Palestine Journal of Mathematics

Vol 15(Special Issue I)(2026) , 29–40

A comprehensive treatment on functions and integrals involving confluent hypergeometric function

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MSC 2010 Classifications: Primary 33C15; Secondary 33C10, 33E12, 26D15, 44A20, 35K05.

Keywords and phrases: Confluent hypergeometric function, Pathway model, Bessel function, Mittag-Leffler function, Turán-type inequalities, Log-convexity, Integral transforms, one-dimensional bio-heat equation.

The authors would like to thank the University of Kerala for providing all facilities.

Abstract In this article, a new mathematical/statistical model associated confluent hypergeometric function along with its associated integral is considered. The model's differential equation involves solving the bio-heat equation specifically for the dermal layer. The associated integral is presented along with its properties and applications in several experimental scenarios. Representations of the integral and its special cases in terms of generalized special functions and series forms are detailed. Applicability of the model in astrophysics, statistics and applied analysis are highlighted. Due to the flexibility of the model and the associated integral, the model is applicable across various STEM disciplines.

1 Introduction

The confluent hypergeometric function is an important transcendental function in mathematics, playing a significant role across multiple scientific disciplines [10, 18, 20, 31]. One of the most relevant applications of the confluent hypergeometric function lies in its emergence as a solution to numerous useful ordinary and partial differential equations. In [30], Sharma and Saxena used confluent hypergeometric function for solving bio-heat equation of skin. They obtained the solution to the Laplace transform of one dimensional equation for constant thermal conductivity in the dermis layer as a product of power function, exponential function and confluent hypergeometric function. In [30], it can be seen that the one dimensional bio- heat equation for in vivo tissue temperature T in the case of constant thermal conductivity K is given by

$$\rho c\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} + m_b c_b(T_b - T) + S, \tag{1.1}$$

where $\rho, c, t, m_b, c_b, T_b$ and S are respectively tissue density, heat capacity, time, blood mass flow rate, heat capacity of blood, blood temperature and rate of metabolic heat generation at a point. They obtained the solution of (1.1) for the dermis layer in the following form:

$$\overline{T} = z^{-\frac{1}{2}}\left[C_3 M_{-\frac{p}{4},\frac{1}{4}}\left(z^2\right) + C_4 M_{-\frac{p}{4},-\frac{1}{4}}\left(z^2\right)\right], \tag{1.2}$$

where C_3, C_4 are constants, $\overline{T}$ is the Laplace transform of $\theta = \frac{T_b - T}{T_b}$ and

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}\mathrm{e}^{-\frac{z}{2}}\ {}_1F_1(\mu - \lambda + \frac{1}{2}; 2\mu + 1; z) \tag{1.3}$$

is the Whittaker function [21]. Here ${}_1F_1(\cdot)$ is the confluent hypergeometric function defined as

$${}_1F_1(p; q; x) = \sum_{k=0}^{\infty} \frac{(p)_k x^k}{(q)_k k!}, \tag{1.4}$$

with $q \neq 0, -1, -2, \ldots$, which converges for all finite x, and $(p)_k$ is the Pochhammer symbol given by

$$(p)_k = \frac{\Gamma(p+k)}{\Gamma(p)} = \begin{cases} 1 & \text{if } k = 0, p \in \mathbb{C} \setminus \{0\}, \\ p(p+1)\ldots(p+k-1) & \text{if } k \in \mathbb{N}, p \in \mathbb{C}, \end{cases}$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{C}$ denotes set of complex numbers.
In many such applications, particularly in quantum mechanics, heat conduction problems, financial modeling (e.g.Black-Scholes model), and numerous other fields [1, 9, 14, 19], the confluent hypergeometric function appears in combinations with exponential functions, power functions, or a combination of both. Moreover, in experimental situations, one may require multiple modifications of a model for fitting the practical data or sometimes want to move from one specific model to another. In order to tackle such situations, one may require models containing power function, exponential and confluent hypergeometric functions or which can switch among these functions. Thus the prime objective of the present paper is to study the properties and applications of a mathematical/statistical model defined as

$$A_{a,\gamma,\delta,x}^{\alpha,\rho,\eta}(t) = c_1 t^{\alpha-1} \mathrm{e}^{-at^{\rho}} \,{}_1F_1\left(\gamma; \delta; -xt^{\eta}\right), \tag{1.5}$$

where $\Re(\alpha) > 0, a > 0, \delta > 0, \rho > 0, \gamma \in \mathbb{R}, \eta \in \mathbb{R}$, ${}_1F_1(\cdot)$ is the confluent hypergeometric function given in (1.4) and c_1 is the normalizing constant given by

$$c_1 = \frac{\rho a^{\frac{\alpha}{\rho}} \Gamma(\gamma)}{\Gamma(\delta) \,{}_2\Psi_1\left[-\frac{x}{a^{\frac{\eta}{\rho}}} \middle| \begin{array}{l} (\gamma, 1), \left(\frac{\alpha}{\rho}, \frac{\eta}{\rho}\right) \\ (\delta, 1) \end{array} \right]}, \tag{1.6}$$

for $\eta < \rho$, if we consider (1.5) as a statistical density. Here ${}_p\psi_q(\cdot)$ is the generalized Wright hypergeometric function [16] which is defined as

$${}_p\psi_q(z) = {}_p\psi_q\left[z \middle| \begin{array}{l} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 k) \ldots \Gamma(a_p + \alpha_p k) z^k}{\Gamma(b_1 + \beta_1 k) \ldots \Gamma(b_q + \beta_q k) k!} \tag{1.7}$$

where $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}, i = 1, \ldots, p; j = 1, \ldots, q$ with $\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1$.
By considering $A_{a,\gamma,\delta,x}^{\alpha,\rho,\eta}(t)$ in equation (1.5) as a mathematical model, the following differential equation is obtained.

