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Classes of Completely Monotone functions and Pick Functions

Deepshikha Mishra and A. Swaminathan

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Abstract This paper provides a brief overview of complete monotone functions and their applications. We highlight an important class associated with complete monotonicity, referred to as the class of Pick functions, and examine recent advancements in the related literature. Additionally, we re-establish a well-known result stating that logarithmic complete monotonicity implies complete monotonicity through a different approach, with a restriction on the interval of complete monotonicity. Furthermore, we explore the complete monotonicity of certain functions involving the gamma function.

(Dedicated to the memory of Prof. Arun Verma)

1 Introduction

During the years 1894–1895, Stieltjes [33], explored the analytic theory of continued fractions and introduced several fundamental mathematical concepts, including the Stieltjes integral and the problem of moments. Over time, the theory of the problem of moments became a cornerstone in mathematical research. Stieltjes adopted the term “problem of moments" from mechanics. The problem involves finding a bounded, non-decreasing function $\psi(x)$ defined on the interval $[0,\infty)$ such that its moments, expressed as $\int_0^\infty x^n\, d\psi(x)$, for $n=0,1,2,\ldots$, satisfy a given set of values [32, 33]

$$\mu_n = \int_0^\infty x^n\, d\psi(x), \quad n = 0,1,2,\ldots.$$

In other words, the moment problem can be described as an inverse problem, where the objective is to recover the measure based on its moments. Hausdorff later studied it for bounded intervals, known as the Hausdorff moment problem [16]. In 1923, it was proved that the Hausdorff moment problem on the interval $[0,1]$ has a solution if, and only if, the sequence $\{\mu_n\}$ is completely monotonic [1, 17]. This result led to introducing the concept of a completely monotonic sequence. Many authors call it totally monotone instead of completely monotone. Later, Hausdorff introduced complete monotone functions as a continuous analogue of complete monotone sequences and defined them as follows.

Definition 1.1. [36] A function f is said to be completely monotone on (a,b) if it is non-negative and possesses derivatives of all orders that satisfy the condition

$$(-1)^n f^{(n)}(x) \geq 0, \quad \forall\, n \in \mathbb{N}, \quad \forall\, x \in (a,b). \tag{1.1}$$

A remarkable interest arose in the study of complete monotone functions when, in 1928, Bernstein characterized completely monotonic functions f as Laplace transforms of non-negative measures on $[0,\infty)$ [11]. This characterization can be stated as follows

Theorem 1.2. *[31] A function $f : (0,\infty) \to \mathbb{R}$ is completely monotone if, and only if, there exists a unique measure ν on $[0,\infty)$ such that*

$$f(x) = \int_0^\infty e^{-xt}\, d\nu(t).$$

This theorem is widely known as Bernstein's theorem in the literature [36]. Complete monotone functions have been studied for their numerous applications across different branches of mathematics such as probability theory [15, 21, 30], mathematical physics [13], numerical analysis [37], asymptotic analysis [18, 19], combinatorics [3] and potential theory [4]. Many researchers have studied the properties of complete monotonicity and its connections to other important function classes, including Pick functions, Bernstein functions, and specific subclasses of Bernstein functions, such as Thorin-Bernstein and complete Bernstein functions. Among these, we focus on the class of Pick functions in this manuscript.

The class of Pick functions consists of holomorphic functions defined on the upper half-plane having non-negative imaginary part [14]. Along with its definition, there are several well-known and useful characterizations of Pick functions, see [14]. One such characterization of Pick functions is associated with completely monotone sequences, where the class of generating functions for completely monotone sequences is represented as a subclass of Pick functions denoted by $P(-\infty, 1)$ [20, 29]. The subclass $P(-\infty, 1)$ consists of those Pick functions that admit an analytic continuation across the interval $(-\infty, 1)$ into the lower half-plane, where the continuation is by reflection [14]. Ruscheweyh, Salinas, and Sugawa explored the relationship between Pick functions and completely monotone sequences in [28, 29]. In [28], they introduced the classes of universally convex, starlike, and prestarlike functions in the slit domain $\mathbb{C} \setminus [1,\infty)$, demonstrating a strong connection between these functions and completely monotone sequences, as well as Pick functions. They also identified interesting classes of Pick functions involving the polylogarithm in their study. To learn about the polylogarithm and generalized polylogarithm, we refer to [25, 22]. Subsequently, the second author posed an open problem regarding the class of Pick functions involving generalized polylogarithms in [9]. This open problem concerns finding the conditions on the parameters of the generalized polylogarithm such that the ratio of generalized polylogarithms belongs to the class of Pick functions.

