Palestine Journal of Mathematics

Vol. 15(Special Issue I)(2026) , 281–291

The extended Voigt function with generalized hypergeometric function and their associated properties

Naveen Kumar

Keywords and phrases: Voigt function, Hypergeometric function, Bessel-Maitland function.

Abstract In this study, our aim is to study the generalized extended Voigt function, which involves the Bessel Maitland function and the generalized hypergeometric function. Further, its useful property as series expression and partly bilateral and partly Unilateral of given $\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}$ function. We establish a number of intriguing recurrence relations between the given Voigt function and the Kampé de Fériet function, Srivastava function, and Daoust function, as well as various generating functions that are partially bilateral and partially unilateral.

1 Introduction and definitions

The sets of positive integers, negative integers, and complex numbers, respectively, are denoted by the symbols $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ throughout the work.

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$

The literature on *Special Functions* contains several generalizations of the Gamma function $\Gamma(z)$, the Beta function $B(\alpha,\beta)$, the hypergeometric functions ${}_1F_1$ and ${}_2F_1$, and the generalized hypergeometric functions ${}_rF_s$ with r numerator and s denominator parameters [13, 14] *and* the references cited in each of these papers).
In 1899, Voigt developed and studied the well-known Voigt functions $K(x,y)$ and $L(x,y)$. primarily because they are used in a variety of scientific fields, including plasma physics, statistical communication theory, neutron physics, and astrophysical spectroscopy. However, in neutron reactions, the Doppler broadened Breit-Wigner resonances are nearly identical to the Voigt functions. Furthermore, the functions

$$K(x,y) + iL(x,y)$$

is, except for a numerical factor, identical to the so-called plasma dispersion function which is tabulated by [16] and [17].
A numerical or analytical development of the Voigt functions is necessary for many specified physical situations. For a thorough examination of many mathematical aspects and computational approaches related to the Voigt functions, see, [18, 19, 8, 7, 1, 2, 4, 3]. For the purposes of our present study, we begin by recalling here the following representations due to [20]:

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-yt-\frac{1}{4}t^2} \cos(xt)\, dt \tag{1.1}$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-yt-\frac{1}{4}t^2} \sin(xt)\, dt \tag{1.2}$$

$$(x \in \mathbb{R};\ \ y \in \mathbb{R}^+),$$

so that

$$K(x,y) + iL(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(y-\iota x)t-\frac{1}{4}t^2} \cos(xt)\, dt \tag{1.3}$$

and

$$K(x,y) - iL(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(y-\iota x)t - \frac{1}{4}t^2} \sin(xt)\, dt \tag{1.4}$$

[8] introduced and studied rather systematically a unification(generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form:

$$V_{\mu,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu e^{-yt-\frac{1}{4}t^2} J_\nu(xt)\, dt \tag{1.5}$$

$$(x, y \in \mathbb{R}^+;\ \Re(\mu+\nu) > -1),$$

where it is well known that

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z \tag{1.6}$$

then

$$K(x,y) = V_{\frac{1}{2},\frac{1}{2}}(x,y), \quad L(x,y) = V_{\frac{1}{2},-\frac{1}{2}}(x,y) \tag{1.7}$$

Subsequently, following the work of [8] closely, [21] proposed a unification of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form:

$$\Omega_{\mu,\nu}[x,y,z] = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu e^{-yt-zt^2} J_\nu(xt) dt \tag{1.8}$$

$$(x, y \in \mathbb{R}^+;\ \Re(\mu+\nu) > -1),$$

In fact, it is easily verified by comparing (1.5) and (1.8) that

$$V_{\mu,\nu}(x,y) = (2\sqrt{z})^{\mu+\frac{1}{2}} \Omega_{\mu,\nu}(2x\sqrt{z}, 2y\sqrt{z}, z) \tag{1.9}$$

or, equivalently, that

$$\Omega_{\mu,\nu}[x,y,z] = (2\sqrt{z})^{-\mu-\frac{1}{2}} V_{\mu,\nu}\left(\frac{x}{2\sqrt{z}}, \frac{y}{2\sqrt{z}}\right) \tag{1.10}$$

Further [[5] p. 53, Eq. (1.27)] defined the generalized Voigt function in the following form:

$$\Omega^{\mu}_{\eta,\nu,\lambda}[x,y,z] = \sqrt{\frac{x}{2}} \int_0^\infty t^\eta e^{-yt-zt^2} J^{\mu}_{\nu,\lambda}(xt) dt \tag{1.11}$$

$$(x, y, z, \mu \in \mathbb{R}^+; \Re(\eta+\nu+2\lambda) > -1),$$

where $J^{\mu}_{\nu,\lambda}(z)$ is well-know Bessel- Maitland function defined as follow [6].

$$J^{\mu}_{\nu,\lambda}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{\nu+2\lambda+2m}}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)} \tag{1.12}$$