$$\begin{aligned} t^2 \frac{\mathrm{d}^2 A}{\mathrm{d}t^2} &+ \left[2(1-\alpha) + 2a\rho t^{\rho} + 1 + \eta\left(\delta + xt^{\eta} - 1\right) \right] t \frac{\mathrm{d}A}{\mathrm{d}t} \\ &+ \Big\{ (1-\alpha)\left[2a\rho t^{\rho} + 1 - \alpha + \eta\left(\delta + xt^{\eta} - 1\right)\right] + a\rho(\rho-1)t^{\rho} + a^2\rho^2 t^{2\rho} \\ &+ x\gamma\eta^2 t^{\eta} + \left[1 + \eta\left(\delta + xt^{\eta} - 1\right)\right] a\rho t^{\rho} \Big\} A = 0. \end{aligned} \tag{1.8}$$

If $a = \frac{1}{2}, \rho = \eta = 2, \alpha = 2, \gamma = \frac{p+3}{4}, \delta = \frac{3}{2}$ and $x = -1$, the differential equation (1.8), reduces to

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} - \left(t^2 + p\right) A = 0. \tag{1.9}$$

The differential equation for $\overline{T}$ given in (1.2) corresponding to the dermis part is same as that of the differential equation in (1.9). Hence the solution given in equation (1.2) for the dermis layer can be written in a simplified form using the model given in (1.5) as

$$\overline{T} = C_3 A_{\frac{1}{2},\frac{p+3}{4},\frac{3}{2},-1}^{2,2,2}(z) + C_4 A_{\frac{1}{2},\frac{p+1}{4},\frac{1}{2},-1}^{1,2,2}(z), \tag{1.10}$$

The solution for T can be effectively substituted by the model given in equation (1.5), even when the system is in a steady state. Thus the model in equation (1.5) provides a simplified approach

for solving such heat transfer equations, yielding the solutions with greater ease.
Behera and Laha [5] has used an integral associated with the model in (1.5) for finding the solution of the radical Schrödinger wave equation for the Columb plus Graz separable potential. This prompted us to explore the properties and applications of the integrals of the form

$$I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-at^\rho} \, {}_1F_1\left(\gamma;\delta;-xt^\eta\right) \mathrm{d}t, \tag{1.11}$$

where $\Re(\alpha) > 0, a > 0, \delta > 0, \rho > 0, \gamma \in \mathbb{R}, \eta \in \mathbb{R}$ and ${}_1F_1(\cdot)$ is the confluent hypergeometric function given in (1.4).
The integral (1.11) can also be represented in terms of the three parameter Mittag - Leffler function as

$$I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-at^\rho} \Gamma(\delta) \, E_{1,\delta}^{\gamma}(-xt^\eta) \mathrm{d}t,$$

where $t > 0, \Re(\alpha) > 0, a > 0, \delta > 0, \gamma > 0, \rho > 0$, and $E_{a,b}^{c}(z) = \sum_{k=0}^{\infty} \frac{(c)_k}{\Gamma(b+ak)} \frac{z^k}{k!}$ with $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0$ is the generalized Mittag-Leffler function [27].
The pathway model introduced by Mathai [22, 25] for real positive scalar case is the following:
For $p < 1$,

$$f_1(x) = k_1 x^{\alpha-1} \left[1 - (1-p)ax^\eta\right]^{\frac{\mu}{1-p}}$$

with $a > 0, \eta > 0, \mu > 0, x \geq 0, [1 - (1-p)ax^\eta] > 0$, belongs to generalized type-1 beta family of densities. For $p > 1$, replacing $1 - p$ by $-(p - 1)$, the function

$$f_2(x) = k_2 x^{\alpha-1} \left[1 + (p-1)ax^\eta\right]^{-\frac{\mu}{p-1}}$$

with $a > 0, \eta > 0, \mu > 0, x \geq 0$, belongs to generalized type-2 beta family of densities. Also for $x \geq 0, a > 0, \eta > 0, \mu > 0$,

$$\lim_{p\to 1_-} f_1(x) = \lim_{p\to 1_+} f_2(x) = k_3 x^{\alpha-1} \mathrm{e}^{-a\mu x^\eta},$$

which belongs to the generalized gamma family of densities. Here k_1, k_2 and k_3 are the normalizing constants. By incorporating the pathway model to the integral in (1.11), one can obtain two integrals of the form:

$$P_{a,\gamma,\delta}^{\alpha,\rho,\eta}(c) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-at^\rho} \, {}_1F_1\left(\frac{1}{p-1};\delta;-c(p-1)t^\eta\right) \mathrm{d}t, \; p > 1 \tag{1.12}$$

and

$$Q_{a,\gamma,\delta}^{\alpha,\rho,\eta}(c) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-at^\rho} \, {}_1F_1\left(\gamma;\frac{1}{q-1};-\frac{c}{q-1}t^\eta\right) \mathrm{d}t, \;\; q > 1, \tag{1.13}$$

where p and q are the pathway parameters for $c > 0$. The integrals in equation (1.12) covers integrals containing exponential and Bessel functions, whereas the integral in (1.13) covers integrals containing exponential and binomial functions.
The integrals arising in physical problems contain combinations of exponential and power function with binomial or confluent or Bessel function. This paper aims to provide a unified framework for tackling such integrals containing these combinations and investigate their interesting properties.

