boundness indicates the restriction of natural growth because of scarce resources, while positivity indicates the population's survival.

Table 1. The list of parameters, biological description, values and sources.

Parameter	Description	Value per day	source
$\aleph$	Constant birth rate	15 people	Kumar et al. [17]
ϑ	Infection rate	0.000455 per person	Kumar et al. [17]
ς	Individuals die	0.01	Kumar et al. [17]
τ	Infected individuals die	0.25	Kumar et al. [17]
ϱ	Treatment cure rate	0.3 per person	Assumed
γ	Individuals are vaccinated	0.02	Assumed
$\mathfrak{H}_{\mathfrak{o}}$	hospital bed capacity	100	Kumar et al. [17]
$\mathfrak{K}_{\mathfrak{1}}$	Occupancy of hospital beds	0.1 per person	Kumar et al. [17]
ζ	Infected individuals recover	0.021	Kumar et al. [17]
ξ	disease-induced death	0.0001	Kumar et al. [17]
$\mathfrak{a}$	Care intensity	0.02	Kumar et al. [17]
$\mathfrak{b}$	Resource insufficiency	0.02 per person	Kumar et al. [17]

4 Positivity and boundedness of solutions of the model

In this section, we prove FOM (3.1), for which we show that all variables are positive and bounded for all $\mathfrak{t} > 0$. However, this compartmental model indicates that $\mathfrak{R}$ does not affect variables $\mathfrak{S}, \mathfrak{I}$ or $\mathfrak{H}$. For a more thorough investigation, consider the following fractional order model:

$$\begin{aligned}
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{S} &= \aleph^{\kappa} - \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \gamma^{\kappa})\mathfrak{S} + \varrho^{\kappa}\mathfrak{I},\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{I} &= \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})\mathfrak{I},\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{H} &= \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{a}^{\kappa}\mathfrak{H}}{1 + \mathfrak{b}^{\kappa}\mathfrak{H}}.
\end{aligned} \tag{4.1}$$

Theorem 3: The closed region $\Gamma = \{(\mathfrak{S}(\mathfrak{t}), \mathfrak{I}(\mathfrak{t}), \mathfrak{H}(\mathfrak{t}), \mathfrak{R}(\mathfrak{t})) : \mathfrak{S}(\mathfrak{t}) \geq 0, \mathfrak{I}(\mathfrak{t}) \geq 0, \mathfrak{H}(\mathfrak{t}) \geq 0, \mathfrak{R}(\mathfrak{t}) \geq 0, \mathfrak{N}(\mathfrak{t}) \leq (\frac{\aleph}{\varsigma})^{\kappa}\}$ is a set of positive invariants in the FOM (4.1) for $\kappa \in (0, 1]$.

Proof: To demonstrate the positivity and boundness of the system FOM (4.1) solution. We possess

$$\begin{aligned}
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{S}(\mathfrak{t})|_{\mathfrak{S}=0} &= \aleph^{\kappa} + \varrho^{\kappa}\mathfrak{I} > 0,\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{I}(\mathfrak{t})|_{\mathfrak{I}=0} &= 0 \geq 0,\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{H}(\mathfrak{t})|_{\mathfrak{H}=0} &= \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{H}_{\mathfrak{o}}\mathfrak{I} > 0.
\end{aligned}$$

Since $\mathfrak{N} = \mathfrak{S} + \mathfrak{I} + \mathfrak{H}$, is the total population, we obtain

$${}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{N}(\mathfrak{t}) = \aleph^{\kappa} - \varsigma^{\kappa}(\mathfrak{S} + \mathfrak{I} + \mathfrak{H}) - \gamma^{\kappa}\mathfrak{S} - (\tau^{\kappa} + \zeta^{\kappa})\mathfrak{I} - \tau^{\kappa}\xi^{\kappa}\mathfrak{H} - \frac{\mathfrak{a}^{\kappa}\mathfrak{H}}{1 + \mathfrak{b}^{\kappa}\mathfrak{H}},$$

$$\begin{aligned}
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{N}(\mathfrak{t}) &\leq \aleph^{\kappa} - \varsigma^{\kappa}(\mathfrak{S} + \mathfrak{I} + \mathfrak{H}),\\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathfrak{t}}\mathfrak{N}(\mathfrak{t}) &\leq \aleph^{\kappa} - \varsigma^{\kappa}(\mathfrak{N}),\\
\mathfrak{N}(\mathfrak{t}) &\leq (\frac{\aleph}{\varsigma})^{\kappa}.
\end{aligned}$$

Here, we clearly obtain $\mathfrak{N}(\mathfrak{t})$ is bounded as $0 < \mathfrak{N}(\mathfrak{t}) \leq (\frac{\aleph}{\varsigma})^{\kappa}$. Therefore, this region is positively invariant. This demonstrates that the model's solution is bound. As a result, FOM (4.1) is both epidemiologically significant and mathematically sound.

The positivity of equilibrium points and the fundamental reproduction number are determined in the next section.

5 Equillibria analysis

Equilibrium Points: The equilibrium points show the dynamics of infection in the given system of disease model. Equilibrium points are of two types: **disease-free equilibrium** and **endemic equilibrium**.

- **Disease-free equilibrium** (DFE) points, which means there is no disease in the population and it typically occurs when the number of infected and hospitalized individuals is zero. In this article, we denote the DFE by $\mathbb{E}_{\mathbb{0}} = (\mathfrak{S}_{\mathfrak{o}}, 0, 0)$, where $\mathfrak{S}_{\mathfrak{o}} = \frac{\aleph^{\kappa}}{\varsigma^{\kappa}+\gamma^{\kappa}}$.
  Now we calculate the reproduction number at DFE point $\mathbb{E}_{\mathbb{0}} = (\mathfrak{S}_{\mathfrak{o}}, 0, 0)$.
  Reproduction Number: It is the average number of secondary infections produced by a single infected individual in a completely susceptible population and denoted by R_0. It provides a measure of how contagious or transmissible a disease is. Here, to calculate the reproduction number R_0, we consider the following disease class equations:

$$
\begin{aligned}
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{I} &= \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})\mathfrak{I}, \\
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{H} &= \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{o}^{\kappa}\mathfrak{H}}{1+\mathbb{b}^{\kappa}\mathfrak{H}}.
\end{aligned} \tag{5.1}
$$

  To calculate R_0 first, we compute the disease-associated term matrix $\mathbb{F}(\mathbb{x})$ and non-disease associated term matrix $\mathbb{V}(\mathbb{x})$ as follows:

$$
\mathbb{F}(\mathbb{x}) = \begin{bmatrix} \vartheta^{\kappa}\mathfrak{S} & 0 \\ 0 & 0 \end{bmatrix},
$$

$$
\mathbb{V}(\mathbb{x}) = \begin{bmatrix} -(\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa}) - \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H}) & \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{I} \\ \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H}) & -\mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa}) - \frac{\mathfrak{o}^{\kappa}}{(1+\mathbb{b}^{\kappa}\mathfrak{H})^2} \end{bmatrix}.
$$

  Also, the Jacobian matrices $\mathbb{F}(\mathbb{x})$ and $\mathbb{V}(\mathbb{x})$ at the DFE point $\mathbb{E}_{\mathbb{0}} = (\mathfrak{S}_{\mathfrak{o}}, 0, 0)$ are:

$$
\mathbb{F} = \begin{bmatrix} \frac{\vartheta^{\kappa}\aleph^{\kappa}}{\varsigma^{\kappa}+\gamma^{\kappa}} & 0 \\ 0 & 0 \end{bmatrix}, \mathbb{V} = \begin{bmatrix} -(\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{H}_{\mathfrak{o}} & 0 \\ \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{H}_{\mathfrak{o}} & -(\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa} + \mathfrak{o}^{\kappa}) \end{bmatrix}
$$

  and

$$
\mathbb{F}\mathbb{V}^{-1} = \begin{bmatrix} \frac{\vartheta^{\kappa}\aleph^{\kappa}}{(\varsigma^{\kappa}+\gamma^{\kappa})((\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa})\mathfrak{I}+\mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})} & 0 \\ 0 & 0 \end{bmatrix}.
$$

  Now, the reproduction number is the spectral radius of the next-generation matrix gives by Driessche et al. [18] and Chowell et al. [19], i.e.

$$
R_0 = \frac{\vartheta^{\kappa}\aleph^{\kappa}}{(\varsigma^{\kappa} + \gamma^{\kappa})(\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa} + \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})}. \tag{5.2}
$$

  The reproduction number indicates that the disease does not spread throughout the population if $R_0 \leq 1$. On the other hand, the disease continues to exist in the population if $R_0 \geq 1$.