Setting $z = \frac{1}{4}$ in equation (1.11) retrieves the generalized Voigt function described by Srivastava and Chen [7]. When we put $\lambda = 0$ and $\mu = 1$ in equation (1.11), it reduces to the generalized Voigt function presented by Klusch [6, 26, 25], which further reduces to the Voigt function established by Srivastava and Miller [7]. Recently an extension of Voigt function in the modified form involving the confluent hypergeometric function defined as [24]

$$\Lambda^{\mu,\alpha,\beta}_{\eta,\nu,\lambda}[x,y,z] = \sqrt{\frac{x}{2}} \int_0^\infty t^\eta e^{-zt^2} {}_1F_1(\alpha;\beta;-yt) J^{\mu}_{\nu,\lambda}(xt) dt \tag{1.13}$$

$$(x, y, z, \mu, \alpha, \beta \in \mathbb{R}^+; \Re(\eta+\nu+2\lambda) > -1),$$

where ${}_1F_1(\alpha;\beta;-yt)$ is the confluent hypergeometric function defined as follows [9] where ${}_1F_1(\alpha;\beta;-yt)$ represents the confluent hypergeometric function defined as follows [9]

$${}_1F_1(a;b;z) = \sum_{m=0}^{\infty} \frac{(a)_m}{m!\,(b)_m} z^m, \tag{1.14}$$

If we set $\alpha = \beta$ in (1.13) we get the result defined in (1.11). The Srivastava and Daoust hypergeometric function [23, 22] which is defined by

$$F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}[z_1,\dots,z_n]$$
$$= F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\ [(b'):\phi'];\dots;[(b)^{(n)}:\phi^{(n)}]: \\ [(c):\psi',\dots,\psi^{(n)}],\ [(d'):\delta'];\dots;[(d)^{(n)}:\delta^{(n)}]:\end{matrix}\quad z_1,\dots,z_n\right]$$
$$= \sum_{k_1,\dots,k_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta'_j+\dots+k_n\theta^{(n)}{}_j}}{\prod_{j=1}^{C}(c_j)_{k_1\psi'_j+\dots+k_n\psi^{(n)}_j}} \frac{\prod_{j=1}^{B'}(b'_j)_{k_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b^{(n)}_j)_{k_n\phi^{(n)}_j}}{\prod_{j=1}^{D'}(d'_j)_{k_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d^{(n)}_j)_{k_n\delta^{(n)}_j}} \frac{z_1^{k_1}}{k_n!}\cdots\frac{z_1^{k_1}}{k_n!}, \tag{1.15}$$

the coefficients

(i) $\theta^m_j(j = 1,\dots,A), \phi^m_J(j = 1,\dots,B^{(m)})$,

(ii) $\psi^m_j(j = 1,\dots,C)$,

(iii) $\delta^m_j(j = 1,\dots,D^{(m)})$,

$\forall\ m \in \{1,\dots,n\}$ are real and positive numbers, and (a) represents the array of A parameters $a_1,\dots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters $b^{(m)}_j(j = 1,\dots,B^{(m)}), \forall\ m \in \{1,\dots,n\}$, with corresponding implications similar interpretations for (c) and $(d^m)(m = 1,\dots,n)$. $(y)_a$ is the Pochhammer symbol:

$$(\delta)_a = \frac{\Gamma(\delta + a)}{\Gamma(a)} \quad \delta, a \in \mathbb{C}. \tag{1.16}$$

The multiple series (1.15) converges (absolutely) either $\Delta_i > 0\ \ (i = 1,\dots,n),\ \forall\ z_1,\dots,z_n \in \mathbb{C}$ or $\Delta_i = 0\ \ |z| < \varrho_i\ \ (i = 1,\dots,n)$, and divergent when $\Delta_i < 0$ except for the trival case $z_1 = \dots z_n = 0$, where

$$\Delta_i = 1 + \sum_{j=1}^{C}\psi^{(i)}_j + \sum_{j=1}^{D^{(i)}}\delta^{(i)}_j - \sum_{j=1}^{A}\theta^{(i)}_j - \sum_{j=1}^{B^{(i)}}\phi^{(i)}_j \ \ (i = 1,\dots,n), \tag{1.17}$$

$$\varrho_i = \min_{\mu_1,\dots,\mu_n>0}(E_i)\ \ (i = 1,\dots,n), \tag{1.18}$$

with

$$E_i = (\mu_i)^{1+\sum_{j=1}^{D^{(i)}}\delta^i_j - \sum_{j=1}^{B^{(i)}}\phi^i_j} \frac{\left\{\prod_{j=1}^{C}\left(\sum_{i=1}^{n}\mu\psi^{(i)}_j\right)^{\psi^{(i)}_j}\right\}\left(\prod_{j=1}^{D^{(i)}}(\delta^{(i)}_j)^{\delta^{(i)}_j}\right)}{\left\{\prod_{j=1}^{A}\left\{\sum_{i=1}^{n}\mu\theta^{(i)}_j\right\}^{\theta^{(i)}_j}\right\}\left\{\prod_{j=1}^{B^{(i)}}(\phi^{(i)}_j)^{\phi^{(i)}_j}\right\}}. \tag{1.19}$$

The special instances of (1.15) reduces to the hypergeometric function of one variable and Kampé de Fériet function $F^{p:q;k}_{l':m';n'}$ through the generalized hypergeometric series of two variables are provided as [23, 22]

$$F^{\mathfrak{p}:\mathfrak{q};k}_{l':m';n'}\left[\begin{matrix}(a_{\mathfrak{p}}):(b_{\mathfrak{q}});(c_k); \\ (\alpha_{\mathfrak{p}}):(\beta_{\mathfrak{q}});(\gamma_k);\end{matrix}\ \mathfrak{X},\mathfrak{Y}\right] = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{\mathfrak{p}}(a_j)_{r+s}\prod_{j=1}^{\mathfrak{q}}(b_j)_r\prod_{j=1}^{k'}(c_j)_s}{\prod_{j=1}^{l'}(\alpha_j)_{r+s}\prod_{j=1}^{m'}(\beta_j)_r\prod_{j=1}^{n'}(\gamma_j)_s}\frac{\mathfrak{X}^r}{r!}\frac{\mathfrak{Y}^s}{s!}. \tag{1.20}$$

