**Palestine Journal of Mathematics**

**Vol 15(Special Issue I)(2026)** , 55–67 

# Some properties of the Generalized Jacobi polynomial via fractional calculus

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MSC 2010 Classifications: 33E20, 26A33, 26A99, 33C45.

Keywords and phrases: Generalized Jacobi polynomial, Fractional integral operators, Fractional differential operators, Lebesgue measurable functions.

*The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

**Abstract** *The present work aims towards the study of some new properties of the generalized Jacobi polynomial by employing the integral and differential operators of fractional calculus on the complex valued Lebesgue measurable functions. The results so obtained will be useful in the theories of special functions, where Jacobi polynomial arises naturally.*

## 1 Introduction and Preliminaries

*One of the most celebrated mathematician of modern era, namely Carl Gustav Jacobi (1804-1851) has introduced a well explored and useful polynomial in terms of Gauss hypergeometric function; having numerous applications in science and engineering [3, 5, 38], as a solution of differential equation* $(1-x^2)\,\frac{d^2y}{dx^2} + [\beta-\alpha-(2+\alpha+\beta)\,x]\,\frac{dy}{dx}, +n\,(1+\alpha+\beta+n)\,y\,(x) = 0\;(x,\alpha,\beta\in\mathbb{C};\,n\in\mathbb{N},\mathrm{Re}\,(\alpha)>(-1)\,,\,\mathrm{Re}\,(\beta)>0)$ *and which was denoted by* $P_n^{(\alpha,\beta)}(x)$*, defined as in* (1.1)*.*

$$P_n^{(\alpha,\beta)}(x) = \left[\frac{(1+\alpha)_n}{n!}\right] {}_2F_1\left(-n,\,\alpha+\beta+n+1;\,1+\alpha;\frac{1-x}{2}\right), \\ (x,\alpha,\beta\in\mathbb{C};\,n\in\mathbb{N},\mathrm{Re}\,(\alpha)>(-1)\,,\,\mathrm{Re}\,(\beta)>0)\,. \tag{1.1}$$

*Recently Waghela et al.[39, 40, 41], defined and studied Generalized Jacobi polynomial* $P_{n,\,\tau}^{(\alpha,\,\gamma,\,\beta)}(x)$ *:*

$$P_{n,\,\tau}^{(\alpha,\,\gamma,\,\beta)}(x) = \left[\frac{(\alpha+1)_n}{n!}\right] {}_2R_1\left(-n,\,n+\alpha+\beta+1;\,\gamma+1;\tau;\frac{1-x}{2}\right); \tag{1.2}$$

$x,\alpha,\beta,\gamma\in\mathbb{C}$ *and* $\tau>0$; $n\in\mathbb{N}$, $\mathrm{Re}\,(\alpha)\,,\,\mathrm{Re}\,(\gamma)>(-1)$ *and* $\mathrm{Re}\,(\beta)>0$ ; *which has its particular case as* (1.1)*, when* $\gamma=\alpha$ *and* $\tau=1$.

*The recent past after 1960's can be considered as an era of fractional calculus, which solves a real world problems with more efficacy as compared to conventional integer ordered calculus [1, 2, 10, 11, 12, 16, 17, 20, 21, 22]. Various fractional calculus operators have been studied in great details along with applications by many mathematicians [4, 6, 7, 8, 9, 10, 13, 14, 15, 18, 19, 23, 24, 25, 29, 30, 31, 33, 34, 36, 37].*
*As a part of useful contribution with integral operator involving newly defined Generalized Ja-*

*cobi fractional polynomial* (1.2)*, the present study introduces following operator* $\left(J^{\xi\,;\,\alpha,\beta}_{a+\,;\,\tau\,,\,\gamma}\,f\right)(x)$.

$$\left(J^{\xi\,;\,\alpha,\beta}_{a+\,;\,\tau\,,\,\gamma}\,f\right)(x) = J^{\xi\,;\,\alpha\,,\beta}_{a+\,;\,\tau,\,\gamma}\,f\,(x)$$
$$= \int\limits_a^x (x-t)^{\gamma} P^{\alpha,\gamma,\beta}_{n,\tau}\left(1-2\xi(x-t)^{\tau}\right) f\,(t)\,dt, \quad (x>a)\,,$$

