Palestine Journal of Mathematics

Vol. 15(Special Issue I)(2026) , 292–297

A SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION BY APPLYING NE TRANSFORM METHOD

R. Modi and R. Sharma

AMS MSC 2020:26A33, 34A08.

Keywords and phrases: Fractional calculus, Generalized Mittag-Leffler function, Fractional differential equations, NR Transform .

Abstract Resolution of fractional differential equations is a burgeoning discipline for today's research as these kind of equations appears in various kind of applied fields. In this study, we will use a recognized transformation technique, NE integral Transform Method to the field of fractional differential equations. Some new results are introduced here with their proofs. Also some examples are accomplished for various kinds of problems having Bagley–Torvik equation. The outcomes derived are in good concurrence with the presented in literature already.

1 Introduction

The most captivating field of fractional calculus has tremendous growth in the various areas of research Recently, various events have awaked in fluid mechanics, control theory, psychology, biology, chemistry [1] and many fields of science modeled by the use of derivatives of fractional order. The field of Fractional calculus is significantly about finding the derivatives and integrals of arbitrary order in any mathematical processes whereas the traditional calculus does not capture extensive freedom of action.

There are various approaches of differentiation of fractional orders e.g. Riemann–Liouville, Caputo and Grünwald–Letnikow Approach .Riemann–Liouville fractional derivative is commonly used in fractional order problems, still this method is not appropriate for real world problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [1]. Unlike the Riemann–Liouville approach, which derives its definition from repeated integration, the Grünwald–Letnikow formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

Fractional differential equations made an impression in various fields as engineering, signal processing, physics, biology, control theory and finance. There is different type of techniques to find the solution of fractional differential equations. The most modestly used methods are; Power Series Method [2], Adomian decomposition method (ADM) [4, 5],Variational Iteration Method (VIM) [4, 7] and Integral Transform Techniques Among Integral Transform solution techniques, The most frequently used technique is Laplace transform technique succeeding by other techniques as Fourier transform, Mellin transform, Sumudu transform, Shehu transform [14], Hankel transform, Natural transform, Elzaki transform, Mohand transform, Laplace-Carson transform[11], Aboodh transform, Raj transform[6].There is one more NE integral Transform technique [8] also developed to find solutions of differential equations. This method provides solution in exact form. The Bagley-Torvik equation (BTE) [9] was formulated in 1983 to fulfill the aim of viscoelastically damped structures. After some period the equation is used by applying fractional calculus to study the behavior of real materials [9].This equations is widely used in various problems in applied sciences. To get solutions of fractional order differential equation, the technique of integral transform plays noteworthy role.

Thus, the present study is dedicated towards this goal. The theory of Integral transform is referred to as operational calculus and now it becomes an essential part of the mathematical background which plays an important role for engineers, scientists' mathematicians, physicists and other streams. These methods take measures an effective ways for the solution of many challenges which are occurring in the various fields of science and engineering. It is great tool for solving ordinary differential equations and fractional order differential equations also and has become popular within the scope.

2 Mathematical prefatory

In order to conclude the main scope of this research article, it is worthy of attention to know some basic information about non-integer order calculus and also some knowledge about the NE transform.

2.1 The NE Integral transform [8]

The NE Integral transform is a generalization as in stereotype to Laplace and other operational transforms for functions of exponential order. This integral transform is the intermingling of Natural integral transform and Aboodh integral transform. The real function g(t)$>$ 0 and g(t)=0 for t$<$ 0 is section wise continuous, exponential order and defined in the set B.
For all t$\geq$ D, positive constants D and L exist *such that,* $|g(t)| \leq Le^{(t/n)}$. and function g (t) is of exponential order $1/n$.
Let B be the set of single transformable functions that is-

$$B = \{g(t) \mid \exists L, n_1, n_2 > 0,\ |g(t)| \leq L \exp\left(\frac{t}{n_1}\right) \text{ if } t \in (-1)^i \times [0, \infty[\} \tag{2.1}$$