- **Endemic equilibrium** (EE) is the state where the disease persists in the population at a constant level. Here we represent the EE as $\mathbb{E}_{\mathbb{1}} = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{H}^*)$.

By equating the right side of system (4.1) to zero, we get the endemic equilibrium (EE) points $\mathbb{E}_{\mathbb{1}} = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{H}^*)$ as follows:

$$
\mathfrak{S}^* = \frac{\aleph^{\kappa} + \varrho^{\kappa}\mathfrak{I}}{\vartheta^{\kappa}\mathfrak{I} + (\varsigma^{\kappa} + \gamma^{\kappa})} \tag{5.3}
$$

$$
\mathfrak{I}^* = \frac{(\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} + \frac{\mathfrak{o}^{\kappa}\mathfrak{H}}{1+\mathbb{b}^{\kappa}\mathfrak{H}}}{\mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})}, \tag{5.4}
$$

and

$$\mathfrak{H}^* = [(\varsigma^\kappa + \tau^\kappa + \zeta^\kappa + \varrho^\kappa + \mathfrak{K}_1{}^\kappa \mathfrak{H}_\mathfrak{o} - \frac{\vartheta^\kappa(\aleph^\kappa + \varrho^\kappa \mathfrak{I})}{\vartheta^\kappa \mathfrak{I} + \varsigma^\kappa + \gamma^\kappa}]\frac{1}{\mathfrak{K}_1{}^\kappa}. \tag{5.5}$$

Now we obtain a cubic equation in $\mathfrak{H}^*$ by substituting the $\mathfrak{I}^*$ value from Eq. (5.4) into Eq. (5.5).

$$\mathscr{M}_1\mathfrak{H}^3 + \mathscr{M}_2\mathfrak{H}^2 + \mathscr{M}_3\mathfrak{H} + \mathscr{M}_4 = 0, \tag{5.6}$$

where

$$\mathscr{M}_1 = \mathfrak{b}^\kappa \vartheta^\kappa \mathfrak{M} - \mathfrak{b}^\kappa \mathfrak{K}_1{}^\kappa \mathfrak{Q},$$

$$\mathscr{M}_2 = \frac{\mathfrak{b}^\kappa \vartheta^\kappa \mathfrak{M}}{\mathfrak{K}_1{}^\kappa}(\mathfrak{E} - \varrho^\kappa \mathfrak{K}_1{}^\kappa) - \mathfrak{b}^\kappa \vartheta^\kappa(\mathfrak{o}^\kappa \mathfrak{H}_\mathfrak{o} + \aleph^\kappa \mathfrak{K}_1{}^\kappa) + \vartheta^\kappa \mathfrak{o}^\kappa + \mathfrak{K}_1{}^\kappa \mathfrak{Q}(2\mathfrak{b}^\kappa \mathfrak{H}_\mathfrak{o} - 1) + \mathfrak{b}^\kappa \mathfrak{Q}\mathfrak{E},$$

$$\mathscr{M}_3 = (\mathfrak{E} + \mathfrak{K}_1{}^\kappa \mathfrak{H}_\mathfrak{o})(\frac{\vartheta^\kappa}{\mathfrak{K}_1{}^\kappa}(\mathfrak{M} + \mathfrak{o}^\kappa) - (\mathfrak{b}^\kappa \mathfrak{H}_\mathfrak{o} - 1)\mathfrak{Q}) + \mathfrak{K}_1{}^\kappa \mathfrak{Q}\mathfrak{H}_\mathfrak{o} + 2\vartheta^\kappa \varrho^\kappa \mathfrak{M} + \vartheta^\kappa \aleph^\kappa \mathfrak{K}_1{}^\kappa(\mathfrak{b}^\kappa \mathfrak{H}_\mathfrak{o} - 1) - \mathfrak{o}^\kappa,$$

$$\mathscr{M}_4 = \mathfrak{Q}(\mathfrak{E} + \mathfrak{K}_1{}^\kappa \mathfrak{H}_\mathfrak{o})(R_0 - 1)$$

and

$$\mathfrak{Q} = \varsigma^\kappa + \gamma^\kappa,$$

$$\mathfrak{E} = \varsigma^\kappa + \tau^\kappa + \zeta^\kappa + \varrho^\kappa,$$

$$\mathfrak{M} = \varsigma^\kappa + \tau^\kappa \xi^\kappa.$$

$\mathfrak{M}, \mathfrak{Q}$ and $\mathfrak{E}$ are always positive for the parameter value that is provided in Table 1. Notably, $\mathscr{M}_1$ is always positive if, $\mathfrak{b}^\kappa \vartheta^\kappa \mathfrak{M} > \mathfrak{b}^\kappa \mathfrak{K}_1{}^\kappa \mathfrak{Q}, \mathscr{M}_4 > 0$ if $R_0 > 1$ and $\mathscr{M}_4 < 0$ if $R_0 < 1$. Every possible outcome of the cubic equation having a specific number of positive roots in accordance with Descartes' sign rule is covered in Table 2.

We talk about persistence in the next section. In mathematical modeling, persistence often refers to a system's

Table 2. Different scenarios illustrating the number of positive solutions of Eq. (5.6).

Cases	$\mathscr{M}_1$	$\mathscr{M}_2$	$\mathscr{M}_3$	$\mathscr{M}_4$	Number of feasible positive roots
	+	+	+	-	1
	+	+	-	-	1
	+	-	-	-	1
$R_0 < 1$	-	+	+	-	2,0
	-	+	-	-	2,0
	+	-	+	-	3,1
	-	-	-	-	0
	-	-	+	-	2,0
	+	+	+	+	0
	+	+	-	+	2,0
	+	-	-	+	2,0
$R_0 > 1$	-	+	+	+	1
	-	-	+	+	1
	-	+	-	+	3,1
	+	-	+	+	2,0
	-	-	-	+	1

or process's capacity to hold onto particular attributes, like population sizes, throughout time or under various circumstances.

6 Persistence

In this part, we now demonstrate the system's persistence using the average Lyapunov function approach [20]. Simply said, persistence means that no population will eventually become extinct and that all populations will endure.The concept of persistence describes how the consequences of previous states or events continue to affect a system's current state even after those states or events have ended.

Theorem 4. The uniform persistence of system (3.1) is guaranteed if $R_0 > 1$.

Proof: The average Lyapunov function for system (3.1) can be viewed as $\beth(\mathfrak{S},\mathfrak{I},\mathfrak{H}) = \mathfrak{S}^{\mathfrak{q}_1}\mathfrak{I}^{\mathfrak{q}_2}\mathfrak{H}^{\mathfrak{q}_3}$, where the constants $\mathfrak{q}_i, i = 1,2,3$ are positive.