*i.e.*

$$\left(J^{\xi\,;\,\alpha,\beta}_{a+\,;\,\tau\,,\,\gamma}\,f\right)(x) = J^{\xi\,;\,\alpha\,,\,\beta}_{a+\,;\,\tau,\,\gamma}\,f\,(x)$$
$$= \frac{(\alpha+1)_n}{n!}\int\limits_a^x (x-t)^{\gamma}{}_2R_1\left(-n,n+\alpha+\beta+1;\gamma+1;\tau;\xi(x-t)^{\tau}\right) f\,(t)\,dt, \tag{1.3}$$

*where* $x,\alpha,\beta,\gamma,b,\xi\in\mathbb{C}$ *and* $\tau>0$; $n\in\mathbb{N}$, $\mathrm{Re}\,(\alpha)\,,\ \mathrm{Re}\,(\gamma)>(-1)$ *and* $\mathrm{Re}\,(\beta)>0$ .

*On substituting* $\tau=1,\gamma=\alpha$, (1.3) *reduces to the following integral operator as in* (1.4) *(involving Jacobi polynomial* (1.1)*, which we denote it here by* $K^{\xi\,;\,\alpha,\beta}_{a+\,;\,\gamma}\,f\,(x)$.

$$K^{\xi;\alpha,\beta}_{a+;\gamma}\,f\,(x)$$
$$= \frac{(\alpha+1)_n}{n!}\int\limits_a^x (x-t)^{\gamma-1}{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\xi(x-t)^{\tau}\right) f\,(t)\,dt, \tag{1.4}$$

$x>a.$

$$K^{\xi;\alpha,\,\beta}_{a+;\gamma}\,f(x) = \frac{(\alpha+1)_n}{n!}\int_b^x (x-t)^{\gamma-1}P_n^{(\alpha,\beta)}\left(1-2\xi(x-t)^{\tau}\right) f(t)\,dt, x>a.$$

*As a part of preliminaries here we give following notations and definitions:*

- $$(\zeta)_n = \begin{cases} 1\ ,\,if\ n=0, \\ \zeta\,(\zeta+1)\,.....\,(\zeta+n-1)\,,\ if\ \ n=1\,,2\,,3\,,..... \\ \dfrac{\Gamma(n+\zeta)}{\Gamma(\zeta)}\,,\ (\zeta\in\mathbb{C}-\{0,-1,-2,...\}\,,\ \textit{with}\ \mathrm{Re}\,(\zeta)>0)\,, \\ \dfrac{(-1)^n(-\zeta)!}{(-n-\zeta)!}\,,\ if\ \ (\zeta=0,-1,-2,....\ \textit{and}\ 0\leqslant n\leqslant(\ -\zeta\,))\,, \\ 0\,,\ if\ \ (\zeta=0,-1,-2,....\ \textit{and}\ n>(-\zeta))\,; \end{cases} \tag{1.5}$$

  $(\zeta)_n$, *denotes the Pochhammer symbol [35].*

- *Left sided and right sided R-L integral operators are defined [17, 32] as below and denoted by* $I^{\mu}_{\alpha+}$ *and* $I^{\mu}_{\beta-}$ *respectively. If* $f\,(x)\in L\,(\alpha,\,\beta)\,,\ \mu\in\mathbb{C},\ \mathrm{Re}\,(\mu)>0$, *then:*

$$ {}_{\alpha}D_x^{-\mu}f\,(x) = {}_{\alpha}I_x^{\mu}f\,(x) = I^{\mu}_{\alpha+}f\,(x) = \left(I^{\mu}_{\alpha+}f\right)(x)$$
$$= \frac{1}{\Gamma\,(\mu)}\int\limits_{\alpha}^{x}\frac{f\,(t)}{(x-t)^{1-\mu}}dt,\ \ (x>\alpha) \tag{1.6}$$

*and*

$$ {}_{x}D_{\beta}^{-\mu}f\,(x) = {}_{x}I_{\beta}^{\mu}f\,(x) = I^{\mu}_{\beta-}f\,(x) = \left(I^{\mu}_{\beta-}f\right)(x)$$
$$= \frac{1}{\Gamma\,(\mu)}\int\limits_{x}^{\beta}\frac{f\,(t)}{(t-x)^{1-\mu}}dt,\ \ (x<\beta)\,. \tag{1.7}$$