The NE integral transform denoted by the operator E (.) is defined by the integral equation:

$$E(p, v) = N\{g(t)\} = \frac{1}{pv} \int_0^\infty e^{-\frac{pt}{v}} g(t)\, dt, \quad \left|\frac{v}{p}\right| < q \tag{2.2}$$

$$E(p, v) = N\{g(t)\} = \frac{1}{p} \int_0^\infty e^{-pt} g(vt)\, dt \tag{2.3}$$

2.2 The NE transform of $(n+1)^{th}$ derivatives of g(t) [8].

$$\begin{aligned} N\left[g^{n+1}(t)\right] &= E_{n+1}(p, v) \\ &= \frac{pE_n(p, v)}{v} - \frac{g_n(0)}{vp} \end{aligned} \tag{2.4}$$

$$E_{n+1}(p, v) = \frac{p^{n+1}}{v^{n+1}} E(p, v) - \sum_{i=0}^{n} \frac{p^{n-(i+1)}}{v^{n-i+1}} g^i(0) \tag{2.5}$$

$$N\left[g^n(t)\right] = \frac{p^n E(p, v)}{v^n} - \frac{p^{n-2}}{v^n} g(0) - \frac{p^{n-3}}{v^{n-1}} g'(0) - \cdots - \frac{g^{(n-1)}(0)}{pv} \tag{2.6}$$

2.3 Caputo fractional derivative [1]

The non-integer order derivative known as the Caputo derivative in regards to function h(t) with order $\lambda > 0$ is justified and determined through the integral as given below

$$ {}^cD_{0,t}^{\lambda} h(t) = \frac{1}{\Gamma(r-\lambda)} \int_0^t \frac{h^{(r)}(\mu)}{(t-\mu)^{\lambda+1-r}}\, d\mu, \quad r-1 < \lambda \leq r \in \mathbb{N} \tag{2.7}$$

2.4 Generalized Mittag-Leffler function

A new generalization of Mittag-Leffler function[12] defined by

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(\phi) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{\phi^{\alpha n}}{(\delta)_{pn}} \tag{2.8}$$

Where $\alpha, \beta, \gamma, \delta, \in \mathbb{C}, min\, \{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta)\} > 0, \quad p, q > 0, \quad q \leq \mathrm{Re}(\alpha) + p$

3 Research methodology

This study mainly aims to solve non-integer order Caputo type initial value problems that have fractional-order $\delta > 0$. Considering the perspective, we have derived the NE transorm of generalized Mittaf-Leffler function. We obtained Inverse NE transform formula .After that, this property along with some fundamental concepts are presented in section 2 are used to prove a new theorems that will be used to get the exact solutions for the numerical examples under consideration.

NE integral transform of generalized Mittag-Leffler function.

Theorem 3.1. *If* $\alpha, \beta, \gamma, \delta, \in C; min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0; p, q > 0, q \leq Re(\alpha) + p$ *then the NE transform of the generalized Mittag-Leffler function*

$$NE\left[(t-a)^{\beta+\mu-1}\right] E^{\gamma,\delta,q}_{\alpha,\beta+\mu,p}(t-a)^{\alpha} = e^{-\frac{pa}{v}} \frac{\Gamma(\delta)}{\Gamma(\gamma)} \frac{v^{\beta+\mu-1}}{p^{\beta+\mu+1}} \, {}_2\Psi_1 \left\{ \begin{array}{l} (\gamma, q)\,(1,1) \\ (\delta, m) \end{array} \right| p^{-\alpha} \tag{3.1}$$

Proof. Making use of the definition 2.2 and 2.8, and interchanging the order of summation and integration we get-

$$\text{NE}\left[(t-a)^{\beta+\mu-1}\right] E^{\gamma,\delta,q}_{\alpha,\beta+\mu,p}\left((t-a)^{\alpha}\right) = \frac{1}{pv}\int_0^{\infty} e^{-\frac{pt}{v}} \left[(t-a)^{\beta+\mu-1}\right] E^{\gamma,\delta,q}_{\alpha,\beta+\mu,m}(t-a)^{\alpha} dt$$

$$= \frac{1}{pv}\int_0^{\infty} e^{-\frac{pt}{v}} \left[(t-a)^{\beta+\mu-1}\right] \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta+\mu)} (t-a)^{\alpha n} \frac{1}{(\delta)_{mn}} dt$$

$$= \frac{1}{pv}\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta+\mu)} \frac{1}{(\delta)_{mn}} \int_0^{\infty} e^{-\frac{pt}{v}} \left[(t-a)^{\alpha n+\beta+\mu-1}\right] dt$$