If $\mathfrak{p}+\mathfrak{q} < l'+m'+1$ and $\mathfrak{p}+k' < l'+n'+1$, this double hypergeometric series is convergent (absolutely) for all values of $\mathfrak{X}$ and $\mathfrak{Y}$. Furthermore, if if $\mathfrak{p}+\mathfrak{q} = l'+m'+1$ and $\mathfrak{p}+k = l'+n'+1$, coupled with any one of the following sets of conditions:

$$(i)\ \ \mathfrak{p} \le l,\ \max\{|\mathfrak{X}|,|\mathfrak{Y}|\} < 1;$$

$$(ii)\ \ \mathfrak{p} > l,\ \ |\mathfrak{X}|^{\frac{1}{(\mathfrak{p}-l)}} + |\mathfrak{Y}|^{\frac{1}{(\mathfrak{p}-l)}} < 1.$$

2 Generalized Extended Voigt function

Here we derive the new generalization of extended Voigt function by involving the generaliged extended hypergeometric function defined as

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}[x,y,z] = \sqrt{\frac{x}{2}} \int_0^\infty t^\eta e^{-zt^2} {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;-yt) J_{\nu,\lambda}^{\mu}(xt)dt \quad (2.1)$$

$$(x,y,z,\mu,\alpha_i,\beta_i \in \mathbb{R}^+; \Re(\eta+\nu+2\lambda) > -1), (i = 1,...,p, j = 1,...,q)$$

where $J_{\nu,\lambda}^{\mu}(z)$ is well-know Bessel- Maitland function and ${}_pF_q$ are generalized extended hypergeometric function ([10], chapter 5, eq.2) defined as

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{k=1}^{\infty} \frac{(\alpha_1)_k,\ldots,(\alpha_p)_k}{k!\,(\beta_1)_k,\ldots,(\beta_q)_k} z^k \quad (2.2)$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ signifies the Pochhammer symbol and $\alpha_i \in \mathbb{C}, \beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-, i = \overline{1,p}, j = \overline{1,q}$. The fundamental ratio test, when applied to the power series on the right side of the preceding equation, reveals that

- If $p \le q$; for all finite z, the series converges,
- If $p = q+1$, then for $|z| = 1$ and $|z| > 1$, the series converges and diverges, respectively.
- Additionally, for $p = q+1$, the series (2.2) is
  - If $Re(\kappa) > 0$, then all points are absolutely convergent to the circle $|z| = 1$,

$$\kappa = \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k$$

  - conditionally convergent when $-1 < \Re(\kappa) \neq 0$ for $|z| = 1, x \neq 1$ and
  - If $\Re(\kappa) \le -1$ then divergent for $|z| = 1$
- If $p > q+1$, Only the defined function is used when the series finishes, and the series does not converge until $z = 0$.

It is absolutly convergent series for all $z \in \mathbb{C}$ when $p \le q$. when $p = q = 1$ equation (2.1) equal to known result.

3 Representation in Series form

We use the series representation of the confluent hypergeometric function and the Bessel-Maitland function defined by (1.12) and equation (2.2), respectively, to derive the explicit representation of our extended Voigt function in terms of the familiar special functions of mathematical physics. We obtain by reversing the sequence of summation and integration,

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}[x,y,z] \quad (3.1)$$

$$= \left(\frac{x}{2}\right)^{\nu+2\lambda+\frac{1}{2}} \sum_{m,n=0}^{\infty} \frac{(-1)^m(\alpha_1)_n\ldots(\alpha_p)_n\,(-y)^n(\frac{x}{2})^{2m}}{(\beta_1)_n\ldots(\beta_q)_n\,\Gamma(\lambda+m+1)\,\Gamma(\nu+\lambda+\mu m+1)\,n!}$$

$$\times \int_0^\infty t^{\eta+\nu+2\lambda+2m+n} e^{-zt^2} dt \quad (3.2)$$

Using the following integral, which can be easily deduced from the well-known Euler gamma function,

$$\int_0^\infty t^\lambda e^{-zt^2} dt = \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right) z^{-\left(\frac{\lambda+1}{2}\right)}, \quad (3.3)$$

$$(\Re(z) > 0, \Re(\lambda) > -1),$$

adding the integral in equation (3.2), we get

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}[x,y,z] = \frac{z^{-\left(\frac{\eta+\nu+2\lambda+1}{2}\right)} x^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+\frac{3}{2}}} \times \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n \left(\frac{-x^2}{4z}\right)^{2m} \left(\frac{-y}{\sqrt{z}}\right)^n \Gamma\left(\frac{\eta+\nu+2\lambda+1+2m+n}{2}\right)}{(\beta_1)_n \ldots (\beta_q)_n\, \Gamma(\lambda+m+1)\, \Gamma(\nu+\lambda+\mu m+1)\, n!} \tag{3.4}$$