We now have

$$\rho(\mathfrak{S},\mathfrak{I},\mathfrak{H}) = \frac{{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}(\beth(\mathfrak{S},\mathfrak{I},\mathfrak{H}))}{\beth(\mathfrak{S},\mathfrak{I},\mathfrak{H})},$$

$$= \mathfrak{q}_1\frac{{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{S}}{\mathfrak{S}} + \mathfrak{q}_2\frac{{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{I}}{\mathfrak{I}} + \mathfrak{q}_3\frac{{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{H}}{\mathfrak{H}},$$

$$= \frac{\mathfrak{q}_1}{\mathfrak{S}}(\aleph^{\kappa} - \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa}+\gamma^{\kappa})\mathfrak{S} + \varrho^{\kappa}\mathfrak{I}) + \frac{\mathfrak{q}_2}{\mathfrak{I}}(\vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})\mathfrak{I})$$

$$+\frac{\mathfrak{q}_3}{\mathfrak{H}}(\mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})\mathfrak{I} - (\varsigma^{\kappa}+\tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{o}^{\kappa}\mathfrak{H}}{1+\mathbb{b}^{\kappa}\mathfrak{H}}).$$

For any choice of $\mathfrak{q}_i > 0$, $i = 1,2,3$, it is necessary to verify that $\rho(\mathfrak{S},\mathfrak{I},\mathfrak{H})$ is positive at each boundary equilibrium. This condition guarantees the uniform persistence of the system.

$$\rho(\mathbb{E}_{\mathbb{0}} = (\frac{\aleph^{\kappa}}{\varsigma^{\kappa}+\gamma^{\kappa}},0,0)) = \mathfrak{q}_1\frac{\varsigma^{\kappa}+\gamma^{\kappa}}{\aleph}[\aleph^{\kappa}-\aleph^{\kappa}] + \mathfrak{q}_2\frac{\vartheta^{\kappa}\aleph^{\kappa}}{\varsigma^{\kappa}+\gamma^{\kappa}}(\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa}+\mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}}) + \mathfrak{q}_3.0,$$

$$\rho(\mathbb{E}_{\mathbb{0}}) = \mathfrak{q}_2(\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa}+\mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})[R_0-1].$$

Thus $\rho(\mathbb{E}_{\mathbb{0}}) > 0$ if $R_0 > 1$.

As a result, system (3.1) is consistently persistent.

In the next section, we analyze stability, including both local and global stability. Stability is essential for dynamic systems to be understood and their behavior predicted. Local stability is crucial for understanding system behavior under minor perturbations, whereas global stability is required to ensure robustness and reliability over the full range of potential scenarios.

7 Stability analysis

This section discusses the analytical criteria for measuring the DFE and EE's stability both locally and globally. Local stability in fractional calculus looks at how the system responds close to an equilibrium point, whereas global stability takes into account how the system behaves across the whole state space and for every initial condition. First, we locate the necessary Jacobian matrix and after that, we ascertain the eigenvalues sign.

$$\mathfrak{J} = \begin{pmatrix} -\vartheta^{\kappa}\mathfrak{I} - (\varsigma^{\kappa}+\gamma^{\kappa}) & -\vartheta^{\kappa}\mathfrak{S} + \varrho^{\kappa} & 0 \\ \vartheta^{\kappa}\mathfrak{I} & -(\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa}+\mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})) & \mathfrak{K}_1{}^{\kappa}\mathfrak{I} \\ 0 & \mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H}) & -\mathfrak{K}_1{}^{\kappa}\mathfrak{I} - (\varsigma^{\kappa}+\tau^{\kappa}\xi^{\kappa}) - \frac{\mathfrak{o}^{\kappa}}{(1+\mathbb{b}^{\kappa}\mathfrak{H})^2} \end{pmatrix}. \tag{7.1}$$

Local Stability

Theorem 5. For $\kappa \in (0,1]$, $\mathbb{E}_{\mathbb{0}} = (\frac{\aleph^{\kappa}}{\varsigma^{\kappa}+\gamma^{\kappa}},0,0)$ is stable under small perturbations in Γ if $R_0 \leq 1$, and otherwise equilibrium losses stability.

Proof: We calculate the matrix $\mathfrak{J}$ at the DFE point $\mathbb{E}_0 = (\frac{\aleph^\kappa}{\varsigma^\kappa+\gamma^\kappa}, 0, 0)$.

$$\mathfrak{J}_{\mathbb{E}_0} = \begin{pmatrix} -(\varsigma^\kappa+\gamma^\kappa) & -\vartheta^\kappa\mathfrak{S}+\varrho^\kappa & 0 \\ 0 & -(\varsigma^\kappa+\tau^\kappa+\zeta^\kappa+\varrho^\kappa+\mathfrak{K}_1{}^\kappa\mathfrak{H}_\mathfrak{o}) & 0 \\ 0 & \mathfrak{K}_1{}^\kappa\mathfrak{H}_\mathfrak{o} & -(\varsigma^\kappa+\tau^\kappa\xi^\kappa+\mathfrak{o}^\kappa) \end{pmatrix}. \tag{7.2}$$

The eigenvalues of Eq. (7.2) are

$$\lambda_1 = -(\varsigma^\kappa+\gamma^\kappa), \lambda_2 = -(\varsigma^\kappa+\tau^\kappa+\zeta^\kappa+\varrho^\kappa+\mathfrak{K}_1{}^\kappa\mathfrak{H}_\mathfrak{o}), \lambda_3 = -(\varsigma^\kappa+\tau^\kappa\xi^\kappa+\mathfrak{o}^\kappa).$$

The above equation contains three eigenvalues with negative real components such that the condition $|\arg(\lambda_i)| = \pi > \frac{\kappa\pi}{2}$ is satisfied for $i = 1, 2, 3$. Therefore according to Theorem (2) we can say, if $R_0 \leq 1$, then $\mathbb{E}_0$ is stable under small perturbations.

Theorem 6. For $\kappa \in (0, 1]$, $\mathbb{E}_1 = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{H}^*)$ is locally asymptotically stable in Γ if $R_0 \geq 1$, and otherwise unstable.

Proof: The Jacobian matrix FOM (3.1) at the endemic equilibrium point $\mathbb{E}_\mathbb{1} = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{H}^*)$ is

$$\mathfrak{J}(\mathbb{E}_\mathbb{1}) = \begin{pmatrix} -\vartheta^\kappa\mathfrak{I}^*-(\varsigma^\kappa+\gamma^\kappa) & -\vartheta^\kappa\mathfrak{S}^*+\varrho^\kappa & 0 \\ \vartheta^\kappa\mathfrak{I}^* & -(\varsigma^\kappa+\tau^\kappa+\zeta^\kappa+\varrho^\kappa+\mathfrak{K}_1{}^\kappa(\mathfrak{H}_\mathfrak{o}-\mathfrak{H}^*)) & \mathfrak{K}_1{}^\kappa\mathfrak{I}^* \\ 0 & \mathfrak{K}_1{}^\kappa(\mathfrak{H}_\mathfrak{o}-\mathfrak{H}^*) & -\mathfrak{K}_1{}^\kappa\mathfrak{I}^*-(\varsigma^\kappa+\tau^\kappa\xi^\kappa)-\frac{\mathfrak{o}^\kappa}{(1+\mathfrak{b}^\kappa\mathfrak{H}^*)^2} \end{pmatrix}. \tag{7.3}$$