This paper is organized into seven sections: The next section deals with the special cases of the introduced integral. Section 3 presents the series representation and H - function representation of the proposed integral and its special cases. Properties like log-convexity, Turán type inequalities for the integral are proved in section 4. Section 5 covers some integral transforms like Laplace transform and Mellin transform of $A_{a,\gamma,\delta,x}^{\alpha,\rho,\eta}(t)$. Applications of the integral are presented in section 6. Concluding remarks are given in section 7.

2 Some Special Cases of $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$

This section demonstrates the versatility of the integral $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$ in equation (1.11) by showcasing its reducibility to various integrals encountered in applied fields.

2.1 Connection to Bessel type integral

If $x = \frac{y^2}{4\gamma}, \eta = 2, \delta = n+1$, where n is not a negative integer, then equation (1.11) reduces to the following form:

$$\begin{aligned} B_{a,n+1}^{\alpha,\rho}(y) &= \lim_{\gamma\to\infty} I_{a,\gamma,n+1}^{\alpha,\rho,2}\left(\frac{y^2}{4\gamma}\right) \\ &= \left(\frac{2}{y}\right)^n \Gamma(n+1) \int_0^\infty t^{\alpha-n-1} \mathrm{e}^{-at^\rho} J_n(yt)\mathrm{d}t, \end{aligned} \tag{2.1}$$

where $J_n(x)$ is the Bessel function of first kind of order n, defined for complex $x \in \mathbb{C}(x \neq 0)$ and is given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{\Gamma(n+k+1)k!}.$$

By the property of Bessel function, the integral in equation (2.1) can be reduced to sine and cosine type integrals for specific values of n as given below:

- When $n = \frac{1}{2}$, then $B_{a,\frac{3}{2}}^{\alpha,\rho}(y) = \frac{1}{y}\int_0^\infty t^{\alpha-2}\mathrm{e}^{-at^\rho} \sin(yt)\mathrm{d}t$.
- When $n = -\frac{1}{2}$, then $B_{a,\frac{1}{2}}^{\alpha,\rho}(y) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho} \cos(yt)\mathrm{d}t$.

2.2 Connection to pathway type integrals

By implementing the pathway concept into the integral $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$, the following four forms are obtained:

- In equation (1.11), if $\gamma = \frac{1}{p-1}, x = y^\tau(p-1)\delta, \eta = -\tau$, then

$$\begin{aligned} C_{a,p}^{\alpha,\rho,\tau}(y) &= \lim_{\delta\to\infty} I_{a,\frac{1}{p-1},\delta}^{\alpha,\rho,-\tau}(y^\tau(p-1)\delta) \\ &= \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho}\left[1+y^\tau(p-1)t^{-\tau}\right]^{-\frac{1}{p-1}}\mathrm{d}t, \end{aligned} \tag{2.2}$$

  where $p > 1, y > 0, \rho > 0, \tau > 0$.

- By putting $t = \frac{1}{z}$ in equation (2.2), we have the following integral

$$^{*}C_{a,p}^{\alpha,\rho,\tau}(y) = \int_0^\infty z^{-\alpha-1}\mathrm{e}^{-az^{-\rho}}\left[1+y^\tau(p-1)z^{\tau}\right]^{-\frac{1}{p-1}}\mathrm{d}z. \tag{2.3}$$

- In equation (1.11), when $\gamma = \frac{1}{p-1}, x = y^\tau(p-1)\delta, \eta = \tau$, then

$$\begin{aligned} D_{a,p}^{\alpha,\rho,\tau}(y) &= \lim_{\delta\to\infty} I_{a,\frac{1}{p-1},\delta}^{\alpha,\rho,\tau}(y^\tau(p-1)\delta) \\ &= \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho}\left[1+y^\tau(p-1)t^{\tau}\right]^{-\frac{1}{p-1}}\mathrm{d}t, \end{aligned} \tag{2.4}$$

  where $p > 1, y \geq 0, \rho > 0, \tau > 0$.

- By putting $t = \frac{1}{z}$ in equation (2.4), we have the following integral

$$^{*}D_{a,p}^{\alpha,\rho,\tau}(y) = \int_0^\infty z^{-\alpha-1}\mathrm{e}^{-az^{-\rho}}\left[1+y^\tau(p-1)z^{-\tau}\right]^{-\frac{1}{p-1}}\mathrm{d}z. \tag{2.5}$$

Remark 2.1. The integrals (2.2), (2.3), (2.4) and (2.5) can be seen in [24].