- *Left and right sided R-L derivative operators are denoted by* $\left(D^{\mu}_{\alpha+}f\right)(x)$ *and* $\left(D^{\mu}_{\alpha-}f\right)(x)$ *respectively and defined by* (1.8), (1.9).
  *For* $\mu \in \mathbb{C}$ *with* $\mathrm{Re}(\mu) > 0$ *and* $n = [\mathrm{Re}(\mu)] + 1$,

$$\left(D^{\mu}_{\alpha+}f\right)(x) = \left(\frac{d}{dx}\right)^{n} \left(I^{n-\mu}_{\alpha+}f\right)(x). \tag{1.8}$$

$$\left(D^{\mu}_{\alpha-}f\right)(x) = \left(-\frac{d}{dx}\right)^{n} \left(I^{n-\mu}_{\alpha-}f\right)(x). \tag{1.9}$$

*A more general form of the Riemann-Liouville fractional derivative operator* $D^{\mu}_{\alpha+}$ *from* (1.6) *has been proposed by defining the fractional derivative operator* $D^{\mu\,,\nu}_{\alpha\,+}$, *of order* $0 < \mu < 1$ *and type* $0 \leq \nu \leq 1$, *with respect to* $x$ *as follows [10]:*

$$\left(D^{\mu,\,\nu}_{\alpha+}f\right)(x) = \left(I^{\nu\,(1-\mu)}_{\alpha+}\,\frac{d}{dx}\,\left(I^{(1-\nu)\ (1-\mu)}_{\alpha+}\,f\right)\right)(x). \tag{1.10}$$

*The equation* (1.10) *make more comprehensible to the classical Riemann-Liouville fractional derivative operator* $D^{\mu}_{\alpha+}$ *when* $\nu = 0$. *Furthermore, in the particular case where* $\nu = 1$, *equation* (1.10) *becomes the Caputo fractional derivative operator.*

- *According to [28], if* $a,\, b,\, c \in\ \mathbb{C}$; $\mathrm{Re}\,(a) > 0$, $\mathrm{Re}\,(b) > 0$, $\mathrm{Re}\,(c) > 0$ *then*

$$\begin{aligned}&\left(\frac{d}{dz}\right)^{m}\left[z^{c-1}{}_2R_1\left(a,\, b;\, c;\, \tau;\, \xi\, z^{\tau}\right)\right]\\ &= z^{c-m-1}\frac{\Gamma\,(c)}{\Gamma\,(c-m)}{}_2R_1\left(a,\, b;\, c-m;\, \tau;\, \xi\, z^{\tau}\right).\end{aligned} \tag{1.11}$$

- *Gauss multiplication theorem [26]: If* $m \in \mathbb{Z}^{+}$ *and* $z \in \mathbb{C}$ *then*

$$\prod_{t=1}^{m}\Gamma\left(z+\frac{t-1}{n}\right) = (2\pi)^{(m-1)/2}\quad m^{\frac{1}{2}-m\,z}\;\Gamma\,(mz). \tag{1.12}$$

- *The following result [37] holds true for the fractional derivative operator* $D^{\mu,\,\nu}_{\alpha+}f$ *defined by* (1.8) *as,*

$$\left(D^{\mu,\,\nu}_{\alpha+}\left[(t-\alpha)^{\lambda-1}\right]\right)(x) = \frac{\Gamma\,(\lambda)}{\Gamma\,(\lambda-\mu)}(x-\alpha)^{\lambda-\mu-1}, \tag{1.13}$$

  *where* $x > \alpha, 0 < \mu < 1; 0 \leqslant \nu \leqslant 1; \mathrm{Re}\,(\lambda) > 0$.