A significant amount of research has been done on Pick functions to investigate the properties and behavior of functions on the real line. These studies aim to determine whether a function belongs to specific classes such as Stieltjes, Bernstein, completely monotone, and complete Bernstein functions, among others. Moreover, Pick functions are studied to investigate various properties such as monotonicity, convexity, and other related characteristics. For related papers on these topics, we refer the interested readers to [5, 7, 8, 10, 20, 24, 29]. Certain classes of Pick functions involving the gamma function have attracted attention due to their connection with the volume of the unit ball in n-space. As a result, various interesting properties of the sequence representing the volumes of the unit ball in n-space have been explored. Berg and Pedresan investigated such Pick functions in [6, 7]. In 2002, Berg and Pedresan discussed such a function in [6], where the function is expressed as follows

$$f(z) = \frac{\log \Gamma(z+1)}{z \log z}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{1.2}$$

Subsequently, Pedersen introduced another Pick function in [23] by replacing the gamma function with the double gamma function in the same f defined in (1.2). Furthermore, Das, Pedersen, and Swaminathan observed that the function $f(z)$ is not a Pick function when $\Gamma(z)$ is replaced with the triple gamma function in (1.2), unlike the case with the double gamma function [12]. Moreover, they addressed this issue in [12] by modifying f not only with the triple gamma function but also by introducing an additional factor involving the derivative of the triple gamma function.

In the next section, we begin by explaining the importance of studying the complete monotonicity of functions. Then, we explore the complete monotonicity of functions involving the gamma function. Finally, we present a complete monotone function expressed as the sum of two functions, each of which is not completely monotone.

2 Complete monotonicity of functions involving the gamma function

The study of complete monotonicity of functions has been a subject of interest, as its consequences offer valuable insights into the behavior of functions. For example, complete monotonicity guarantees monotonicity and convexity, while also providing important information about inequalities, bounds, and other functional properties, as discussed in [2, 5, 7, 8, 26, 34]. However, in this paper, we specifically focus on the complete monotonicity of the function defined as

$$g(x) = 1 - \log x + \frac{1}{x} \log \Gamma(x+1), \quad x \in (0, \infty), \tag{2.1}$$

which has been discussed in [27]. Here $\Gamma(x)$ is the Euler's gamma function.

In Theorem 1 of [27], Qi and Chen established that a logarithmically completely monotonic function is also completely monotonic. They demonstrated this equivalence by assuming $f(x)$ to be completely monotonic and then proving that $\exp(f(x))$ retains the complete monotonicity. In this paper, we prove the same result by assuming that $\log(f(x))$ is completely monotonic. However, the method we use is restricted to the interval $(0, \infty)$. The proof is presented in Theorem 2.1.

Theorem 2.1. *A logarithmic completely monotonic function on $(0, \infty)$ is also completely monotonic on $(0, \infty)$.*

Proof. Let f be a logarithmically completely monotonic function on $(0, \infty)$. Then, by Theorem 1.2, there exists a unique measure μ on $[0, \infty)$ such that

$$\log f(x) = \int_0^\infty e^{-xt}\, d\mu(t). \tag{2.2}$$

Also, by Definition 1.1, we have

$$(-1)^n \left(\int_0^\infty e^{-xt}\, d\mu(t) \right)^{(n)} \geq 0, \quad \forall\, n \in \mathbb{N} \cup \{0\}, \quad \forall\, x \in (0, \infty). \tag{2.3}$$

From (2.2) we get

$$f(x) = \exp\left(\int_0^\infty e^{-xt}\, d\mu(t) \right). \tag{2.4}$$

Clearly, f in (2.4) is always non-negative. Now our aim is to show that f defined in (2.4), satisfies (1.1) and we prove the result by induction.