2.3 Connection to binomial type integrals

- In equation (1.11), if $\Re(\alpha) > 0, \rho > 0, x = -\delta y, y > 0$ and $\gamma = -\tau$ for $\tau > 0$

$$E_{a,\tau}^{\alpha,\rho,\eta}(y) = \lim_{\delta\to\infty} I_{a,-\tau,\delta}^{\alpha,\rho,\eta}(-\delta y) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho}\left(1 - yt^\eta\right)^\tau \mathrm{d}t.$$

- In equation (1.11), if $\Re(\alpha) > 0, \rho > 0, x = \delta y$, for $y > 0$ and $\tau > 0$

$$F_{a,\gamma}^{\alpha,\rho,\eta}(y) = \lim_{\delta\to\infty} I_{a,-\tau,\delta}^{\alpha,\rho,\eta}(-\delta y) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho}\left(1 + yt^\eta\right)^{-\gamma} \mathrm{d}t.$$

3 Series and Contour Integral Representations

This section covers the series representations and H - function representation of the integral $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$ and its special cases.

3.1 Series representations

In applied disciplines, series representations are crucial as it helps in identifying the asymptotic behavior of the function. It also aids in identifying the behavior of solutions of differential and integral equations and providing approximations for the function. By considering the importance of series representation in applied fields, here we are presenting the series representations of the introduced integral and its special cases.

(i) The next theorem shows that integral $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$ can be obtained as a series of generalized Wright hypergeometric type defined in (1.7).

Theorem 3.1. *For* $t > 0, \Re(\alpha) > 0, a > 0, \rho > 0, \Re(\delta) > 0$ *and* $\eta < \rho$,

$$I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x) = \frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}\, {}_2\Psi_1\left[-\frac{x}{a^{\frac{\eta}{\rho}}}\,\middle|\, \begin{matrix}(\gamma,1), \left(\frac{\alpha}{\rho}, \frac{\eta}{\rho}\right)\\ (\delta, 1)\end{matrix}\right],$$

where ${}_2\Psi_1(\cdot)$ *is the Wright hypergeometric function defined in (1.7).*

Proof. Using (1.11) and (1.4) and changing the order of integration and summation,

$$\begin{aligned} I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x) &= \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\delta)_k}\frac{(-x)^k}{k!}\int_0^\infty \mathrm{e}^{-at^\rho}t^{\alpha+\eta k-1}\mathrm{d}t \\ &= \frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\Gamma\left(\frac{\alpha+\eta k}{\rho}\right)}{\Gamma(\delta+k)\,k!}\left(-\frac{x}{a^{\frac{\eta}{\rho}}}\right)^k \\ &= \frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}\, {}_2\Psi_1\left[-\frac{x}{a^{\frac{\eta}{\rho}}}\,\middle|\, \begin{matrix}(\gamma,1), \left(\frac{\alpha}{\rho}, \frac{\eta}{\rho}\right)\\ (\delta, 1)\end{matrix}\right]. \end{aligned}$$

□

(ii) If $\eta = \rho$, then

$$I_{a,\gamma,\delta}^{\alpha,\rho,\rho}(x) = \frac{\Gamma\left(\frac{\alpha}{\rho}\right)}{\rho a^{\frac{\alpha}{\rho}}}\, {}_2F_1\left(\gamma, \frac{\alpha}{\rho}; \delta; -\frac{x}{a}\right),$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function defined as

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!}, \quad |z| < 1.$$

(iii) If $\rho = 1$, then the integral $B^{\alpha,\rho}_{a,n+1}(y)$ given in (2.1) can be obtained in terms of associated Legendre function as

$$B^{\alpha,1}_{a,n+1}(y) = \left(\frac{2}{y}\right)^n \Gamma(n+1) \int_0^\infty t^{\alpha-n-1} \mathrm{e}^{-at} J_n(yt) \mathrm{d}t$$
$$= \left(\frac{2}{y}\right)^n \Gamma(n+1)\Gamma(\alpha) \left(a^2+y^2\right)^{-\frac{\alpha-n}{2}} P^{-n}_{\alpha-n-1}\left[a\left(a^2+y^2\right)^{-\frac{1}{2}}\right], \tag{3.1}$$

where $P^{\mu}_{\nu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$, where $\arg \frac{z+1}{z-1} = 0$ if z is real and greater than 1, is the associated Legendre function of the first kind or spherical function of the first kind [34].

(iv) If $\rho = 2$, then the integral $B^{\alpha,\rho}_{a,n+1}(y)$ given in (2.1) can be obtained in terms of Whittaker function as

$$B^{\alpha,2}_{a,n+1}(y) = \left(\frac{2}{y}\right)^n \Gamma(n+1) \int_0^\infty t^{\alpha-n-1} \mathrm{e}^{-at^2} J_n(yt) \mathrm{d}t$$
$$= \frac{2^n \Gamma\left(\frac{\alpha}{2}\right)}{y^{n+1} a^{\frac{\alpha-n-1}{2}}} \mathrm{e}^{\left(-\frac{y^2}{8a}\right)} M_{\frac{\alpha-n-1}{2}, \frac{n}{2}}\left(\frac{y^2}{4a}\right), \tag{3.2}$$

where $\Re(a) > 0, \Re(\alpha) > 0$, and $M_{\lambda,\mu}(z)$ is the Whittaker function given in equation (1.3).