The characteristic polynomial of $\mathfrak{J}(\mathbb{E}_\mathbb{1})$ can be written as follows:

$$P(\lambda) = \lambda^3 + \mathfrak{D}_1\lambda^2 + \mathfrak{D}_2\lambda + \mathfrak{D}_3, \tag{7.4}$$

where

$$\mathfrak{D}_1 = (\vartheta^\kappa + \mathfrak{K}_1{}^\kappa)\mathfrak{I}^* + \mathfrak{Q} + \mathfrak{E} + \mathfrak{F} + \mathfrak{M} + \mathfrak{G},$$

$$\mathfrak{D}_2 = \mathfrak{E}(\mathfrak{K}_1{}^\kappa\mathfrak{I}^* + \mathfrak{Q} + \mathfrak{G}) + \mathfrak{F}(\mathfrak{M} + \mathfrak{G} + (\vartheta^\kappa\mathfrak{I}^* + \mathfrak{Q})) + (\vartheta^\kappa\mathfrak{I}^* + \mathfrak{Q})(\mathfrak{K}_1{}^\kappa\mathfrak{I}^* + \mathfrak{M} + \mathfrak{G}) - \vartheta^\kappa\mathfrak{I}^*(\varrho^\kappa - \vartheta^\kappa\mathfrak{S}^*),$$

$$\mathfrak{D}_3 = (\vartheta^\kappa\mathfrak{I}^* + \mathfrak{Q})((\mathfrak{E} + \mathfrak{F})(\mathfrak{K}_1{}^\kappa\mathfrak{I}^* + \mathfrak{M} + \mathfrak{G}) - \mathfrak{K}_1{}^\kappa\mathfrak{I}^*\mathfrak{Q}) + \vartheta^\kappa\mathfrak{I}^*((\varrho^\kappa - \vartheta^\kappa\mathfrak{S}^*)\mathfrak{K}_1{}^\kappa\mathfrak{I}^* + \mathfrak{M} + \mathfrak{G})$$

and

$$\mathfrak{Q} = \varsigma^\kappa + \gamma^\kappa,$$

$$\mathfrak{E} = \varsigma^\kappa + \tau^\kappa + \zeta^\kappa + \varrho^\kappa,$$

$$\mathfrak{F} = \mathfrak{K}_1{}^\kappa(\mathfrak{H}_\mathfrak{o} - \mathfrak{H}^*),$$

$$\mathfrak{M} = \varsigma^\kappa + \tau^\kappa\xi^\kappa,$$

$$\mathfrak{G} = \frac{\mathfrak{o}^\kappa}{(1 + \mathfrak{b}^\kappa\mathfrak{H}^*)^2}.$$

Here, we can see that $\mathfrak{D}_1 > 0, \mathfrak{D}_2 > 0$ if $\mathfrak{E}(\mathfrak{K}_1{}^\kappa\mathfrak{I}^*+\mathfrak{Q}+\mathfrak{G})+\mathfrak{F}(\mathfrak{M}+\mathfrak{G}+(\vartheta^\kappa\mathfrak{I}^*+\mathfrak{Q}))+(\vartheta^\kappa\mathfrak{I}^*+\mathfrak{Q})(\mathfrak{K}_1{}^\kappa\mathfrak{I}^*+\mathfrak{M}+\mathfrak{G}) > \vartheta^\kappa\mathfrak{I}^*(\varrho^\kappa - \vartheta^\kappa\mathfrak{S}^*)$ and $\mathfrak{D}_3 > 0$ if $(\mathfrak{E} + \mathfrak{F})(\mathfrak{K}_1{}^\kappa\mathfrak{I}^* + \mathfrak{M} + \mathfrak{G}) > \mathfrak{K}_1{}^\kappa\mathfrak{I}^*\mathfrak{Q}$ when $R_0 \geq 1$. Thus, the eigenvalues of a matrix $\mathfrak{J}(\mathbb{E}_\mathbb{1})$ have negative real portions in accordance with the Routh-Hurwitz criterion [21]. Therefore $\kappa \in (0, 1)$, the endemic equilibrium point of FOM (3.1) is locally asymptotically stable in Γ.

Global Stability

Now, we examine the global stability at disease-free equilibrium (DFE) and endemic equilibrium (EE) points by using the Lasalle invariance theorem as given by Zhang et al. [22].

Theorem 7. The disease-free equilibrium $\mathbb{E}_0$ is globally asymptotically stable, if $R_0 \leq 1, \mathfrak{K}_1 = 0$ and $\varrho = 0$ hold.

Proof: Here, we used the method outlined by Akuka et al. [23]. Consider a Lyapunov function $\mathfrak{B}(\mathbb{t})$ as follows:

$$\mathfrak{B}(\mathbb{t}) = (\mathfrak{S} - \mathfrak{S}_{\mathfrak{o}} - \mathfrak{S}_{\mathfrak{o}} ln\frac{\mathfrak{S}}{\mathfrak{S}_{\mathfrak{o}}}) + \mathfrak{I} + \mathfrak{H}), \mathfrak{S}_{\mathfrak{o}} = \frac{\aleph^{\kappa}}{\varsigma^{\kappa} + \gamma^{\kappa}}. \tag{7.5}$$

Applying Lemma (2) and the Caputo fractional derivative in the above Eq. (7.5) gives

$${}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{B}(\mathbb{t}) \leq (1 - \frac{\mathfrak{S}_0}{\mathfrak{S}}){}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{S}(\mathbb{t}) + {}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{I}(\mathbb{t}) + {}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{H}(\mathbb{t}). \tag{7.6}$$

When we enter the value from the model (4.1) into the equation above, we obtain

$$\begin{aligned}
{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{B}(\mathbb{t}) &\leq -(\varsigma^{\kappa} + \gamma^{\kappa})\frac{(\mathfrak{S} - \mathfrak{S}_0)^2}{\mathfrak{S}} + \vartheta^{\kappa}\mathfrak{S}_0\mathfrak{I} - \varrho^{\kappa}\mathfrak{I}\frac{\mathfrak{S}_0}{\mathfrak{S}} + \varrho^{\kappa}\mathfrak{I} + \mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}}\mathfrak{I} \\
&\quad -(\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa} + \mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{o}^{\kappa}\mathfrak{H}}{(1 + \mathbb{b}^{\kappa}\mathfrak{H})}, \\
&= -(\varsigma^{\kappa} + \gamma^{\kappa})\frac{(\mathfrak{S} - \mathfrak{S}_0)^2}{\mathfrak{S}} - \varrho^{\kappa}\mathfrak{I}\frac{\mathfrak{S}_0}{\mathfrak{S}} + \varrho^{\kappa}\mathfrak{I} + \mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}}\mathfrak{I} + (R_0 - 1)(\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa} + \mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})\mathfrak{I} \\
&\quad -(\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{o}^{\kappa}\mathfrak{H}}{(1 + \mathbb{b}^{\kappa}\mathfrak{H})}.
\end{aligned} \tag{7.7}$$

Every solution finally tends to the biggest positive invariant subset of the collection $\mathbb{E} = ((\mathfrak{S}, \mathfrak{I}, \mathfrak{H})|{}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{B}(\mathbb{t}) = 0)$, by the Lasalle invariance principle. It follows that

$${}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{B}(\mathbb{t}) < 0 \quad \text{if} \quad R_0 \leq 1,\ \mathfrak{K}_1 = 0, \varrho = 0 \text{ and } \mathfrak{S}, \mathfrak{I}, \mathfrak{H} > 0,$$

while

$${}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{B}(\mathbb{t}) = 0 \quad \text{when} \quad \mathfrak{S} = \mathfrak{S}_{\mathfrak{o}}, \mathfrak{I} = \mathfrak{I}_{\mathfrak{o}} = 0, \mathfrak{H} = \mathfrak{H}_{\mathfrak{o}} = 0.$$

On this set, the states are given by

$$\mathfrak{S}_{\mathfrak{o}} = \frac{\aleph^{\kappa}}{\varsigma^{\kappa} + \gamma^{\kappa}}, \quad \mathfrak{I} = 0, \quad \mathfrak{H} = 0.$$

Therefore, all trajectories asymptotically approach the disease-free equilibrium (DFE).