On splitting the n-series into even and odd terms, we obtain

$$\begin{aligned} \Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}[x,y,z] &= \frac{z^{-\left(\frac{\eta+\nu+2\lambda+1}{2}\right)} x^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+\frac{3}{2}}} \\ &\times \Bigg[\sum_{m,n=0}^{\infty} \frac{(\alpha_1)_{2n} \ldots (\alpha_p)_{2n} \left(\frac{-x^2}{4z}\right)^{m} \left(\frac{y^2}{4z}\right)^n \Gamma\left(\frac{\eta+\nu+2\lambda+1}{2}+m+n\right)}{(\beta_1)_{2n} \ldots (\beta_q)_{2n} \left(\frac{1}{2}\right)_n \Gamma(\lambda+m+1)\, \Gamma(\nu+\lambda+\mu m+1)\, n!} \\ &- \frac{y}{\sqrt{z}} \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_{2n+1} \ldots (\alpha_p)_{2n+1} \left(\frac{-x^2}{4z}\right)^{m} \left(\frac{y^2}{4z}\right)^n \Gamma\left(\frac{\eta+\nu+2\lambda+2}{2}+m+n\right)}{(\beta_1)_{2n+1} \ldots (\beta_q)_{2n+1} \left(\frac{3}{2}\right)_n \Gamma(\lambda+m+1)\, \Gamma(\nu+\lambda+\mu m+1)\, n!} \Bigg] \end{aligned} \tag{3.5}$$

By using the Kampe de Feriet function (1.20) in equation (3.5) is

$$\begin{aligned} \Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}[x,y,z] &= \frac{x^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+\frac{3}{2}} z^P\, \Gamma(\lambda+1)\Gamma(\lambda+\nu+1)} \\ &\times \Bigg[\Gamma(P) F_{0:\mu+1;2q+1}^{1:1;2p} \left[\begin{matrix} P : 1; \triangle\,(2;\alpha_1) \cdots \triangle\,(2;\alpha_p); \\ - : \lambda+1, \triangle\,(\mu;\nu+\lambda+1); \triangle\,(2;\beta_1) \cdots \triangle\,(2;\beta_q), \frac{1}{2}; \end{matrix} \frac{-x^2}{4z\mu^\mu}, \frac{y^2}{4z} \right] \\ &- \frac{\alpha_1 \cdots \alpha_p\, y}{\beta_1 \cdots \beta_q \sqrt{z}} \Gamma\left(P+\frac{1}{2}\right) \\ &\times F_{0:\mu+1;2q+1}^{1:1;2p} \left[\begin{matrix} P+\frac{1}{2} : 1; \triangle\,(2;\alpha_1+1) \cdots \triangle\,(2;\alpha_p+1); \\ - : \lambda+1, \triangle\,(\mu;\nu+\lambda+1); \triangle\,(2;\beta_1+1) \cdots \triangle\,(2;\beta_q+1), \frac{3}{2}; \end{matrix} \frac{-x^2}{4z\mu^\mu}, \frac{y^2}{4z} \right] \Bigg] \end{aligned} \tag{3.6}$$

where $P = \frac{\eta+\nu+2\lambda+1}{2}$ and $\Delta(m;a)$ abbreviates the array of m parameters $\frac{a}{m}, \frac{a+1}{m}, \cdots \frac{a+m+1}{m}, m \geq 1$. For $p = q$, $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_p = \beta_1 = \beta_2 = \cdots = \beta_q$, equation (3.5) reduces to the explicit representation of Voigt function defined by Srivastava et al. [5, p. 55, Eq. (2.4)]. For the same conditions, (3.6) reduces to a slightly modified version of the representation defined by

[5] (i.e., in terms of the parameters μ and λ).

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\underbrace{\alpha,\ldots,\alpha,\alpha,\ldots,\alpha}_{2p=p+q\text{times}}}[x,y,z]$$
$$= \Omega^{\mu}_{\eta,\nu,\lambda}[x,y,z] = \frac{x^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+\frac{3}{2}}z^{P}\,\Gamma(\lambda+1)\Gamma(\lambda+\nu+1)}$$
$$\times\Bigg[\Gamma(P)F^{1:1;0}_{0:\mu+1;1}\left[\begin{matrix}P:1;-;; \\ -:\lambda+1,\triangle\,(\mu;\nu+\lambda+1);\frac{1}{2};\end{matrix}\ \frac{-x^2}{4z\mu^{\mu}},\frac{y^2}{4z}\right]$$
$$-\frac{y}{\sqrt{z}}\Gamma(P+\frac{1}{2})F^{1:1;0}_{0:\mu+1;1}\left[\begin{matrix}P+\frac{1}{2}:1;-;; \\ -:\lambda+1,\triangle\,(\mu;\nu+\lambda+1);\frac{3}{2};\end{matrix}\ \frac{-x^2}{4z\mu^{\mu}},\frac{y^2}{4z}\right]\Bigg] \tag{3.7}$$

$$(x,y,z,\mu\in\mathbb{R}^{+};\Re(\lambda)>-1,\Re(\nu+\lambda)>-1,\Re(P)>0).$$

When $y=0$ and $p=q,\alpha=\alpha_1=\alpha_2=\cdots=\alpha_p=\beta_1=\beta_2=\cdots=\beta_q$ in equation (3.6), we get the result:

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\underbrace{\alpha,\ldots,\alpha,\alpha,\ldots,\alpha}_{2p=p+q\text{times}}}[x,0,z]$$
$$= \Omega^{\mu}_{\eta,\nu,\lambda}[x,0,z] = \frac{x^{\nu+2\lambda+\frac{1}{2}}\Gamma(P)}{2^{\nu+2\lambda+\frac{3}{2}}z^{P}\,\Gamma(\lambda+1)\Gamma(\lambda+\nu+1)}$$
$$\times\,{}_2F_{\mu+1}\left[\begin{matrix}P:1; \\ \lambda+1,\triangle\,(\mu;\nu+\lambda+1);\end{matrix}\ \frac{-x^2}{4z\mu^{\mu}}\right] \tag{3.8}$$

$$(x,z,\mu\in\mathbb{R}^{+};\Re(\lambda)>-1,\Re(\nu+\lambda)>-1,\Re(P)>0).$$

where ${}_pF_q$ is the generalized hypergeometric function [10]

4 Partly bilateral and partly unilateral of $\Lambda^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}_{\eta,\nu,\lambda}$ representation

We proceed by stating the following well-known finding [11]:

$$\exp\left[s+t+\frac{xt}{s}\right]=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{s^m}{m!}\frac{t^p}{p!}{}_1F_1[-p;m+1;x], \tag{4.1}$$

If $s,t,$ and x are replaced by $s\xi^2,t\xi^2,$ and $x\xi^2$, respectively, and both sides of the resultant identity are multiplied by $\xi^{\eta}e^{-z\xi^2}{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;-w\xi)J^{\mu}_{\nu,\lambda}(q\xi)$ and integrating both sides of the final derived identity with regard to ξ ranging from 0 to ∞ and exchanging the summations and integration yields

$$\int_0^{\infty}\xi^{\eta}\exp\left[-\left(z-s-t+\frac{xt}{s}\right)\xi^2\right]{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;-w\xi)J^{\mu}_{\nu,\lambda}(q\xi)d\xi$$
$$=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{s^m}{m!}\frac{t^p}{p!}\int_0^{\infty}\xi^{\eta+2m+2p}e^{-z\xi^2}{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;-w\xi)$$
$$\times J^{\mu}_{\nu,\lambda}(q\xi)\,{}_1F_1[-p;m+1;x\xi^2]\,d\xi. \tag{4.2}$$

On comparing equation (4.2) and equation (2.1), we get

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}[q,w,z-s-t+\frac{xt}{s}]$$
$$=\sqrt{\frac{q}{2}}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{s^m}{m!}\frac{t^p}{p!}\int_0^{\infty}\xi^{\eta+2m+2p}e^{-z\xi^2}{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;-w\xi)$$
$$\times J_{\nu,\lambda}^{\mu}(q\xi){}_1F_1[-p;m+1;x\xi^2]\,d\xi \tag{4.3}$$

Expanding the generalized hypergeometric function ${}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;-w\xi)$ and the Bessel-Maitland function, switching the integration and summations, and then integrating the involved integral with the aid of the known integral formula below.

$$\int_0^{\infty}x^{\delta-1}e^{-\zeta x^2}{}_1F_1(a;b;\omega x^2)dx=\frac{1}{2}\zeta^{-\frac{\delta}{2}}\Gamma\left(\frac{\delta}{2}\right){}_2F_1\left(a;\frac{\delta}{2};b;\frac{\omega}{\zeta}\right) \tag{4.4}$$

using the above result, we obtain

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}\left[q,w,z-s-t+\frac{xt}{s}\right]$$
$$=\frac{q^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+\frac{1}{2}}z^P}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{s^m}{m!}\frac{t^p}{p!}z^{-m-p}\sum_{i,j=0}^{\infty}\frac{(\alpha_1)_i,\cdots,(\alpha_p)_i}{(\beta_1)_i,\cdots,(\beta_q)_i}\frac{\Gamma\left(P+m+p+j+\frac{i}{2}\right)}{\Gamma(\lambda+j+1)\,\Gamma(\nu+\lambda+\mu j+1)\,i!}$$
$$\times\left(\frac{-w}{\sqrt{z}}\right)^i\left(\frac{-q^2}{4z}\right)^j{}_2F_1[-p,P+m+p+j+\frac{i}{2};m+1;\frac{x}{z}] \tag{4.5}$$

where $P=\frac{\eta+\nu+2\lambda+1}{2}$, For $\alpha=\alpha_1=\alpha_2=\cdots=\alpha_p=\beta_1=\beta_2=\cdots=\beta_q$ equation (4.5) reduces to the known result of Srivastava et al. [5, p. 59, Eq. (3.5)]. Now

$$\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}\left[q,w,z-s-t+\frac{xt}{s}\right]$$
$$=\frac{q^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+\frac{1}{2}}z^P\,\Gamma(\lambda+1)\Gamma(\nu+\lambda+1)}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{s}{z}\right)^m}{m!}\frac{\left(\frac{t}{z}\right)^p}{p!}\Bigg[\Gamma(P+m+p)$$
$$\times F_{0:q+1;2;1}^{1:p;1;1}\left[\begin{matrix}(P+m+p:1,1,1):(\alpha_1,2),\cdots,(\alpha_p,2);(1,1);(-p,1);\\ -:(\beta_1,2),\cdots,(\beta_q,2),\left(\frac{1}{2},1\right);(\lambda+1,1),(\nu+\lambda+1,\mu);(m+1,1);\end{matrix}\ \frac{w^2}{4z},\frac{-q^2}{4z},\frac{x}{z}\right]$$
$$\times\frac{w}{\sqrt{z}}\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\Gamma(P+p+m+\frac{1}{2})$$
$$\times F_{0:q+1;2;1}^{1:p;1;1}\left[\begin{matrix}\left(P+m+p+\frac{1}{2}:1,1,1\right):(\alpha_1+1,2),\cdots,(\alpha_p+1,2);(1,1);(-p,1);\\ -:(\beta_1+1,2),\cdots,(\beta_q+1,2),\left(\frac{3}{2},1\right);(\lambda+1,1),(\nu+\lambda+1,\mu);(m+1,1);\end{matrix}\ \frac{w^2}{4z},\frac{-q^2}{4z},\frac{x}{z}\right]\Bigg] \tag{4.6}$$

where $F_{e:f;g;h}^{a:b;c;d}$ is the well-known Srivastava and Daoust function [9].