Differentiating (2.4), we obtain

$$f'(x) = f(x) \left(\int_0^\infty e^{-xt}(-t)\, d\mu(t) \right)$$

Since $f(x)$ is always non-negative and the integral $\int_0^\infty e^{-xt}(-t)\, d\mu(t)$ is negative (using (2.3)), it follows that $f'(x) \leq 0$.

Let us assume that the complete monotonicity of f given in (2.4) holds for some $k \in \mathbb{N}$, i.e.,

$$(-1)^n \left(\exp\left(\int_0^\infty e^{-xt}\, d\mu(t) \right) \right)^{(n)} \geq 0, \quad \forall\, n \leq k. \tag{2.5}$$

Now we take

$$\begin{aligned} (-1)^{k+1} f^{(k+1)}(x) &= (-1)^{k+1} \frac{d^k}{dx^k} f'(x) \\ &= (-1)^{k+1} \frac{d^k}{dx^k} \left(\exp\left(\int_0^\infty e^{-xt} d\mu(t) \right) \left(\int_0^\infty e^{-xt}\, d\mu(t) \right)' \right), \end{aligned} \tag{2.6}$$

where $\left(\int_0^\infty e^{-xt}\, d\mu(t) \right)'$ denotes the derivative of the integral $\left(\int_0^\infty e^{-xt}\, d\mu(t) \right)$.

Using Leibniz's rule for the derivative of a product in (2.6), we have

$$(-1)^{k+1} f^{(k+1)}(x) = (-1)^{k+1} \sum_{j=0}^{k} \binom{k}{i} \left(\int_0^\infty e^{-xt} d\mu(t) \right)^{(j+1)} \left(\exp \left(\int_0^\infty e^{-xt} d\mu(t) \right) \right)^{(k-j)} \tag{2.7}$$

Rewriting (2.7) leads to

$$(-1)^{k+1} f^{(k+1)}(x)$$
$$= \sum_{j=0}^{k} \binom{k}{j} \underbrace{\left((-1)^{j+1} \left(\int_0^\infty e^{-xt} d\mu(t) \right)^{(j+1)} \right)}_{C_1} \underbrace{\left((-1)^{k-j} \left(\exp(\int_0^\infty e^{-xt} d\mu(t)) \right)^{(k-j)} \right)}_{C_2}$$

Clearly, from (2.3), C_1 is non-negative for all values of i, and C_2 is also non-negative, as per our assumption in (2.5). Therefore, we conclude that

$$(-1)^{k+1} f^{(k+1)}(x) \geq 0.$$

Thus, by induction, (1.1) is true and the proof is complete. □

Apart from the result that logarithmic complete monotonicity implies complete monotonicity, Qi and Chen examined the complete monotonicity of the function g, given in (2.1), on $(0, \infty)$ as Theorem 2 in [27] which is given below.

Theorem 2.2. *[27] The function g defined in (2.1) is completely monotonic on $(0, \infty)$.*

If we modify the function g defined in (2.1) by adding the term $-\frac{1}{2x} \log 2\pi x$, the modified function remains completely monotonic on $(0, \infty)$ which we have shown in following Theorem.

Theorem 2.3. *The function defined by*

$$\phi(x) = 1 - \log x + \frac{1}{x} \log \Gamma(x+1) - \frac{1}{2x} \log 2\pi x, \quad x \in (0, \infty), \tag{2.8}$$

is completely monotonic.

Proof. Writing the integral representation of $\log \Gamma(x)$ given in [[35] p.258] as

$$\log \Gamma(x) = \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log 2\pi + \underbrace{\int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tx}}{t} dt}_{I(x)}$$

Putting the value of $\log \Gamma(x)$ in (2.8), we obtain

$$\phi(x) = \frac{1}{x} I(x) \tag{2.9}$$

Since $\frac{1}{x}$ is completely monotonic on $(0, \infty)$, to show the complete monotonicity of ϕ, it is enough to show the complete monotonicity of $I(x)$, as the set of completely monotonic functions is closed under multiplication.