The rate of hospital bed occupancy and treatment cure rate are zero when the disease does not continue to exist in the system (4.1). As a result, the DFE has global asymptotic stability.

Theorem 8. If the following inequality is true, then the unique endemic equilibrium $\mathbb{E}_{\mathbb{1}} = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{H}^*)$, if it exists, is globally asymptotically stable:

$$(-\frac{(\vartheta^{\kappa}\mathfrak{S}^* + \varrho^{\kappa})}{\mathfrak{S}})^2 < \mathfrak{X}_1\mathfrak{X}_2,$$

$$\frac{\mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})}{\mathfrak{H}} < \mathfrak{X}_2\mathfrak{X}_3,$$

where

$$\mathfrak{X}_1 = \frac{2(\vartheta^{\kappa}\mathfrak{I} + \varsigma^{\kappa} + \gamma^{\kappa})}{\mathfrak{S}},$$

$$\mathfrak{X}_2 = 2R_0((\mathfrak{S} - \mathfrak{S}^*) + \frac{\mathfrak{K}_1^{\kappa}}{\vartheta^{\kappa}}(\mathfrak{H} - \mathfrak{H}^*))$$

and

$$\mathbb{K} = \frac{\aleph^{\kappa}(\mathfrak{I} - \mathfrak{I}^*)}{(\varsigma^{\kappa} + \gamma^{\kappa})(\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa} + \mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})}.$$

Proof: Here, we used the method outlined by Kumar et al. [24]. The Goh–Volterra form Vargas-De-Leon [25] of the Lyapunov function is described as follows:

$$\mathbb{L} = (\mathfrak{S} - \mathfrak{S}^* - \mathfrak{S}^* ln\frac{\mathfrak{S}}{\mathfrak{S}^*}) + \mathbb{K}(\mathfrak{I} - \mathfrak{I}^* - \mathfrak{I}^* ln\frac{\mathfrak{I}}{\mathfrak{I}^*}) + (\mathfrak{H} - \mathfrak{H}^* - \mathfrak{H}^* ln\frac{\mathfrak{H}}{\mathfrak{H}^*}), \tag{7.8}$$

where $\mathbb{K}$ is an appropriately chosen positive constant.
Applying Lemma (2) and the Caputo fractional derivative in the above Eq. (7.8) gives

$$ {}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathbb{L}(\mathbb{t}) \leq (1-\frac{\mathfrak{S}^*}{\mathfrak{S}}){}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{S}(\mathbb{t}) + \mathbb{K}(1-\frac{\mathfrak{I}^*}{\mathfrak{I}}){}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{I}(\mathbb{t}) + (1-\frac{\mathfrak{H}^*}{\mathfrak{H}}){}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathfrak{H}(\mathbb{t}). \tag{7.9} $$

Applying system (4.1) in the above equation, we get

$$ {}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathbb{L}(\mathbb{t}) \leq (1-\frac{\mathfrak{S}^*}{\mathfrak{S}})(\aleph^{\kappa} - \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa}+\gamma^{\kappa})\mathfrak{S} + \varrho^{\kappa}\mathfrak{I}) + \mathbb{K}(1-\frac{\mathfrak{I}^*}{\mathfrak{I}}) $$

$$ (\vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})\mathfrak{I}) + (1-\frac{\mathfrak{H}^*}{\mathfrak{H}})(\mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})\mathfrak{I} - (\varsigma^{\kappa}+\tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{o}^{\kappa}\mathfrak{H}}{1+\mathbb{b}^{\kappa}\mathfrak{H}}). $$

$$ {}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathbb{L}(\mathbb{t}) \leq \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3. $$

The following are the values of $\mathfrak{B}_i's$ by basic algebraic calculations:

$$ \mathfrak{B}_1 = -\frac{(\vartheta^{\kappa}\mathfrak{I}+\varsigma^{\kappa}+\gamma^{\kappa})}{\mathfrak{S}}(\mathfrak{S}-\mathfrak{S}^*)^2 - \frac{(\vartheta^{\kappa}\mathfrak{S}^*+\varrho^{\kappa})}{\mathfrak{S}}(\mathfrak{S}-\mathfrak{S}^*)(\mathfrak{I}-\mathfrak{I}^*), $$

$$ \mathfrak{B}_2 = \mathbb{K}(\vartheta^{\kappa}(\mathfrak{S}-\mathfrak{S}^*)(\mathfrak{I}-\mathfrak{I}^*) + \mathfrak{K}_1^{\kappa}(\mathfrak{H}-\mathfrak{H}^*)(\mathfrak{I}-\mathfrak{I}^*)), $$

$$ \mathfrak{B}_3 = \frac{\mathfrak{K}_1{}^{\kappa}}{\mathfrak{H}}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})(\mathfrak{H}-\mathfrak{H}^*)(\mathfrak{I}-\mathfrak{I}^*) - \frac{(\mathfrak{K}_1{}^{\kappa}\mathfrak{I}+\varsigma^{\kappa}+\tau^{\kappa}\xi^{\kappa})}{\mathfrak{H}}(\mathfrak{H}-\mathfrak{H}^*)^2 - \frac{\mathfrak{o}^{\kappa}(\mathfrak{H}-\mathfrak{H}^*)^2}{\mathfrak{H}(1+\mathbb{b}^{\kappa}\mathfrak{H})(1+\mathbb{b}^{\kappa}\mathfrak{H}^*)}. $$

Then ${}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathbb{L}(\mathbb{t})$ can be written as

$$ {}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathbb{L}(\mathbb{t}) = -\frac{1}{2}\mathfrak{b}_{11}(\mathfrak{S}-\mathfrak{S}^*)^2 + \mathfrak{b}_{12}(\mathfrak{S}-\mathfrak{S}^*)(\mathfrak{I}-\mathfrak{I}^*) - \frac{1}{2}\mathfrak{b}_{22}(\mathfrak{I}-\mathfrak{I}^*)^2 $$

$$ -\frac{1}{2}\mathfrak{b}_{11}(\mathfrak{S}-\mathfrak{S}^*)^2 + \mathfrak{b}_{13}(\mathfrak{S}-\mathfrak{S}^*)(\mathfrak{H}-\mathfrak{H}^*) - \frac{1}{2}\mathfrak{b}_{33}(\mathfrak{H}-\mathfrak{H}^*)^2 $$

$$ -\frac{1}{2}\mathfrak{b}_{22}(\mathfrak{I}-\mathfrak{I}^*)^2 + \mathfrak{b}_{23}(\mathfrak{I}-\mathfrak{I}^*)(\mathfrak{H}-\mathfrak{H}^*) - \frac{1}{2}\mathfrak{b}_{33}(\mathfrak{H}-\mathfrak{H}^*)^2, $$

where

$$ \mathfrak{b}_{11} = \frac{2(\vartheta^{\kappa}\mathfrak{I}+\varsigma^{\kappa}+\gamma^{\kappa})}{\mathfrak{S}}, \mathfrak{b}_{12} = -\frac{(\vartheta^{\kappa}\mathfrak{S}^*+\varrho^{\kappa})}{\mathfrak{S}} $$

$$ \mathfrak{b}_{22} = 2R_0((\mathfrak{S}-\mathfrak{S}^*) + \frac{\mathfrak{K}_1^{\kappa}}{\vartheta^{\kappa}}(\mathfrak{H}-\mathfrak{H}^*)), \mathfrak{b}_{13} = 0, $$

$$ \mathfrak{b}_{23} = \frac{\mathfrak{K}_1{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}}-\mathfrak{H})}{\mathfrak{H}}, \mathfrak{b}_{33} = \frac{2}{\mathfrak{H}}((\mathfrak{K}_1{}^{\kappa}\mathfrak{I}+\varsigma^{\kappa}+\tau^{\kappa}\xi^{\kappa}) + \frac{\mathfrak{o}^{\kappa}}{(1+\mathbb{b}^{\kappa}\mathfrak{H})(1+\mathbb{b}^{\kappa}\mathfrak{H}^*)}). $$