## 2 Operators for fractional integration and differentiation associated with polynomial $P^{(\alpha,\,\gamma,\,\beta)}_{n,\,\tau}(1-2x)$

**Remark 2.1.** Riemann-Liouville integrals of order $\mu$ concerning $P^{(\alpha,\,\gamma,\,\beta)}_{n,\,\tau}(1-2x)$ and $P^{(\alpha,\,\beta)}_{n}(1-2x)$ can immediately be obtained as

$$\begin{aligned}&I^{\mu}_{0+}\left\{P^{(\alpha,\,\gamma,\,\beta)}_{n,\,\tau}(1-2x)\right\}\\ &= \frac{(1+\alpha)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r\,\Gamma\,(1+\alpha+\beta+n+\tau r)\,\Gamma\,(1+\gamma)}{\Gamma\,(1+\alpha+\beta+n)\,\Gamma\,(1+\gamma+\tau r)}\frac{1}{\Gamma\,(1+\mu+r)}x^{r+\mu}\end{aligned}$$

and

$$\begin{aligned}&I^{\mu}_{0+}\left\{P^{(\alpha,\,\beta)}_{n}(1-2x)\right\}\\ &= \frac{(1+\alpha)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r}\frac{1}{\Gamma\,(1+\mu+r)}x^{r+\mu},\end{aligned}$$

by applying following Result [35] :

A function $f(z)$ is analytic within the disc $|z| < R$ can be expressed as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for ( |z| < R) then

$$_0D_z^{-\mu}\left\{z^{\zeta-1}f(z)\right\} = \frac{\Gamma(\zeta)}{\Gamma(\zeta+\mu)}z^{\zeta+\mu-1}\sum_{n=0}^{\infty}\frac{a_n(\zeta)_n}{(\zeta+\mu)_n}z^n,$$

for $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\zeta) > 0$.

**Theorem 2.2.** *If* $x, \alpha, \beta, \gamma \in \mathbb{C}$ *and* $\tau > 0; n \in \mathbb{N}$, $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\gamma) > (-1)$ *and* $\mathrm{Re}(\beta) > 0; a \in \mathbb{R}_+ = [0, \infty)$, $\mu, \xi \in \mathbb{C}, \mathrm{Re}(\mu) > 0$, $\tau > 0$, $x > \alpha, \xi \in \mathbb{C}$, *and* $\tau$, $\left|\xi(x-\alpha)^{\tau}\right| < 1$.

$$I_{a+}^{\mu}\left[(x-a)^{\gamma}P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right] = \frac{(x-a)^{\mu+\gamma}\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)}P_{n,\tau}^{(\alpha,\gamma+\mu,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right). \tag{2.1}$$

$$D_{a+}^{\mu}\left[(x-a)^{\gamma}P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right] = \Gamma(1+\gamma)\left\{\frac{(x-a)^{\gamma-\mu}}{\Gamma(1+\gamma-\mu)}P_{n,\tau}^{(\alpha,\gamma-\mu,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right\}. \tag{2.2}$$

*If* $0 < \mu < 1,\ 0 \leqslant \nu \leqslant 1$, *then*

$$\left(D_{a+}^{\mu,\nu}\left[(x-a)^{\gamma}P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right]\right) = (x-a)^{\gamma-\mu}\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\mu)}P_{n,\tau}^{(\alpha,\gamma-\mu,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right). \tag{2.3}$$

*Proof.*

$$\begin{aligned}
&I_{a+}^{\mu}\left[(x-a)^{\gamma}P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right]\\
&=\left[\frac{(\alpha+1)_n}{n!}\right]\frac{1}{\Gamma(\mu)}\frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha+\beta+n)}\sum_{r=0}^{n}\frac{(-n)_r\Gamma(1+\alpha+\beta+n+\tau r)}{\Gamma(1+\gamma+\tau r)r!}\\
&\quad\cdot\xi^r\left(\int_a^x\frac{(t-a)^{\gamma}}{(x-t)^{1-\mu}}(t-a)^{\tau r}dt\right)\\
&=\left[\frac{(\alpha+1)_n}{n!}\right]\frac{1}{\Gamma(\mu)}\frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha+\beta+n)}\\
&\quad\cdot\sum_{r=0}^{n}\frac{(-n)_r\Gamma(1+\alpha+\beta+n+\tau r)}{\Gamma(1+\gamma+\tau r)r!}\cdot\xi^r\left(\int_a^x\frac{(t-a)^{\gamma+\tau r}}{(x-t)^{1-\mu}}dt\right)\\
&=\left[\frac{(\alpha+1)_n}{n!}\right]\frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha+\beta+n)}\\
&\quad\cdot\sum_{r=0}^{n}\frac{(-n)_r\Gamma(1+\alpha+\beta+n+\tau r)}{\Gamma(1+\gamma+\tau r)r!}\xi^r I_{a+}^{\mu}\left[(x-a)^{\gamma+\tau r}\right].
\end{aligned}$$