We have,

$$I(x) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tx}}{t} dt.$$

Consider the function

$$\alpha(t) = \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}.$$

By rewriting $\alpha(t)$, we obtain

$$\alpha(t) = \frac{(t-2)e^t + t + 2}{2t(e^t - 1)}.$$

Next, define the function

$$\beta(t) = (t-2)e^t + t + 2, \quad t \in [0, \infty).$$

Since $t = 0$ is a minimum for $\beta(t)$, it follows that

$$\beta(t) \geq \beta(0) = 0 \quad \forall\, t \in [0, \infty).$$

Consequently, $\beta(t) > 0$ for $t \in (0, \infty)$, which implies that $\alpha(t) > 0$ for $t \in (0, \infty)$. Therefore, $I(x)$ is the Laplace transform of a non-negative measure. By Theorem 1.2, it follows that $I(x)$ is completely monotonic on $(0, \infty)$, and hence, the function ϕ is also completely monotonic. In fact, we can find the measure with respect to which ϕ is completely monotone. Equation (2.9) can be written as

$$\phi(x) = \left(\int_0^\infty e^{-xt} dt\right)\left(\int_0^\infty e^{-xt}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{1}{t} dt\right).$$

This implies

$$\phi(x) = \mathcal{L}\left\{\frac{1}{x}\right\} * \mathcal{L}\left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right\}.$$

By using the convolution of Laplace transform, we can write ϕ as

$$\phi(x) = \int_0^\infty e^{-xt}\left(\int_0^t \left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u - 1}\right) du\right) dt. \tag{2.10}$$

which shows that ϕ is the Laplace transform of a non-negative measure. The positivity of the measure has already been established. □

From Theorem 2.3, it can bee seen that function ϕ can also be written in the form of g given in (2.1) as

$$\phi(x) = g(x) - \frac{1}{2x}\log 2\pi x, \quad x \in (0, \infty) \tag{2.11}$$

From Theorem 2.2, we know that g is completely monotonic on $(0, \infty)$, and in Theorem 2.3, we have proved that ϕ is also completely monotonic on $(0, \infty)$. However, the function $-\frac{1}{2x}\log 2\pi x$ is not completely monotonic on $(0, \infty)$. This provides one of the examples (although there are many) of the fundamental concept that the complete monotonicity of a function (expressed as the sum of two functions) on $(0, \infty)$ does not necessarily imply that each individual summand is completely monotonic on the same interval.

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References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, translated by N. Kemmer, Hafner Publishing Co., New York, 1965.

[2] H. Alzer and C. Berg, Some classes of completely monotonic functions. II, Ramanujan J. **11** (2006), no. 2, 225–248.

[3] K. M. Ball, Completely monotonic rational functions and Hall's marriage theorem, J. Combin. Theory Ser. B **61** (1994), no. 1, 118–124.

[4] C. Berg and G. Forst, *Potential theory on locally compact abelian groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 87, Springer, New York, 1975.

[5] C. Berg and H. L. Pedersen, A completely monotone function related to the gamma function, J. Comput. Appl. Math. **133** (2001), no. 1-2, 219–230.

[6] C. Berg and H. L. Pedersen, Pick functions related to the gamma function, Rocky Mountain J. Math. **32** (2002), no. 2, 507–525.

[7] C. Berg and H. L. Pedersen, A one-parameter family of Pick functions defined by the gamma function and related to the volume of the unit ball in n-space, Proc. Amer. Math. Soc. **139** (2011), no. 6, 2121–2132.

[8] C. Berg and H. L. Pedersen, A completely monotonic function used in an inequality of Alzer, Comput. Methods Funct. Theory **12** (2012), no. 1, 329–341.

[9] C. Berg, Open problems, Integral Transforms Spec. Funct. **26** (2015), no. 2, 90–95.

[10] C. Berg, A complete Bernstein function related to the fractal dimension of Pascal's pyramid modulo a prime, Expositiones Mathematicae, 2024.

[11] S. Bernstein, Sur les fonctions absolument monotones, Acta Math. **52** (1929), no. 1, 1–66.

[12] S. Das, H. L. Pedersen and A. Swaminathan, Pick functions related to the triple Gamma function, J. Math. Anal. Appl. **455** (2017), no. 2, 1124–1138.