(v) If $\rho = 2$ and $\alpha = n+1$, then the integral $B^{\alpha,\rho}_{a,n+1}(y)$ given in (2.1) can be obtained in terms of modified Bessel function of the first kind as

$$B^{n+1,2}_{a,n+1}(y) = \left(\frac{2}{y}\right)^n \Gamma(n+1) \int_0^\infty \mathrm{e}^{-at^2} J_n(yt) \mathrm{d}t$$
$$= \frac{2^{n-1}\sqrt{\pi}\Gamma(n+1)}{y^n \sqrt{a}} \mathrm{e}^{-\left(\frac{y^2}{8a}\right)} I_{\frac{n}{2}}\left(\frac{y^2}{8a}\right), \tag{3.3}$$

where $y > 0, \Re(n) > -1$ and $I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}$ is the modified Bessel function of the first kind [28].

(vi) If $\gamma = \delta, \rho = 2$ and $\eta = 1$, then the integral $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ given in (1.11) has the following series representation in terms of the parabolic function.

$$G^{\alpha,2,1}_a(x) = I^{\alpha,2,1}_{a,\gamma,\delta}(x) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-at^2-xt} \mathrm{d}t$$
$$= (2a)^{-\frac{\alpha}{2}} \Gamma(\alpha) \mathrm{e}^{\frac{x^2}{8a}} R_{-\alpha}\left(\frac{x}{\sqrt{2a}}\right), \tag{3.4}$$

where $\Re(\alpha) > 0, \Re(a) > 0$, and

$$R_p(z) = 2^{\frac{p}{2}} \exp\left(-\frac{z^2}{4}\right)$$
$$\times \left[\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} {}_1F_1\left(-\frac{p}{2}; \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi z}}{\Gamma\left(-\frac{p}{2}\right)} {}_1F_1\left(\frac{1-p}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right]$$

is the parabolic cylinder function [21].

Remark 3.2. Equations (3.1), (3.2), (3.3), (3.4) can be seen in [11].

3.2 H - function representation

The integral $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$ and its special cases can be obtained in terms of contour integrals via H-function, where H - function is a Mellin-Barnes type contour integral defined as

$$H_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_q,B_q)\end{array}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{C}}\Theta(s)z^{-s}\mathrm{d}s, \tag{3.5}$$

where $i=(-1)^{1/2}, z\neq 0$, and $z^{-s}=\exp\left[-s(\ln|z|+i\arg z)\right]$, where $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg|z|$ is not necessarily the principal value. Here

$$\Theta(s)=\frac{\left[\prod_{j=1}^{m}\Gamma(b_j+B_js)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\right]\left[\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right]}.$$

An empty product is always interpreted as unity; $m,n,p,q\in N_0$ with $0\leq n\leq p, 1\leq m\leq q, A_j, j=1,\cdots,p$ and $B_j, j=1,\cdots,q$ are real positive numbers, $a_j, j=1,\cdots,p$ and $b_j, j=1,\cdots,q$ are complex numbers, $\mathcal{C}$ is a contour separating the poles of $\Gamma(b_j+B_js), j=1,\cdots,m$ from those of $\Gamma(1-a_j-A_js), j=1,\cdots,n.$ More details about the existence conditions, properties and applications can be seen in [15, 26].

(i) $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$ has the following H - function representation.

Theorem 3.3. *For* $\Re(\alpha)>0, a>0, \delta>0, \rho>0, \gamma\in\mathbb{R}, \eta\in\mathbb{R}$,

$$I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)=\frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}H_{2,2}^{1,2}\left[\frac{x}{a^{\frac{\eta}{\rho}}}\left|\begin{array}{l}(1-\gamma,1)\left(1-\frac{\alpha}{\rho},\frac{\eta}{\rho}\right)\\(0,1),(1-\delta,1)\end{array}\right.\right],$$

where $\eta<\rho$.

Proof. By Theorem 3.1,

$$\begin{aligned}I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)=&\frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}\,{}_2\Psi_1\left[-\frac{x}{a^{\frac{\eta}{\rho}}}\left|\begin{array}{l}(\gamma,1),\left(\frac{\alpha}{\rho},\frac{\eta}{\rho}\right)\\(\delta,1)\end{array}\right.\right]\\=&\frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\,\Gamma\left(\frac{\alpha+\eta k}{\rho}\right)}{\Gamma(\delta+k)\,k!}\left(-\frac{x}{a^{\frac{\eta}{\rho}}}\right)^k.\end{aligned}$$

By using the Mellin - Barnes integral representation [28] for $I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)$, we have

$$\begin{aligned}I_{a,\gamma,\delta}^{\alpha,\rho,\eta}(x)=&\frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(s)\Gamma(\gamma-s)\Gamma\left(\frac{\alpha}{\rho}-\frac{\eta s}{\rho}\right)}{\Gamma(\delta-s)}\left(\frac{x}{a^{\frac{\eta}{\rho}}}\right)^{-s}\mathrm{d}s\\=&\frac{\Gamma(\delta)}{\rho a^{\frac{\alpha}{\rho}}\Gamma(\gamma)}H_{2,2}^{1,2}\left[\frac{x}{a^{\frac{\eta}{\rho}}}\left|\begin{array}{l}(1-\gamma,1)\left(1-\frac{\alpha}{\rho},\frac{\eta}{\rho}\right)\\(0,1),(1-\delta,1)\end{array}\right.\right].\end{aligned}$$