$$= \frac{1}{pv}\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{(\delta)_{mn}} e^{-\frac{pa}{v}} \frac{1}{\left(\frac{p}{v}\right)^{\alpha n+\beta+\mu}}$$

$$= \frac{1}{pv} e^{-\frac{pa}{v}} \left(\frac{v}{p}\right)^{\beta+\mu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{(\delta)_{mn}} \frac{1}{p^{\alpha n}}$$

$$= \frac{1}{pv} e^{-\frac{pa}{v}} \frac{\Gamma(\delta)}{\Gamma(\gamma)} \left(\frac{v}{p}\right)^{\beta+\mu} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\Gamma(\gamma+qn)}{\Gamma(\delta+mn)} \frac{p^{(-\alpha)n}}{n!}$$

$$= e^{-\frac{pa}{v}} \frac{\Gamma(\delta)}{\Gamma(\gamma)} \frac{v^{\beta+\mu-1}}{p^{\beta+\mu+1}} \, {}_2\Psi_1 \left\{ \begin{array}{l} (\gamma, q)\,(1,1) \\ (\delta, m) \end{array} \right| p^{-\alpha}$$

This is the result. □

Theorem 3.2. *If* $\phi > 0$, $j \in \mathbb{R}$, *then we have the inverse NE transform formula:*

$$NE^{-1}\left[\frac{\left(\frac{p}{v}\right)^{\phi-\sigma}}{\left(\frac{p}{v}\right)^{\phi} - j}\right] = \rho^{\sigma-1} E_{\phi,\sigma}\left(j\rho^{\phi}\right), \quad \left|\frac{j}{\left(\frac{p}{v}\right)^{\phi}}\right| < 1 \tag{3.2}$$

Proof.

$$\begin{aligned}
\text{NE}\left[\rho^{\sigma-1}E_{\phi,\sigma}\left(j\rho^{\phi}\right)\right] &= \frac{1}{pv}\int_0^{\infty} e^{-\frac{p\rho}{v}}\rho^{\sigma-1}E_{\phi,\sigma}\left(j\rho^{\phi}\right)\,d\rho \\
&= \frac{1}{pv}\int_0^{\infty} e^{-\frac{p\rho}{v}}\rho^{\sigma-1}\sum_{i=0}^{\infty}\frac{(j\rho^{\phi})^i}{\Gamma(i\phi+\sigma)}\,d\rho, \quad i>0 \\
&= \frac{1}{pv}\sum_{i=0}^{\infty}\frac{j^i}{\Gamma(i\phi+\sigma)}\int_0^{\infty} e^{-\frac{p\rho}{v}}\rho^{\sigma+\phi i-1}\,d\rho \\
&= \frac{1}{pv}\sum_{i=0}^{\infty}\frac{j^i}{\Gamma(i\phi+\sigma)}\frac{\Gamma(i\phi+\sigma)}{\left(\frac{p}{v}\right)^{\sigma+\phi i}} \\
&= \frac{1}{pv\left(\frac{p}{v}\right)^{\sigma}}\sum_{i=0}^{\infty}\frac{j^i}{\left(\frac{p}{v}\right)^{\phi i}} \\
&= \frac{1}{p^{\sigma+1}v^{1-\sigma}}\frac{(\frac{p}{v})^{\phi}}{(\frac{p}{v})^{\phi}-j}, \quad \left|\frac{j}{\left(\frac{p}{v}\right)^{\phi}}\right|<1 \\
&= \frac{1}{pv}\frac{\left(\frac{p}{v}\right)^{\phi-\sigma}}{\left(\frac{p}{v}\right)^{\phi}-j} \\
\text{NE}^{-1}\left[\frac{1}{pv}\frac{\left(\frac{p}{v}\right)^{\phi-\sigma}}{\left(\frac{p}{v}\right)^{\phi}-j}\right] &= \rho^{\sigma-1}E_{\phi,\sigma}(j\rho^{\phi})
\end{aligned}$$

This is the result. □

Now we will solve some fractional differential equations involving caputo fractional differential operator by using NE integral transform technique.