Sufficient requirements for ${}^{\mathbb{C}}_{\mathfrak{e}}D^{\kappa}_{\mathbb{t}}\mathbb{L}(\mathbb{t})$ being negative definite according to the Sylvester criterion are as follows:

$$ \mathfrak{b}_{12}{}^2 < \mathfrak{b}_{11}\mathfrak{b}_{22}, \tag{7.10} $$

$$ \mathfrak{b}_{13}{}^2 < \mathfrak{b}_{11}\mathfrak{b}_{33}, \tag{7.11} $$

$$ \mathfrak{b}_{23}{}^2 < \mathfrak{b}_{22}\mathfrak{b}_{33}. \tag{7.12} $$

It follows that $\mathfrak{b}_{ii} > 0, \forall\, i = 1,2,3,4,5$ since $\mathfrak{X}_i > 0, \forall\, i = 1,2,3,4,5$. Here, $\mathfrak{b}_{13} = 0$ suggests that Eq. (7.11) is true. We can satisfy Eq. (7.10) and (7.12) if we choose $\mathbb{K} = \frac{\aleph^{\kappa}(\mathfrak{I}-\mathfrak{I}^*)}{(\varsigma^{\kappa}+\gamma^{\kappa})(\varsigma^{\kappa}+\tau^{\kappa}+\zeta^{\kappa}+\varrho^{\kappa}+\mathfrak{K}_1{}^{\kappa}\mathfrak{H}_{\mathfrak{o}})}$. This implies that the global asymptotic stability of the endemic equilibrium $\mathbb{E}_{\mathbb{1}} = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{H}^*)$ is present.
The analysis of sensitive parameters, which determines which input parameter is most impacted by the model's output, is covered in the next section. Sensitivity analysis can be used to optimize, quantify uncertainty, and enhance model robustness by determining which factors have the most significant effects on the model's predictions or behavior.

8 Sensitivity analysis

This section examines the sensitivity of the parameters influencing the basic reproduction number R_0, demonstrating that changing one parameter also changes the R_0. In mathematical modeling, sensitivity analysis is a crucial tool. It indicates which input parameters are most affected by the model's output. It allows the researchers to see which parameters in the numerical simulation need more attention. To determine the sensitivity index, we apply the differentiation of the reproduction number approach. Therefore, we can say that when the parameter changes, the sensitivity analysis is utilized to quantify how much the variable changes. To assess each FOM (3.1) we use the normalized forward sensitivity index approach to the parameter's significance about the disease incidence as specified in Rosa et al. [26].

Defination. The parameters for our model (3.1) are determined by the normalized forward sensitivity index of R_0, Chitnis et al. [27].

$$\chi_{\upsilon}^{R_0} = \frac{\upsilon}{R_0}\frac{\partial R_0}{\partial \upsilon}, \tag{8.1}$$

where $\chi_{\upsilon}^{R_0}$ represents the sensitivity index of reproduction number (R_0) concerning any parameter υ.

Thus, the important parameters for altering the fundamental reproduction numbers are indices $\aleph, \vartheta, \varsigma, \tau, \varrho, \gamma, \mathfrak{H}_{\mathfrak{o}}, \mathfrak{K}_{\mathfrak{1}}, \zeta, \xi, \mathfrak{o}$ and $\mathbb{b}$.

By Eq. (5.2) the reproduction number R_0 for order $(\kappa = 1)$ is

$$R_0 = \frac{\vartheta\aleph}{(\varsigma+\gamma)(\varsigma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}})},$$

In the classical case $(\kappa = 1)$, the sensitivity index of R_0 concerning the parameter, $\aleph, \chi_{\aleph}^{R_0}$,is calculated as follows:

$$\chi_{\aleph}^{R_0} = \left[\frac{\partial}{\partial\aleph}\left(\frac{\vartheta\aleph}{(\varsigma+\gamma)(\varsigma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}})}\right)\right]\left(\frac{\varsigma(\varsigma+\gamma)(\varsigma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}})}{\vartheta\aleph}\right) = 1.$$

$$\chi_{\varsigma}^{R_0} = -\frac{\varsigma(2\varsigma+\gamma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}})}{(\varsigma+\gamma)(\varsigma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}})} = -0.33427,$$

$$\chi_{\tau}^{R_0} = -\frac{\tau}{\varsigma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}}} = -0.023629,$$

$$\chi_{\mathfrak{H}_{\mathfrak{o}}}^{R_0} = -\frac{\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}}}{\varsigma+\tau+\zeta+\varrho+\mathfrak{K}_{\mathfrak{1}}\mathfrak{H}_{\mathfrak{o}}} = -0.94517.$$

Similarly, other sensitive parameters $\vartheta, \varrho, \gamma, \mathfrak{K}_{\mathfrak{1}}, \xi, \mathfrak{o}$ and $\mathbb{b}$ can be calculated. Using the parameter values from Table 1, the sensitivity index values of R_0 for compartmental model parameters assessed at DFE for FOM (3.1) are presented in Table 3.

Table 3. Model parameter sensitivity indices for $\kappa = 1$.

Parameter	Value	$\chi_{\upsilon}^{R_0}$
$\aleph$	15	1
ϑ	0.000455	1
ζ	0.021	-0.00019848
$\mathfrak{H}_{\mathfrak{o}}$	100	-0.94517
ς	0.01	-0.33427
τ	0.25	-0.023629
$\mathfrak{K}_{\mathfrak{1}}$	0.001	-0.94517
γ	0.02	-0.666667
ϱ	0.3	-0.02835

By Table 3, we get $\chi_{\aleph}^{R_0}$ = 1 and $\chi_{\vartheta}^{R_0}$ = 1; this shows that the value of R_0 is directly correlated with the parameters $\aleph$ and ϑ. For example, a 5 % increase or decrease in the value of $\aleph$, ϑ, will result in a 5 % increase or

decrease in the value of R_0. Conversely, $\chi_{\gamma}^{R_0}$ = -0.666667, which indicates an inverse relationship between R_0 and the parameter γ. This indicates that for the fractional order κ = 1, a slight percentage increase (decrease) in the value of γ will result in the decrement (increment) in R_0. Similarly, we can see another parameter $\zeta, \mathfrak{H}_{\mathfrak{o}}, \varsigma, \tau, \mathfrak{K}_{\mathfrak{1}}, \varrho$.