□

(ii) The integral $C_{a,p}^{\alpha,\rho,\tau}(y)$ given in (2.2) has the following H- function representation.

$$\begin{aligned}C_{a,p}^{\alpha,\rho,\tau}(y)=&\int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho}\left[1+y^\tau(p-1)t^{-\tau}\right]^{-\frac{1}{p-1}}\mathrm{d}t\\=&\frac{1}{\rho\tau a^{\frac{\alpha}{\rho}}\Gamma\left(\frac{1}{p-1}\right)}H_{1,2}^{2,1}\left[ya^{\frac{1}{\rho}}(p-1)^{\frac{1}{\tau}}\left|\begin{array}{l}\left(1-\frac{1}{p-1},\frac{1}{\tau}\right)\\\left(0,\frac{1}{\tau}\right),\left(\frac{\alpha}{\rho},\frac{1}{\rho}\right)\end{array}\right.\right].\end{aligned} \tag{3.6}$$

(iii) The integral ${}^*C^{\alpha,\rho,\tau}_{a,p}(y)$ given in (2.3) also has the same H- function representation as given in equation (3.6).

(iv) The integral $D^{\alpha,\rho,\tau}_{a,p}(y)$ given in (2.4) has the following H- function representation.

$$D^{\alpha,\rho,\tau}_{a,p}(y) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho}\left[1+y^\tau(p-1)t^\tau\right]^{-\frac{1}{p-1}}\,\mathrm{d}t$$
$$= \frac{1}{\rho\tau(p-1)^{\frac{\alpha}{\tau}}y^\alpha\Gamma\left(\frac{1}{p-1}\right)} H^{2,1}_{1,2}\left[y(p-1)^{\frac{1}{\tau}}a^{\frac{1}{\rho}}\,\middle|\,\begin{matrix}\left(1-\frac{\alpha}{\tau},\frac{1}{\tau}\right)\\ \left(0,\frac{1}{\rho}\right),\left(\frac{1}{p-1}-\frac{\alpha}{\tau},\frac{1}{\tau}\right)\end{matrix}\right]. \tag{3.7}$$

(v) The integral ${}^*D^{\alpha,\rho,\tau}_{a,p}(y)$ given in (2.5) has the same H- function representation as given in equation (3.7).

4 Properties of the integral $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$

This section explores the basic inequalities satisfied by the proposed model, which enables the model to connect with diverse mathematical fields.

4.1 Log-convexity

A function $h:(0,1)\to(0,1)$ is said to be log-convex if its natural logarithm is convex. That is, for all $\lambda_1,\lambda_2>0$, and $\mu\in[0,1]$,

$$h[\mu\lambda_1+(1-\mu)\lambda_2]\le[h(\lambda_1)]^\mu[h(\lambda_2)]^{1-\mu}.$$

Reliability and infinitely divisible random variables are two areas where log-convexity is relevant. The log-convexity of $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ with respect to the parameters α and a are proved in the next theorems.

Theorem 4.1. *For* $\Re(\alpha)>0, a>0, \rho>0, \delta>0$, $\alpha\mapsto I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ *is log-convex on* $(0,\infty)$.

Proof.

$$I^{\mu\alpha_1+(1-\mu)\alpha_2,\rho,\eta}_{a,\gamma,\delta}(x) = \int_0^\infty t^{\mu\alpha_1+(1-\mu)\alpha_2-1}\mathrm{e}^{-at^\rho}\,{}_1F_1\left(\gamma;\delta;-xt^\eta\right)\mathrm{d}t$$
$$= \int_0^\infty \left[t^{\alpha_1-1}\mathrm{e}^{-at^\rho}\,{}_1F_1\left(\gamma;\delta;-xt^\eta\right)\right]^\mu \left[t^{\alpha_2-1}\mathrm{e}^{-at^\rho}\,{}_1F_1\left(\gamma;\delta;-xt^\eta\right)\right]^{1-\mu}\mathrm{d}t.$$

By Holders inequality [12], for $p>1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$, given by

$$\int_a^b|h_1(x)h_2(x)|\,\mathrm{d}x\le\left\{\int_a^b|h_1(x)|^p\,\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_a^b|h_2(x)|^q\,\mathrm{d}x\right\}^{\frac{1}{q}},$$

where h_1 and h_2 are real-valued functions defined on $[a,b]$ and $|h_1|^p$ and $|h_2|^p$ are integrable functions on $[a,b]$.

$$I^{\mu\alpha_1+(1-\mu)\alpha_2,\rho,\eta}_{a,\gamma,\delta}(x)\le\left[\int_0^\infty t^{\alpha_1-1}\mathrm{e}^{-at^\rho}\,{}_1F_1\left(\gamma;\delta;-xt^\eta\right)\mathrm{d}t\right]^\mu\left[\int_0^\infty t^{\alpha_2-1}\mathrm{e}^{-at^\rho}\,{}_1F_1\left(\gamma;\delta;-xt^\eta\right)\mathrm{d}t\right]^{1-\mu}.$$

That is,

$$I^{\mu\alpha_1+(1-\mu)\alpha_2,\rho,\eta}_{a,\gamma,\delta}(x)\le\left[I^{\alpha_1,\rho,\eta}_{a,\gamma,\delta}(x)\right]^\mu\left[I^{\alpha_2,\rho,\eta}_{a,\gamma,\delta}(x)\right]^{1-\mu},$$

which is true for all $\mu\in[0,1]$. Hence $\alpha\mapsto I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ is log-convex in $(0,\infty)$. □

Theorem 4.2. *For* $\Re(\alpha)>0, a>0, \rho>0, \delta>0$, $a\mapsto I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ *is log-convex on* $(0,\infty)$.