Theorem 3.3. *A solution of the non-homogeneous fractional differential equation (simple harmonic vibration equation) using the NE transform method is given by:*

$$D^{\tau}[U(t)] + \xi^2 U(t) = q, \quad 1 \le \tau \le 2 \tag{3.3}$$

with the initial conditions:

$$\begin{aligned} U(0) &= l_0, \\ U'(0) &= l_1. \end{aligned} \tag{3.4}$$

Proof. By applying NE transform both the side of equation

$$\mathrm{NE}[D^{\tau}[U(t)]] + \xi^2 \mathrm{NE}[U(t)] = q\,\mathrm{NE}(1),$$

$$\left(\frac{p}{v}\right)^{\tau} E(p,v) - \sum_{i=0}^{\tau-1} \frac{p^{\tau-(i+2)}}{v^{\tau-i}} U^{(i)}(0) + \xi^2 E(p,v) = q\frac{1}{p^2},$$

$$\left(\frac{p}{v}\right)^{\tau} E(p,v) - \frac{p^{\tau-2}}{v^{\tau}} U(0) - \frac{p^{\tau-3}}{v^{\tau-1}} U'(0) + \xi^2 E(p,v) = q\frac{1}{p^2},$$

$$\left[\left(\frac{p}{v}\right)^{\tau} + \xi^2\right] E(p,v) = q\frac{1}{p^2} + \frac{p^{\tau-2}}{v^{\tau}} U(0) + \frac{p^{\tau-3}}{v^{\tau-1}} U'(0),$$

$$\left[\left(\frac{p}{v}\right)^{\tau} + \xi^2\right] E(p,v) = q\frac{1}{p^2} + \frac{p^{\tau-2}}{v^{\tau}} l_0 + \frac{p^{\tau-3}}{v^{\tau-1}} l_1,$$

$$\left[\left(\frac{p}{v}\right)^{\tau} + \xi^2\right] E(p,v) = \frac{1}{v^2}\left[\frac{p^{\tau-2}}{v^{\tau-2}} l_0 + \frac{p^{\tau-3}}{v^{\tau-3}} l_1 + \frac{q}{\frac{p^2}{v^2}}\right].$$

$$E(p,v) = \frac{1}{v^2}\left[\frac{\left(\frac{p}{v}\right)^{\tau-2}}{\left(\frac{p}{v}\right)^{\tau} + \xi^2} l_0 + \frac{\left(\frac{p}{v}\right)^{\tau-3}}{\left(\frac{p}{v}\right)^{\tau} + \xi^2} l_1 + \frac{q}{\frac{p^2}{v^2}\left[\left(\frac{p}{v}\right)^{\tau} + \xi^2\right]}\right]$$

$$E(p,v) = \frac{1}{pv}\left[\frac{\left(\frac{p}{v}\right)^{\tau-1}}{\left(\frac{p}{v}\right)^{\tau} + \xi^2} l_0 + \frac{\left(\frac{p}{v}\right)^{\tau-2}}{\left(\frac{p}{v}\right)^{\tau} + \xi^2} l_1 + \frac{q\xi^2}{\frac{p}{v}\xi^2\left[\left(\frac{p}{v}\right)^{\tau} + \xi^2\right]}\right.$$

Now by taking the inverse NE transform method 3.1, we get

$$U(t) = l_0 E_{\tau,1}(-\xi^2 t^{\tau}) + l_1 t E_{\tau,2}(-\xi^2 t^{\tau}) + \frac{q}{\xi^2}\left[1 - E_{\tau}(-\xi^2 t^{\tau})\right]$$

This is the required solution. □

Theorem 3.4. *Consider the non-homogeneous linear fractional order differential equation, named as the Bagley-Torvik equation [10], with* $\delta = \frac{3}{2}$*:*

$$D^2\phi(\tau) + \delta D_{0,\tau}^{3/2}\phi(\tau) + \phi(\tau) = \tau, \quad \textit{where } \phi(0) = 0,\ \phi'(0) = 1 \tag{3.5}$$