By using: If $\mu,\ \beta \in \mathbb{C},\ \mathrm{Re}(\mu) > 0,\ \mathrm{Re}(\beta) > 0$ then

$$I_{\alpha+}^{\mu}(x-\alpha)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\mu+\beta)}(x-\alpha)^{\mu+\beta-1},$$

as in [20], we have,

$$
\begin{aligned}
& I_{a+}^{\mu}\left[(x-a)^{\gamma} P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right] \\
& =\left[\frac{(\alpha+1)_n}{n!}\right] \frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r) r!} \\
& \quad \cdot \frac{\Gamma(1+\gamma+\tau r)}{\Gamma(1+\gamma+\mu+\tau r)} \xi^{r}(x-a)^{\mu+\gamma+\tau r} \\
& =\left[\frac{(\alpha+1)_n}{n!}\right] \frac{(x-a)^{\mu+\gamma} \Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} \frac{\Gamma(1+\gamma+\mu)}{\Gamma(n+\alpha+\beta+1)} \\
& \quad \cdot \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\mu+\tau r)} \frac{\left(\xi(x-a)^{\tau}\right)^{r}}{r!} \\
& =\left[\frac{(\alpha+1)_n}{n!}\right] \frac{(x-a)^{\mu+\gamma} \Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} \\
& \quad \cdot { }_2 R_1\left(-n, n+\alpha+\beta+1 ; 1+\gamma+\mu ; \tau ; \xi(x-a)^{\tau}\right) \\
& =\frac{(x-a)^{\mu+\gamma} \Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} P_{n,\tau}^{(\alpha,\gamma+\mu,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right).
\end{aligned}
$$

This concludes the proof of (2.1).

Starting from the left-hand side of (2.2) and applying (1.8),

$$
\begin{aligned}
& D_{a+}^{\mu}\left[(x-a)^{\gamma} P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right] \\
& =D_{a+}^{\mu}\left[(x-a)^{\gamma}\left[\frac{(\alpha+1)_n}{n!}\right]{ }_2 R_1\left(-n, n+\alpha+\beta+1 ; 1+\gamma ; \tau ; \xi(x-a)^{\tau}\right)\right] \\
& =\left(\frac{d}{dx}\right)^{m} \\
& \cdot\left\{I_{a+}^{m-\mu}\left[(x-a)^{\gamma} \frac{(\alpha+1)_n}{n!}{ }_2 R_1\left(-n, n+\alpha+\beta+1 ; 1+\gamma ; \tau ; \xi(x-a)^{\tau}\right)\right]\right\},
\end{aligned}
$$

(Here, $m=[\operatorname{Re}(\mu)+1]$) on using (2.1), this results in the following expression:

$$
\begin{aligned}
& D_{a+}^{\mu}\left[(x-a)^{\gamma}\left[\frac{(\alpha+1)_n}{n!}\right]{ }_2 R_1\left(-n, n+\alpha+\beta+1 ; 1+\gamma ; \tau ; \xi(x-a)^{\tau}\right)\right] \\
& =\left(\frac{d}{dx}\right)^{m}\left[\frac{(\alpha+1)_n}{n!}\right] \frac{(x-a)^{m-\mu+\gamma} \Gamma(1+\gamma)}{\Gamma(1+\gamma+m-\mu)} \\
& \qquad \cdot{ }_2 R_1\left(-n, n+\alpha+\beta+1 ; 1+\gamma+m-\mu ; \tau ; \xi(x-a)^{\tau}\right).
\end{aligned}
$$

Applying (1.11) gives

$$
\begin{aligned}
& =\Gamma(1+\gamma) \frac{(x-a)^{\gamma-\mu}}{\Gamma(1+\gamma-\mu)}\left[\frac{(\alpha+1)_n}{n!}\right] \\
& \quad \cdot{ }_2 R_1\left(-n, n+\alpha+\beta+1 ; 1+\gamma-\mu ; \tau ; \xi(x-a)^{\tau}\right) \\
& =\Gamma(1+\gamma)\left\{\frac{(x-a)^{\gamma-\mu}}{\Gamma(1+\gamma-\mu)} P_{n,\tau}^{(\alpha,\gamma-\mu,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right\}.
\end{aligned}
$$

This demonstrates the validity of equation (2.2).