Proof.

$$\begin{aligned} I^{\alpha,\rho,\eta}_{\mu a_1+(1-\mu)a_2,\gamma,\delta}(x) &= \int_0^\infty t^{\alpha-1} e^{-[\mu a_1+(1-\mu)a_2]t^\rho} \,{}_1F_1(\gamma;\delta;-xt^\eta)\,dt \\ &= \int_0^\infty \left[t^{\alpha-1} e^{-a_1 t^\rho} \,{}_1F_1(\gamma;\delta;-xt^\eta)\right]^\mu \left[t^{\alpha-1} e^{-a_2 t^\rho} \,{}_1F_1(\gamma;\delta;-xt^\eta)\right]^{1-\mu} dt \\ &\le \left[\int_0^\infty t^{\alpha-1} e^{-a_1 t^\rho} \,{}_1F_1(\gamma;\delta;-xt^\eta)\,dt\right]^\mu \left[\int_0^\infty t^{\alpha-1} e^{-a_2 t^\rho} \,{}_1F_1(\gamma;\delta;-xt^\eta)\,dt\right]^{1-\mu}. \end{aligned}$$

That is,

$$I^{\alpha,\rho,\eta}_{\mu a_1+(1-\mu)a_2,\gamma,\delta}(x) \le \left[I^{\alpha,\rho,\eta}_{a_1,\gamma,\delta}(x)\right]^\mu \left[I^{\alpha,\rho,\eta}_{a_2,\gamma,\delta}(x)\right]^{1-\mu},$$

which is true for all $\mu \in [0,1]$. Hence $a \mapsto I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ is log-convex on $(0,\infty)$. □

Corollary 4.3. *For* $t > 0, \Re(\alpha) > 0, a > 0, \delta > 0, \eta > 0, \rho > 0,$ $\alpha, a \mapsto I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ *is log-convex on* $(0,\infty)$*. That is,*

$$I^{\mu\alpha_1+(1-\mu)\alpha_2,\rho,\eta}_{\mu a_1+(1-\mu)a_2,\gamma,\delta}(x) \le \left[I^{\alpha_1,\rho,\eta}_{a_1,\gamma,\delta}(x)\right]^\mu \left[I^{\alpha_2,\rho,\eta}_{a_2,\gamma,\delta}(x)\right]^{1-\mu}. \tag{4.1}$$

Proof. The proof follows directly from Theorems 4.1 and 4.2. □

4.2 Turán type inequality

Turán inequality is a mathematical inequality in number theory, named after the Hungarian mathematician Paul Turán [33] and is given by $[P_n(x)]^2 > P_{n-1}(x)P_{n+1}(x)$, where $|x| < 1, n \in 1, 2 \ldots$. It is interesting to see that $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ satisfies Turán type inequality.

Theorem 4.4. *If* $\alpha, \eta, \phi \in \mathbb{R},\ \alpha > 0,\ \rho > 0, \nu > 0, t > 0, \gamma \neq 1,$ *and* $\delta > 0,$ *then the Turán type inequality*

$$\left[I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)\right]^2 - \left[I^{\alpha-\phi,\rho,\eta}_{a-\phi,\gamma,\delta}(x)\right]\left[I^{\alpha+\phi,\rho,\eta}_{a+\phi,\gamma,\delta}(x)\right] < 0 \tag{4.2}$$

holds for $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$.

Proof. The proof follows directly from Corollary 4.3. Equation (4.2) can be obtained by substituting $\mu = \frac{1}{2}, \alpha_1 = \alpha - \phi, \alpha_2 = \alpha + \phi, a_1 = a - \phi$ and $a_2 = a + \phi$ in equation (4.1). □

5 Integral Transforms

One of the most important tools for solving differential equations is the integral transform and hence it plays an essential role in mathematics, physics and engineering. This section focuses on important integral transforms, like Laplace and Mellin transforms, of $A^{\alpha,\rho,\eta}_{a,\gamma,\delta,x}(t)$ defined in equation (1.5).

5.1 Laplace transform

The Laplace transform of $A^{\alpha,\rho,\eta}_{a,\gamma,\delta,x}(t)$, with s as the transform variable, is given by

$$L_A(s) = \frac{c_1}{\rho a^{\frac{\alpha}{\rho}}} \sum_{k=0}^\infty \frac{(\gamma)_k}{(\delta)_k} \frac{\left(-\frac{x}{a^{\frac{\eta}{\rho}}}\right)^k}{k!} H^{1,1}_{1,1}\left[\frac{s}{a^{\frac{1}{\rho}}} \middle| \begin{matrix} \left(1 - \frac{\alpha+\eta k}{\rho}, \frac{1}{\rho}\right) \\ (0,1) \end{matrix}\right],$$

where $\rho > 1, c_1$ is the normalizing constant given in equation (1.6) and $H^{1,1}_{1,1}(\cdot)$ is the H- function defined in equation (3.5).