Proof. By implementing the NE transform technique on both sides

$$\mathrm{NE}[D^2\phi(\tau)] + \mathrm{NE}[\delta D_{0,\tau}^{3/2}\phi(\tau)] + \mathrm{NE}[\phi(\tau)] = \mathrm{NE}\{\tau\}$$

$$\frac{p^2}{v^2}E(p,v) - \frac{1}{v^2}\phi(0) - \frac{1}{pv}\phi'(0) + \frac{p^{3/2}}{v^{3/2}}E(p,v) - \frac{1}{p^{1/2}v^{3/2}}\phi(0) - \frac{1}{p^{3/2}v^{1/2}}\phi'(0) + E(p,v) = \frac{v}{p^3}$$

$$\left(\frac{p^2}{v^2} + \frac{p^{3/2}}{v^{3/2}} + 1\right) E(p,v) - \frac{1}{pv} - \frac{1}{p^{3/2}v^{1/2}} = \frac{v}{p^3}$$

Rearranging for $E(p,v)$:

$$E(p,v) = \frac{\frac{1}{pv} + \frac{1}{p^{3/2}v^{1/2}} + \frac{v}{p^3}}{\frac{p^2}{v^2} + \frac{p^{3/2}}{v^{3/2}} + 1}$$

$$E(p,v) = \frac{v}{p^3}$$

Now by taking Inverse NE transform method

$$\phi(\tau) = t$$

This is the required solution. □

4 Conclusion:

In this theorem, a new method called the inverse fractional NE integral transform method has been successfully applied to linear fractional differential equation. We proved four theorems related to this method. The resolution of some example shows that the inverse NE transform method is more powerful and efficient for finding exact solutions of linear fractional differential equations.

References

[1] Podlubny, I, *Fractional differential equations: an introduction to fractional derivatives,fractional differential equations ,to methods of their solution ans some of their applications*, Elsevier, **198** , (1999).

[2] Kilbas A. A., Srivastava H. M., and Trujillo J. J., *Theory and Applications of Fractional Differential Equations*,San Diego, Calif, USA, Elsevier., **71**,(2006).

[3] Benedict Barnes, *polynomial integral transform for solving differential equations*. European Journal of Pure and Applied Mathematics , **9(2)** , 140-151, (2016).

[4] El-Sayed A. M. A., Rida S. Z., and Arafa A. A. M., *Exact solutions of fractional-order biological population model*, Communications in Theoretical Physics, **52(6)** , 992–996, (2009).

[5] Safari, M., Ganji, D. D., and Moslemi, M., *Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV-Burgers-Kuramoto equation*, *Computers and Mathematics with Applications*, **58**(11–12), 2091–2097, (2009).

[6] C.Jesuraj and A.rajkumar, *A new modified transform called Raj transform to solve differential equations and problems in engineering and science*,International journal on emerging technologies, **11(2)** , 958-964,(2020).

[7] Wu G.-C. and Lee E. W., *Fractional variational iteration method and its application*. Physics Letters A., **374(25)**), 2506–2509, (2010).

[8] E. M. Xhaferraj, *The New Integral Transform: "NE Transform" and Its Applications* , European Journal of Formal Sciences and Engineering , **6(1)**, 22-34, (2023).

[9] Bagley, R.L., Torvik, *P.J. Fractional Calculus-A different approach to the analysis of viscoelastically damped structures*, AIAA J., **21**, 741-748 , (1983).

[10] Podlubny,I., *Fractional Differential equations*, California,USA:Academic Press, 1,(1990).

[11] Dr. Yaghvendra Kumawat, Mohsin Turuk, and Rashmi Modi, *The Study Of Inverse Laplace-Carson Integral Transform and Solution Of Linear Fractional Differential Equation*, Elementary Education Online. **19(4)**,7470–7477,(2023).

[12] T.O. Salim, Ahmad W. Faraj, *A generalization of Mittag-Leffler function and integral operator associated with fractional calculus*, Journal of fractional calculus and applications , **3(5)**,1-13, (2012).

[13] R.Goyal,M. Purohit, *The Double Integral Transforms Of The Incomplete generalized S- Function*, International Journal of Advanced Science and Technology, **29(7)**, 12728-12734, (2020).

[14] M. Purohit,R.Goyal *Shehu Integral Transform of the S Function and solution of fractional differential equation*, NIU International Journal of Human Rights, **8**,81-84, (2021).

Author information

R. Modi, Department of Mathematics, S.S. Jain Subodh P.G. College, Jaipur, 302004, India.
E-mail: rashmimodi9@gmail.com

R. Sharma, Department of Physics, S.S. Jain Subodh P.G. College, Jaipur, 302004, India.
E-mail: rina.physics1@gmail.com