5 Generating Relations

We provide a collection of (presumably) novel generating functions in this section that are partly bilateral and partly unilateral. By extending the L.H.S. of equation (4.6) with its assistance of equation (3.6), it is possible to derive a generating relation between the Srivastava and Daoust function and the Kampé de Fériet function. We have, in fact.

$$\left(\frac{z}{Z}\right)^{P}\Gamma(P)F^{1:1;2p}_{0:\mu+1;2q+1}\left[\begin{matrix} P:1;\triangle(2;\alpha_1),\ldots,\triangle(2;\alpha_p); \\ -:\lambda+1,\triangle(\mu;\nu+\lambda+1);\triangle(2;\beta_1),\ldots,\triangle(2;\beta_q),\frac{1}{2}; \end{matrix} \frac{-q^2}{4Z\mu^{\mu}},\frac{\omega^2}{4Z}\right]$$

$$\times -\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\frac{\omega}{\sqrt{Z}}\Gamma(P+\frac{1}{2})$$

$$\times F^{1:1;2p}_{0:1;2q+1}\left[\begin{matrix} P+\frac{1}{2}:1;\triangle(2;\alpha_1+1)\cdots\triangle(2;\alpha_p+1); \\ -:\lambda+1,\triangle(\mu;\nu+\lambda+1);\triangle(2;\beta_1+1)\cdots\triangle(2;\beta_q+1),\frac{3}{2}; \end{matrix} \frac{-q^2}{4Z\mu^{\mu}},\frac{\omega^2}{4Z}\right]$$

$$=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{s}{z}\right)^m}{m!}\frac{\left(\frac{t}{z}\right)^p}{p!}\Bigg[\Gamma(P+m+p)$$

$$\times F^{1:p;1;1}_{0:q+1;2;1}\left[\begin{matrix} (P+m+p:1,1,1):(\alpha_1,2),\cdots,(\alpha_p,2);(1,1);(-p,1); \\ -:(\beta_1,2),\cdots,(\beta_q,2),\left(\frac{1}{2},1\right);(\lambda+1,1),(\nu+\lambda+1,\mu);(m+1,1); \end{matrix} \frac{w^2}{4z},\frac{-q^2}{4z},\frac{x}{z}\right]$$

$$\times\frac{w}{\sqrt{z}}\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\Gamma(P+p+m+\frac{1}{2})$$

$$\times F^{1:p;1;1}_{0:q+1;2;1}\left[\begin{matrix} (P+m+p+\frac{1}{2}:1,1,1):(\alpha_1+1,2),\cdots,(\alpha_p+1,2);(1,1);(-p,1); \\ -:(\beta_1+1,2),\cdots,(\beta_q+1,2),\left(\frac{3}{2},1\right);(\lambda+1,1),(\nu+\lambda+1,\mu);(m+1,1); \end{matrix} \frac{w^2}{4z},\frac{-q^2}{4z},\frac{x}{z}\right]\Bigg] \tag{5.1}$$

$$\left(q,w,z,Z,\mu,\alpha_i,\beta_i\in\mathbb{R}^+;\Re(\alpha)>-1,\Re(\nu)>0,\Re(P)>0\right)$$

where $Z=z-s-t+\frac{xt}{s}$ and $P=\frac{\eta+\nu+2\lambda+1}{2}$. On setting $q=0$ in equation (5.1), we obtain a (presumably) new relation between the generalized hypergeometric function and Kampé de Fériet function given by

$$\left(\frac{z}{Z}\right)^{C}\left\{\Gamma(P)_{2p+1}F_{2q+1}\left(\begin{matrix} P,\triangle(2;\alpha_1),\ldots,\triangle(2;\alpha_p); \\ \triangle(2;\beta_1),\ldots,\triangle(2;\beta_q),\frac{1}{2}; \end{matrix} \frac{\omega^2}{4z}\right)\right\}$$

$$-\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\frac{\omega}{\sqrt{Z}}\Gamma\left(P+\frac{1}{2}\right){}_{2p+1}F_{2q+1}\left(\begin{matrix} P+\frac{1}{2},\triangle(2;\alpha_1),\ldots,\triangle(2;\alpha_p); \\ \triangle(2;\beta_1),\ldots,\triangle(2;\beta_q),\frac{1}{2}; \end{matrix} \frac{\omega^2}{4z}\right)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{s}{z}\right)^m}{m!}\frac{\left(\frac{t}{z}\right)^p}{p!}\Bigg[\Gamma(P+m+p)$$

$$\times F^{1:2p;1}_{0:2q+1;1}\left[\begin{matrix} P+m+p;\triangle(2,\alpha_1),\cdots,\triangle(2,\alpha_p);-p; \\ -:\triangle(2,\beta_1),\cdots,\triangle(2,\beta_q),\frac{1}{2};m+1; \end{matrix} \frac{w^2}{4z},\frac{x}{z}\right]$$