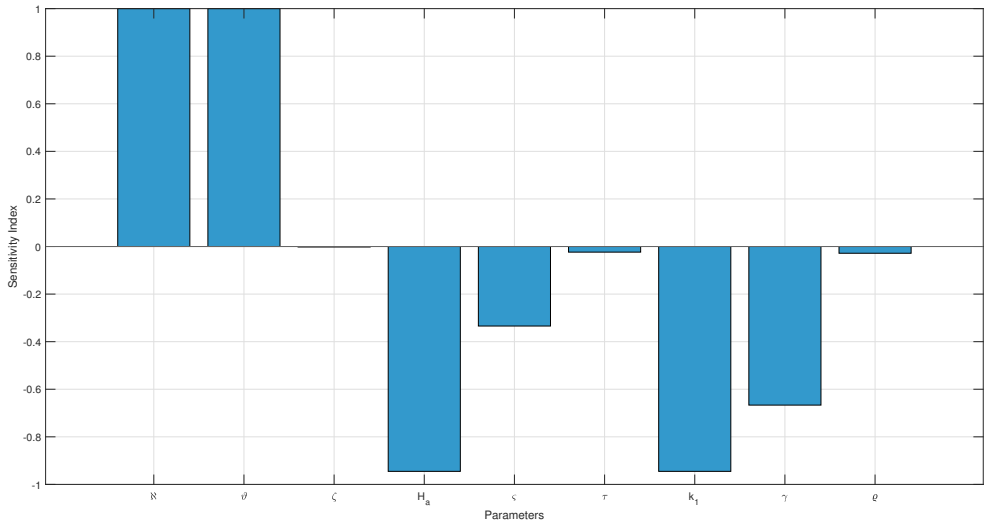

Figure 2: Sensitivity indices of the R_0.

However, the derivative order $\kappa \in (0, 1)$ determines the sensitivity index of FOM (3.1) parameters. The sensitivity index of the model parameters is significantly impacted by the influence of the fractional derivative order $\kappa \in (0, 1)$ on model parameters γ and ϱ. The sensitivity indexes of γ and ϱ show their impact on R_0 concerning κ, being highly sensitive to the fluctuation of κ. Conversely, the other sensitivity indices $\aleph, \vartheta, \tau, \mathfrak{H}_{\mathfrak{o}}, \mathfrak{K}_{\mathfrak{1}}$ and ζ are significantly less susceptible to changes in the fractional-order κ. As the fractional derivative order κ declines, the sensitivity of the fractional order model decreases in absolute value, making it less sensitive than the integer order model.

9 Numerical Simulation and Discussion

In recent years, numerous novel analytical techniques have been developed to resolve the nonlinear fractional derivative models that have been applied as real-world issue models. This section will outline the Euler technique constructed in [28] and then use it to derive numerical solutions for the fractional disease model (3.1) for a range of realistic parameter values and fractional orders.
consider the nonlinear FDE problem

$$
\begin{aligned}
&{}_{\mathbb{t}_{\mathbb{0}}}^{\mathbb{C}} D_{\mathbb{t}}^{\kappa} \mathfrak{Y}(\mathbb{t}) = f(\mathbb{t}, \mathfrak{Y}(\mathbb{t})), 0 \leq \mathbb{t} < \mathbb{T}, \\
&\mathfrak{Y}^{(n)}(0) = \mathfrak{Y}_0^n, n = 0, 1, 2, 3, ..., [\kappa] - 1.
\end{aligned} \tag{9.1}
$$

The numerical scheme for the equation above is constructed as

$$
\mathfrak{Y}_{\mathfrak{n}} = \mathfrak{Y}_{\mathfrak{o}} + \frac{\mathfrak{h}}{\Gamma(\kappa + 1)} \sum_{\mathbb{j}=\mathbb{0}}^{n-1} (\mathfrak{b}_{\mathbb{j},\mathfrak{n}}) f(\mathbb{t}_{\mathbb{j}}, \mathfrak{Y}(\mathbb{t}_{\mathbb{j}})), \tag{9.2}
$$

where $\mathfrak{b}_{\mathfrak{j},n} = (n-\mathfrak{j})^{\kappa} - (n-\mathfrak{j}-1)^{\kappa}$.
The Caputo fractional derivative of FOM (3.1) can be numerically solved as follows:

$$\mathfrak{S}(\mathbb{t}_{\mathfrak{n}}) = \mathfrak{S}_{\mathfrak{o}} + \frac{\mathfrak{h}^{\kappa}}{\Gamma(\kappa+1)} \sum_{\mathfrak{j}=\mathbb{0}}^{n-1} \mathfrak{b}_{\mathfrak{j},\mathfrak{n}} (\aleph^{\kappa} - \vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \gamma^{\kappa})\mathfrak{S} + \varrho^{\kappa}\mathfrak{I}),$$

$$\mathfrak{I}(\mathbb{t}_{\mathfrak{n}}) = \mathfrak{I}_{\mathfrak{o}} + \frac{\mathfrak{h}^{\kappa}}{\Gamma(\kappa+1)} \sum_{\mathfrak{j}=\mathbb{0}}^{n-1} \mathfrak{b}_{\mathfrak{j},\mathfrak{n}} (\vartheta^{\kappa}\mathfrak{S}\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa} + \zeta^{\kappa} + \varrho^{\kappa})\mathfrak{I} - \mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})\mathfrak{I}),$$

$$\mathfrak{H}(\mathbb{t}_{\mathfrak{n}}) = \mathfrak{H}_{\mathfrak{o}} + \frac{\mathfrak{h}^{\kappa}}{\Gamma(\kappa+1)} \sum_{\mathfrak{j}=\mathbb{0}}^{n-1} \mathfrak{b}_{\mathfrak{j},\mathfrak{n}} (\mathfrak{K}_{\mathfrak{1}}{}^{\kappa}(\mathfrak{H}_{\mathfrak{o}} - \mathfrak{H})\mathfrak{I} - (\varsigma^{\kappa} + \tau^{\kappa}\xi^{\kappa})\mathfrak{H} - \frac{\mathfrak{a}^{\kappa}\mathfrak{H}}{1 + \mathfrak{b}^{\kappa}\mathfrak{H}}), \tag{9.3}$$

$$\mathfrak{R}(\mathbb{t}_{\mathfrak{n}}) = \mathfrak{R}_{\mathfrak{o}} + \frac{\mathfrak{h}^{\kappa}}{\Gamma(\kappa+1)} \sum_{\mathfrak{j}=\mathbb{0}}^{n-1} \mathfrak{b}_{\mathfrak{j},\mathfrak{n}} (\delta^{\kappa}\mathfrak{I} + \frac{\mathfrak{a}^{\kappa}\mathfrak{H}}{1 + \mathfrak{b}^{\kappa}\mathfrak{H}} + \gamma^{\kappa}\mathfrak{S} - \varsigma^{\kappa}\mathfrak{R}).$$

The defined model (3.1) was numerically simulated using MATLAB programming. Eq.(9.3), which constructs the numerical simulations, is intended to demonstrate the behaviors for $\kappa = 0.80, 0.90$ and 1.00. For simulation purposes, we take the time range to be [0, 30] and we show the solution graph of FOM (3.1) for step size h = 0.02 with starting population $\mathfrak{S}(0) = 300, \mathfrak{I}(0) = 100$ and $\mathfrak{H}(0) = 70$. These graphs show the stability of the model for the DFE point.
Using the parameters $\aleph = 5, \varrho = 0.2$ and other parameters same as Table 1, we get the disease-free equilibrium point $\mathbb{E}_{\mathbb{0}} = (\mathfrak{S}_{\mathfrak{o}}, \mathfrak{I}_{\mathfrak{o}}, \mathfrak{H}_{\mathfrak{o}}) = (166.66, 0, 0)$ and the fundamental reproduction number $R_0 = 0.77775 < 1$. By applying the parameter values from Table 1 and considering different fractional orders, the eigenvalues of Eq. (7.2) were computed numerically. When $\kappa = 0.80$, the eigenvalues are $\lambda_1 = -0.068853, \lambda_2 = -1.180257$ and $\lambda_3 = -0.0690615$. For $\kappa = 0.90$, the results are $\lambda_1 = -0.045424, \lambda_2 = -0.886408$ and $\lambda_3 = -0.0454967$. At $\kappa = 1.00$, the eigenvalues of the Jacobian matrix $\mathfrak{J}(\mathbb{E}_{\mathbb{0}})$ are $\lambda_1 = -0.03, \lambda_2 = -0.681$ and $\lambda_3 = -0.030025$. Since all eigenvalues are negative in each case, the equilibrium point $\mathbb{E}_{\mathbb{0}}$ is locally asymptotically stable for $\kappa = 0.80$, $\kappa = 0.90$, and $\kappa = 1.00$. Consequently, the system tends toward the disease-free equilibrium, which is illustrated in Figure 3.
Figure 3(a) shows that the susceptible population will be disease free over the time, while figures 3(b) and figure 3(c) show that the infected and hospitalized individuals are becoming extinct with time and 3(d) shows a recovery rate increase with time. In figure 3 the red, blue and green lines are for $\kappa = 0.8, 0.9$ and 1.0, respectively. Also from figure 3, we can observe that the solution graph of the proposed $\mathfrak{SIHR}$ model gives more accurate graph for fractional order and as κ approaches to one, the graph also approaches to the classical order $\kappa = 1$.