For deriving (2.3), observe that

$$\begin{aligned}&\left(D_{a+}^{\mu,\nu}\left[(x-a)^{\gamma}P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right)\right]\right)\\&=D_{a+}^{\mu,\nu}\\&\cdot\frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)}\frac{(\alpha+1)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r\Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r)}\frac{\xi^r}{r!}(x-a)^{\gamma+\tau r}\\&=\frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)}\frac{(\alpha+1)_n}{n!}\\&\cdot\sum_{r=0}^{n}\frac{(-n)_r\Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r)}\frac{\xi^r}{r!}\cdot D_{a+}^{\mu,\nu}\left[(x-a)^{\gamma+\tau r}\right];\end{aligned}$$

and using the following identity (1.13) from [37], for the fractional derivative operator $D_{\alpha+}^{\mu,\nu}f$ as in (1.8),

$$\left(D_{\alpha+}^{\mu,\nu}\left[(t-\alpha)^{\mu-1}\right]\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu-\mu)}(x-\alpha)^{\mu-\mu-1},$$

where $x>\alpha, 0<\mu<1\ , 0\leqslant\nu\leqslant1\ , \operatorname{Re}(\mu)>0)$ yields,

$$\begin{aligned}&D_{a+}^{\mu,\nu}\left[(x-a)^{\gamma}\frac{(\alpha+1)_n}{n!}\,{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(x-a)^{\tau}\right)\right]\\&=\frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)}\frac{(\alpha+1)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r\Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r)}\frac{\xi^r}{r!}\\&\quad\cdot\frac{\Gamma(1+\gamma+\tau r)}{\Gamma(1+\gamma+\tau r-\mu)}(x-a)^{\gamma+\tau r-\mu}\\&=(x-a)^{\gamma-\mu}\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\mu)}\frac{(\alpha+1)_n}{n!}\\&\quad\cdot{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma-\mu;\tau;\xi(x-a)^{\tau})\\&=(x-a)^{\gamma-\mu}\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\mu)}P_{n,\tau}^{(\alpha,\gamma-\mu,\beta)}\left(1-2\left(\xi(t-a)^{\tau}\right)\right).\end{aligned}$$

This establishes the proof of the assertion in equation (2.3).
Hence the Theorem 2.2. □

## 3 Certain properties of the operator $\left(J_{a+;\tau,\gamma}^{\xi;\alpha,\beta}f\right)(x)$

**Theorem 3.1.** *If* $x,\alpha,\beta,\gamma,\mu,\xi,a\in\mathbb{C}$ *and* $\tau>0$; $n\in\mathbb{N}$, $\operatorname{Re}(\alpha)$ , $\operatorname{Re}(\gamma)>(-1)$ *and* $\operatorname{Re}(\beta)>0$ ; *then*

$$\begin{aligned}&\left(J_{a+;\tau,\gamma}^{\xi;\alpha,\beta}(x-a)^{\mu-1}\right)\\&=(x-a)^{\gamma+\mu}\Gamma(\mu)\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)}P_{n,\tau}^{(\alpha,\gamma+\mu,\beta)}\left(1-2\left(\xi(x-a)^{\tau}\right)\right).\end{aligned}\tag{3.1}$$

*Proof.* Based on equation (1.3),

$$\left(J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} f(x)\right)$$
$$= \int_a^x (x-t)^{\gamma} P_{n,\,\tau}^{(\alpha,\,\gamma,\,\beta)} \left(1-2\left(\xi(x-t)^{\tau}\right)\right) f(t)\,dt, \quad (x>a)$$
$$= \frac{(\alpha+1)_n}{n} \int_a^x (x-t)^{\gamma} {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; \xi(x-t)^{\tau}\right) f(t)\,dt,$$
$$(x>a).$$