[13] W. A. Day, On monotonicity of the relaxation functions of viscoelastic materials, Proc. Cambridge Philos. Soc. **67** (1970), 503–508.

[14] W. F. Donoghue Jr., *Monotone matrix functions and analytic continuation*, Die Grundlehren der mathematischen Wissenschaften, Band 207, Springer, New York, 1974.

[15] W. Feller, *An Introduction to Probability Theory and Its Applications. Vol. I*, John Wiley & Sons, Inc., New York, 1950.

[16] F. Hausdorff, Summationsmethoden und Momentfolgen. I, Math. Z. **9** (1921), no. 1-2, 74–109.

[17] F. Hausdorff, Momentprobleme für ein endliches Intervall, Math. Z. **16** (1923), no. 1, 220–248.

[18] S. Koumandos, Remarks on some completely monotonic functions, J. Math. Anal. Appl. **324** (2006), no. 2, 1458–1461.

[19] S. Koumandos, On Ruijsenaars' asymptotic expansion of the logarithm of the double gamma function, J. Math. Anal. Appl. **341** (2008), no. 2, 1125–1132.

[20] J.G. Liu and R. L. Pego, On generating functions of Hausdorff moment sequences, Trans. Amer. Math. Soc. **368** (2016), no. 12, 8499–8518.

[21] K. S. Miller and S. G. Samko, Completely monotonic functions, Integral Transform. Spec. Funct. **12** (2001), no. 4, 389–402.

[22] S. R. Mondal and A. Swaminathan, Geometric properties of generalized polylogarithm, Integral Transforms Spec. Funct. **21** (2010), no. 9-10, 691–701.

[23] H. L. Pedersen, The double gamma function and related Pick functions, J. Comput. Appl. Math. **153** (2003), no. 1-2, 361–369.

[24] H. L. Pedersen, Inverses of gamma functions, Constr. Approx. **41** (2015), no. 2, 251–267.

[25] S. Ponnusamy and S. Sabapathy, Polylogarithms in the theory of univalent functions, Results Math. **30** (1996), no. 1-2, 136–150.

[26] F. Qi and S. L. Xu, The function $(b^x - a^x)/x$: inequalities and properties, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359.

[27] F. Qi and C.P. Chen, A complete monotonicity property of the gamma function, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[28] S. Ruscheweyh, L. C. Salinas and T. Sugawa, Completely monotone sequences and universally prestarlike functions, Israel J. Math. **171** (2009), 285–304.

[29] O. Roth, S. Ruscheweyh and L. C. Salinas, A note on generating functions for Hausdorff moment sequences, Proc. Amer. Math. Soc. **136** (2008), no. 9, 3171–3176.

[30] K. Sato, *Lévy processes and infinitely divisible distributions*, translated from the 1990 Japanese original, Cambridge Studies in Advanced Mathematics, 68, Cambridge Univ. Press, Cambridge, 1999.

[31] R. Schilling, R. Song and Z. Vondraček, *Bernstein functions*, second edition, De Gruyter Studies in Mathematics, 37, de Gruyter, Berlin, 2012.

[32] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society Mathematical Surveys, Vol. I, Amer. Math. Soc., New York, 1943.

[33] T.J. Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. **8** (1894), no. 4, 1–122.

[34] H. Vogt and J. Voigt, A monotonicity property of the Γ-function, JIPAM. J. Inequal. Pure Appl. Math. **3** (2002), no. 5, Article 73, 3 pp.

[35] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, reprint of the fourth (1927) edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1996.

[36] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series, vol. 6, Princeton Univ. Press, Princeton, NJ, 1941.

[37] J. Wimp, *Sequence transformations and their applications*, Mathematics in Science and Engineering, 154, Academic Press, Inc., New York, 1981.

Author information

Deepshikha Mishra, Department of Mathematics, Indian Institute of Technology, Roorkee-247667, Uttarakhand, India.
E-mail: deepshikha_m@ma.iitr.ac.in

A. Swaminathan, Department of Mathematics, Indian Institute of Technology, Roorkee-247667, Uttarakhand, India.
E-mail: a.swaminathan@ma.iitr.ac.in, mathswami@gmail.com