In figure 4 we are using the parameters value $\aleph = 10$ and other parameters are used from Table 1. After computation, we get $R_0 = 1.5695 > 1$, which demonstrates that the illness is widespread throughout the population. In figure 4 (a), the susceptible population doesn't go extinct with time and can be reduced when κ is of fraction order $\kappa = 0.8, 0.9$ (red and green lines in graph), although the susceptible population can not vanish for $R_0 > 1$. Similarly, Figure 4(b) shows that when the order of differentiation lowers, the number of infectious individuals reduces as well. As red line shows the minimum infected population for fractional order ($\kappa = 0.8$) and the blue line shows maximum infected population for classical order ($\kappa = 1.0$). The similar observation can be done for 4 (c) and (d).

10 Conclusion

In the current study, we have examined and researched mathematical models and techniques for infectious illness estimates with nonlinear incidence and saturation treatment cure rates. Fractional models have the

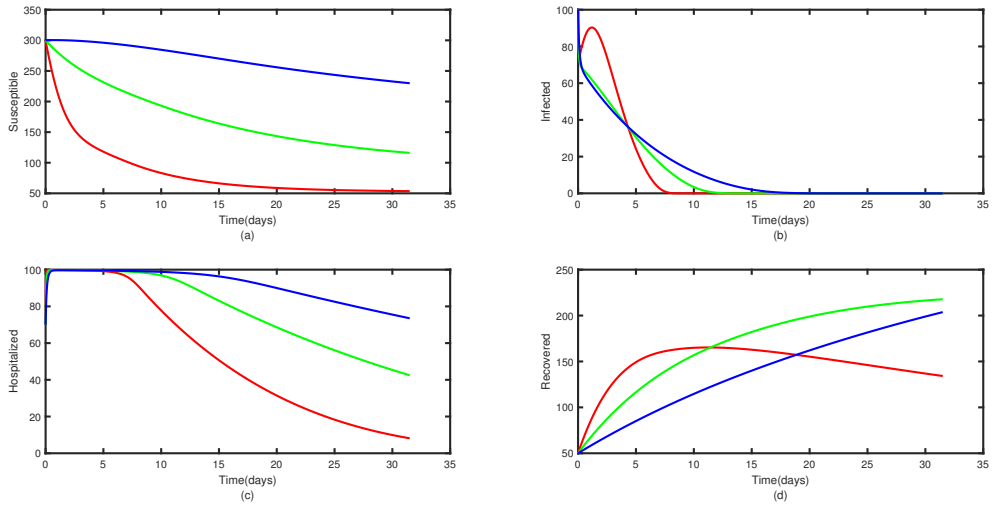

Figure 3: Using parameter values from the evolution of (a) susceptible($\mathfrak{S}$), (b) infected($\mathfrak{I}$), (c) hospitalized($\mathfrak{H}$) and (d) recovered($\mathfrak{R}$) trajectories for FOM (3.1) with varying fractional derivative order κ for $R_0 = 0.77775 < 1$.

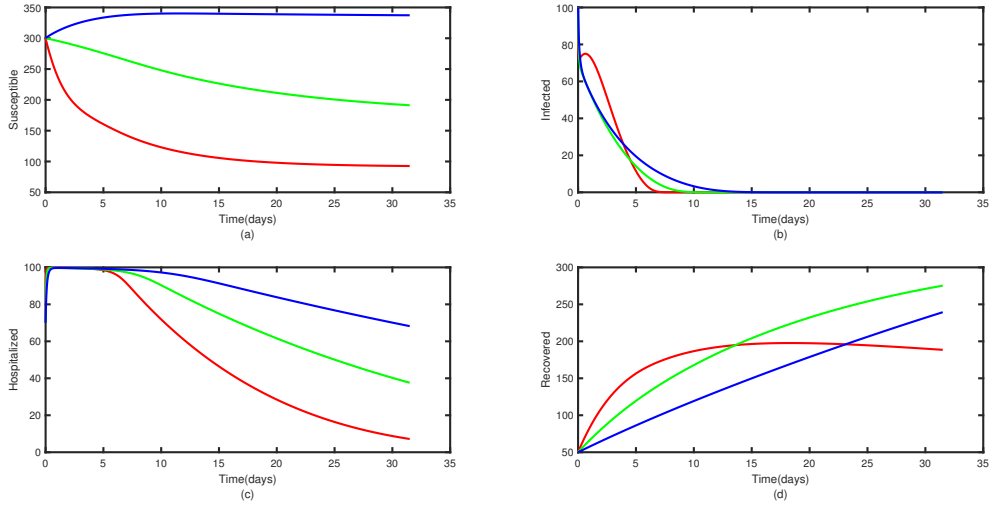

Figure 4: Using parameter values, the dynamic behavior of (a) susceptible($\mathfrak{S}$), (b) infected($\mathfrak{I}$), (c) hospitalized($\mathfrak{H}$) and (d) recovered($\mathfrak{R}$) trajectories for FOM (3.1) with varying fractional derivative order $\kappa = 0.80, 0.90$ and 1 for $R_0 = 1.5547 > 1$.

potential to capture more intricate dynamics in the study of infectious diseases than standard integer-order models. Fractional infectious disease models are therefore crucial for understanding and managing diseases with complex, memory-dependent and nonlinear dynamics. We used a fractional infectious model because it offers a more precise and adaptable framework for researching the spread and management of infectious illnesses. The nonlinear mathematical model of susceptible, infected, hospitalized and recovered classes is examined in this work. The primary benefit of fractional order modeling is that it explores the model's memory and non-local effect. We examined the limited number of hospital beds in the affected area. Knowing whether a solution exists is crucial in all biological models since it establishes the requirements for a non-linear problem's

positivity and ensures the solution is unique. Two non-negative equilibrium points are found in the defined model: a DFE point ($\mathbb{E}_0$) and an EE point ($\mathbb{E}_1$). We calculate R_0 to find how contagious a disease spreads in the population and also use it to estimate the spread of infection. In infectious disease models, the EE $R_0 > 1$ and the DFE $R_0 < 1$ have been examined for local stability because they clarify how solutions respond under various initial conditions. The endemic and disease-free equilibria also contribute to global stability. An infectious disease model's global stability is crucial because it can provide insight into the dynamics of the illness and the circumstances in which it might persist. We calculate sensitive parameters because we can find which input parameter is most affected by the model's output. The behavior of the model is readily visible in the numerical simulation.

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