$$\times\frac{w}{\sqrt{z}}\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\Gamma(P+p+m+\frac{1}{2})\Gamma\left(P+p+m+\frac{1}{2}\right)$$

$$\times F^{1:p;1;1}_{0:q+1;2;1}\left[\begin{matrix} P+m+p+\frac{1}{2};\triangle(2,\alpha_1+1),\cdots,\triangle(2,\alpha_p+1);-p; \\ -:\triangle(2,\beta_1+1),\cdots,\triangle(2,\beta_q+1),\frac{3}{2};m+1; \end{matrix} \frac{w^2}{4z},\frac{x}{z}\right]\Bigg] \tag{5.2}$$

$$(\omega,x,z,\alpha_i,\beta_j\in\mathbb{R}^+,(i=1,\cdots,p,j=1,\cdots,q);\Re(p)>0)$$

Now setting $w = 0$ in equation (5.2) and replacing P into C, for $p = q, \alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_p = \beta_1 = \beta_2 = \cdots = \beta_q$, $\lambda = 0$ and $\mu = 1$

$$\left(\frac{z}{Z}\right)^{C} = \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{\left(\frac{s}{z}\right)^m}{m!} \frac{\left(\frac{t}{z}\right)^p}{p!} (C)_{m+p}\, {}_2F_1 \left[\begin{matrix} C+m+p, -p; & \frac{x}{z} \\ m+1; & \end{matrix}\right] \tag{5.3}$$

Now, by using the definition of Jacobi polynomials $J_n^{(\alpha,\beta)}$ [10, see p.254]

$$J_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, \alpha+\beta+n+1; & \frac{1-x}{z} \\ \alpha+1; & \end{matrix}\right] \tag{5.4}$$

equation (5) reduces to the following known result of Pathan and Yasmeen [12, p. 242, Eq. (2.2)]

$$\left(\frac{z}{Z}\right)^{C} = \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{\left(\frac{s}{z}\right)^m}{m!} \frac{\left(\frac{t}{z}\right)^p}{p!} (C)_{m+p}\, J_P^{(m,C-1)}(1-x) \tag{5.5}$$

$$(x, z, Z \in \mathbb{R}^+; \Re(C) > 0.$$

The generalized hypergeometric function and Kampé de Fériet function now have the new generating relation shown below if we put $w = 0$ in equation (5.1)

$$\begin{aligned} \left(\frac{z}{Z}\right)^{P} {}_2F_{\mu+1} \left[\begin{matrix} P, 1; & \frac{-q^2}{4Z\mu^\mu} \\ \lambda+1, \triangle(\mu; \nu+\lambda+1); & \end{matrix}\right] &= \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{\left(\frac{s}{z}\right)^m}{m!} \frac{\left(\frac{t}{z}\right)^p}{p!} (P)_{m+p} \\ &\times F^{1:1;1}_{0:\mu+1;1} \left[\begin{matrix} P+m+p : 1; -p; & \frac{-q^2}{4z\mu^\mu}, \frac{x}{z} \\ -; \lambda+1, \triangle(\mu; \nu+\lambda+1); m+1; & \end{matrix}\right] \end{aligned} \tag{5.6}$$

$$(q, x, z, Z \in \mathbb{R}^+; \Re(\lambda) > -1, \Re(\nu) > 0, \Re(P) > 0).$$

Initializing $x = 0$ in equation (5.6), For the generalized hypergeometric function, we obtain the following generating function:

$$\begin{aligned} \left(\frac{z}{Z_1}\right)^{P} {}_2F_{\mu+1} \left[\begin{matrix} P, 1; & \frac{-q^2}{4Z_1\mu^\mu} \\ \lambda+1, \triangle(\mu, \nu+\lambda+1); & \end{matrix}\right] &= \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{\left(\frac{s}{z}\right)^m}{m!} \frac{\left(\frac{t}{z}\right)^p}{p!} (P)_{m+p} \\ &\times {}_2F_{\mu+1} \left[\begin{matrix} P+m+p : 1; & \frac{-q^2}{4z\mu^\mu} \\ \lambda+1, \triangle(\mu, \nu+\lambda+1); & \end{matrix}\right] \end{aligned} \tag{5.7}$$

$$(q, z, Z_1 \in \mathbb{R}^+; \Re(\lambda) > -1, \Re(\nu) > 0, \Re(P) > 0).$$

where $Z_1 = z - s - t$ and $P = \frac{\eta+\nu+1}{2}$. We arrive at the new generating function for the confluent hypergeometric function ${}_1F_1$ by substituting $\lambda = 0$ and $\mu = 1$ into equation (5.7) as follows:

$$\begin{aligned} \left(\frac{z}{Z_1}\right)^{P_1} {}_1F_1 \left(P_1; \nu+1; \frac{-q^2}{4Z_1}\right) &= \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{\left(\frac{s}{z}\right)^m}{m!} \frac{\left(\frac{t}{z}\right)^p}{p!} (P_1)_{m+p} \\ &\times {}_1F_1 \left(P_1+m+p; \nu+1; \frac{-q^2}{4Z}\right) \end{aligned} \tag{5.8}$$

$$(q, z, Z_1 \in \mathbb{R}^+; \Re(\nu) > 0, \Re(P_1) > 0).$$

6 Conclusion

In present paper, we introduce the new generalization of extended Voigt–type function which involving the generaliged extended hypergeometric function and Bessel-Maitland function. Further we defined series representation, Partly bilateral and unilateral of $\Lambda_{\eta,\nu,\lambda}^{\mu,\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q}$ of the given function Finally, we studied generating relations between the Kampé de Fériet function and Srivastava and Daoust function.