5.2 Mellin transform

The Mellin transform of $A^{\alpha,\rho,\eta}_{a,\gamma,\delta,x}(t)$, with p as the transform variable, is given by

$$M_A(p) = \frac{c_1\Gamma(\delta)}{\rho a^{\frac{\alpha+p-1}{\rho}}\Gamma(\gamma)}\ {}_2\Psi_1\left[\frac{-x}{a^{\frac{\eta}{\rho}}}\middle|\begin{matrix}\left(1-\frac{\alpha+p}{\rho},\frac{\eta}{\rho}\right),(\gamma,1)\\(\delta,1)\end{matrix}\right],$$

for $\rho > \eta, c_1$ is the normalizing constant given in equation (1.6) and ${}_2\Psi_1(\cdot)$ is the Wright hypergeometric function defined in equation (1.7).

6 Some Applications

This section highlights the relevance and utility of the proposed model in astrophysics, statistical modeling and integral transforms.

6.1 Applications in Astrophysics

For specific values of the parameters, the integral $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ reduce to the standard reaction rate integral for non-resonant thermonuclear reactions in the Maxwell–Boltzmannian case.

Connection with standard reaction rate integral

If $\gamma = \delta$, then $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ in equation (1.11) reduces to

$$G^{\alpha,\rho,\eta}_a(x) = I^{\alpha,\rho,\eta}_{a,\delta,\delta}(x) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-at^\rho - xt^\eta}\mathrm{d}t. \tag{6.1}$$

When $a = 1, \rho = 1$ and $\eta = -\frac{1}{2}$, equation (6.1) reduces to

$$G^{\alpha,1,-\frac{1}{2}}_1(x) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t-xt^{-\frac{1}{2}}}\mathrm{d}t, \tag{6.2}$$

which is the collision probability integral for non-resonant thermonuclear reactions in the Maxwell–Boltzmannian case. A series of papers by Haubold and Mathai [23, 29] showcases the analytical solutions of Maxwell-Boltzmann theory of reaction rate integrals. The standard integral in equation (6.2) is generalized in many different forms for resonant and non-resonant reactions in depleted and cut-off cases. In 2008, Haubould and Kumar [13] extended the reaction rate probability integral in equation (6.2) to cover more unstable and chaotic cases.

6.2 Applications in Statistics

Tsallis statistics [32] is given by

$$h(x) = c\left[1 + (p-1)t\right]^{-\frac{1}{p-1}}. \tag{6.3}$$

If $\alpha = 1, a = 0, \gamma = \frac{1}{p-1}, \delta = \frac{1}{q-1}, x = \frac{p-1}{q-1}$, and $\eta = 1$ in $A^{\alpha,\rho,\eta}_{a,\gamma,\delta,x}(t)$ defined in (1.5), then $\lim_{q\to1} A^{\alpha,\rho,\eta}_{a,\gamma,\delta,x}(t)$ agrees with $h(x)$ given in (6.3). Moreover, if $\alpha = 1, a = 0, \gamma = \frac{1}{p-1}, \delta = \frac{1}{q-1}, x = \frac{p-1}{q-1}$, and $\eta = 1$ then $\lim_{q\to1} A^{\alpha,\rho,\eta}_{a,\gamma,\delta,x}(t)$ is the superstatistics introduced by Beck and Cohen [2, 3, 4, 7].

6.3 Applications in Applied Analysis

A special case of the integral $I^{\alpha,\rho,\eta}_{a,\gamma,\delta}(x)$ serves as the kernel of the Krätzel transform.

Connection with Krätzel transform

Krätzel [17] introduced an integral transform given by

$$K_{\nu}^{(\rho)}f(x) = \int_0^{\infty} M_{\rho}^{\nu}(xt)f(t)\mathrm{d}t,$$

called Krätzel transform, where

$$M_{\rho}^{\nu}(x) = \int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t^{\rho}-xt^{-1}}\mathrm{d}t, \nu \in \mathbb{C}, \rho \in \mathbb{R}, \tag{6.4}$$

is the kernel function.
For $a = 1$ and $\eta = -1$, the integral $G_a^{\alpha,\rho,\eta}(x)$ in (6.1) reduces to the form in (6.4). That is,

$$G_1^{\alpha,\rho,-1}(x) = M_{\rho}^{\nu}(x).$$

The function given in (6.4) is connected to modified Bessel function of the third kind or Mc Donald function [8], which is a very useful function in applied analysis, especially in chemical physics [6].

7 Conclusion

The introduced model offers a unified and simplified approach to solve various ordinary differential equations and partial differential equations, as evidenced by its successful application to the bio-heat equation given in [30]. Hence it serves as a useful tool for developing applications across a variety of domains, offering a flexible framework that makes use of the advantages of confluent hypergeometric function. The representation of the proposed integral and its special cases in terms of numerous significant special functions, such as the H- function, associated Legendre function, Whittaker function, modified Bessel function, and parabolic cylinder function, provides a convenient way to approximate complex integrals. Moreover, the examination of key properties like log-convexity, and Turán-type inequalities ensures its practical utility in various scientific and engineering problems. This versatility allows the model to encompass several existing functions and adapt to diverse applications across multiple disciplines.

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