Acknowledgements

I sincerely acknowledge Gurugram University for providing the necessary resources, support, and an encouraging environment to carry out this research work.

Conflict of Interest

The authors declare no conflict of interest.

References

[1] R. K. Parmar and R. Agarwal and N. Kumar and S. D. Purohit, *Extended elliptic-type integrals with associated properties and Turán-type inequalities*, Advance in difference equation, V(2021), 2021.

[2] R. Agarwal and N. Kumar and R. K. Parmar and S. D. Purohit, *Fractional calculus operators of the product of generalized modified Bessel function of the second kind*, Commun. Korean Math. Soc. , v(36), 557-573, 2021.

[3] R. Agarwal and N. Kumar and R. K. Parmar and S. D. Purohit, *Some families of the general Mathieu-type series with associated properties and functional inequalities*, Mathematical Methods in the Applied Sciences, v(45), 2132-2150, 2022.

[4] R. Agarwal and N. Kumar and R. K. Parmar and S. D. Purohit, *Operators of fractional integrals and derivatives of the (p,q)- extended function and related Jacobi transforms*, JPS Scientific Publications India, v(09), 2021.

[5] H. M. Srivastava and M. A. Pathan and M. Kamarujjama, *Some unified presentations of the generalized Voigt functions*, Commun. Appl. Anal., v(2), 49-64, 1998.

[6] D. Klusch, *Astrophysical spectroscopy and neutron reactions, integral transforms and Voigt functions*, Astrophys Space Sci., v(175), 229-240 1991.

[7] H. M. Srivastava and M. P. Chen *Some unified presentations of the Voigt functions*, Astrophys Space Sci., v(192), 63-74, 1992.

[8] H. M. Srivastava and E. A. Miller, *A unified presentation of the Voigt functions*, Astrophys Space Sci., v(135), 111-118, 1987.

[9] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press New York , 1987.

[10] E. D. Rainville, *Special functions*, Macmillan Co. New York, 1960.

[11] H. M. Srivastava and M. G. Bin-Saad and M. A. Pathan, *A new theorem on multidimensional generating relations and its applications*, Proc. Jangjeon Math. Soc., v(10), 7-22, 2007.

[12] M. A. Pathan and Yasmeen, *On partly bilateral and partly unilateral generating functions*, J. Aust. Math. Soc. Ser. B., v(28), 240-245, 1986.

[13] M. J. Luo and R. K. Raina, *Extended generalized hypergeometric functions and their applications*, Bull. Math. Anal. Appl., v(05), 2013.

[14] M. A. Ozarslan and E. Ozergin, *Some generating relations for extended hypergeometric functions via generalized fractional derivative operator*, Math. Comput. Modelling, v(52), 1825-1833, 2010.

[15] N. Khan and M. Ghayasuddin and W. A. Khan and T. Abdeljawad and K. S. Nisar, *Further extension of Voigt function and its properties*, Advances in Difference Equations, v(2020), 229, 2020.

[16] H. E. Fettis and J. C. Caslin and K. R. Cramer, *An Improved Tabulation of the Plasma Dispersion Function*, Air force system command wright patterson AFB Ohio, 1971.

[17] B. D. Fried and S. D. Conte, *The Plasma Dispersion function* , Academic Press New York, 1961.

[18] B. H. Armstrong and R. N. Nicholls, *Emission Absorption and Transfer of Radiation in Heated Atmospheres* , Pergamon Press New York, 1961.

[19] H. J. Haubold and R. W. John, *Spectral line profiles and neutron cross sections new results concerning the analysis of Voigt functions*, Astrophys space science, v(65), 477-491, 1979.

[20] F. Reiche, *Uber die Emission Absorption and intensitatsverteilung von Spektrallinien*, Ber. Deutsch. phy. Ges., v(15), 3-21, 1913.

[21] D. Klusch *Astrophysical spectroscopy and Neutron reactions Integral transform and Voigt function*, J. Austral. Math. Soc. Ser. B., v(28), 240-245, 1986.

[22] H. M. Srivastava and R. K. Saxena, *Operators of fractionl integration and their applications*, Appl. Math. Comput., v(118), 01-52, 2001.

[23] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series* , Halsted Press John Wiley and Sons New York Chichester Brisbane and Toronto, 1985.

[24] N. Khan and M. Ghayasuddin and W. A. Khan and T. Abdeljawad and K. S. Nisar, *Further extension of Voigt function and its properties*, Advances in Difference Equations, v(2020), 1-13, 2020.

[25] R. K. Parmar and S. Saravanan, *Extended generalized Voigt-type functions and related bounds*, Journal of Classical Analysis, v(21), 45-56, 2023.

[26] R. K. Parmar, *Bounding inequalities for the generalized Voigt function*, The Journal of Analysis, v(28), 191-197, 2020.

Author information

Naveen Kumar, Department of Mathematics, Gurugram University Gurugram, Haryana-122003, India.
E-mail: naveen@gurugramuniversity.ac.in

WARNING: Title too long for running head.
PLEASE supply a shorter form with \headlinetitle