Therefore,

$$\left(J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} (x-a)^{\mu-1}\right)$$
$$= \frac{(\alpha+1)_n}{n!} \int_a^x (x-t)^{\gamma}$$
$$\cdot\ {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; \xi(x-t)^{\tau}\right)(t-a)^{\mu-1} dt$$

$$= \frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)} \left[\frac{(\alpha+1)_n}{n!}\right] \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r) r!} \xi^r$$
$$\cdot \left(\int_a^x (t-a)^{\mu-1}(x-t)^{\tau r+\gamma} dt\right)$$
$$= \frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)} \left[\frac{(\alpha+1)_n}{n}\right] \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r) r!}$$
$$\cdot\ \xi^r \left((x-a)^{\gamma+\tau r+\mu} \beta\left(\gamma+1+\tau r,\, \mu\right)\right)$$
$$= (x-a)^{\gamma+\mu} \Gamma(\mu) \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} \left[\frac{(\alpha+1)_n}{n!}\right]$$
$$\cdot \left\{\frac{\Gamma(1+\gamma+\mu)}{\Gamma(n+\alpha+\beta+1)} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\mu+\tau r)} \frac{\left(\xi(x-a)^{\tau}\right)^r}{r!}\right\}$$
$$\therefore \left(J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} (x-a)^{\mu-1}\right)$$
$$= (x-a)^{\gamma+\mu} \Gamma(\mu) \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} P_{n,\,\tau}^{(\alpha,\,\gamma+\mu,\,\beta)} \left(1-2\left(\xi(x-a)^{\tau}\right)\right).$$

This completes (3.1). □

**Theorem 3.2.** *If* $x, \alpha, \beta, \gamma, \mu, \xi \in \mathbb{C}$ *and* $\tau > 0$; $n \in \mathbb{N}$, $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\gamma) > (-1)$ *and* $\mathrm{Re}(\beta) > 0$ ; *and* $b > a$*, then the operator* $J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta}$ *is bounded on* $L(a,\, b)$ *and*

$$\left\| J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} f \right\|_1 \leqslant B \left\| f \right\|_1 ; \tag{3.2}$$

*where,*

$$B = (b-a)^{\mathrm{Re}(\gamma+1)} \left|\frac{(\alpha+1)n}{n}\right| \sum_{r=0}^{n} \frac{\left|(-n)_r\right| \left|(n+\alpha+\beta+1)_{\tau r}\right|}{\left|(\gamma+1)_{\tau r}\right| \left[\tau r + \mathrm{Re}(\gamma+1)\right]} \frac{\left|\xi(b-a)^{\tau}\right|^r}{r!} . \tag{3.3}$$

*Proof.* From (1.3) and (1.9),

$$\left(J^{\xi\,;\,\alpha,\beta}_{a+;\tau,\gamma}\,(x-a)^{\mu-1}\right)$$
$$=\frac{(\alpha+1)n}{n}\int_a^x (x-t)^{\gamma}{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(x-t)^{\tau}\right)$$
$$\cdot\,(t-a)^{\mu-1}dt$$

$$=\frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)}\left[\frac{(\alpha+1)n}{n!}\right]\sum_{r=0}^{n}\frac{(-n)_r\Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r)r!}\xi^r$$
$$\cdot\left(\int_a^x (t-a)^{\mu-1}(x-t)^{\tau r+\gamma}dt\right)$$
$$=\frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)}\left[\frac{(\alpha+1)n}{n}\right]\sum_{r=0}^{n}\frac{(-n)_r\Gamma(n+\alpha+\beta+1+\tau r)}{\Gamma(1+\gamma+\tau r)r!}$$
$$\cdot\,\xi^r\left((x-a)^{\gamma+\tau r+\mu}\beta\left(\gamma+1+\tau r,\,\mu\right)\right)$$

upon substituting $(x-t)=u$, we have

$$=\int_a^b\left[\int_0^{b-t}(u)^{\text{Re}(\gamma+1)-1}\left|\frac{(\alpha+1)_n}{n!}{}_2R_1\left(n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(u)^{\tau}\right)\right|du\right]$$
$$\cdot\,|f\,(t)|\,dt$$

Applying equation (1.2) and performing further simplification yields,

$$\leqslant\int_a^b\left|\frac{(\alpha+1)_n}{n!}\right|\sum_{r=0}^{n}\frac{|(-n)_r|\,|(n+\alpha+\beta+1)_{\tau r}|}{|(1+\gamma)_{\tau r}|}\frac{|\xi^r|}{r!}$$
$$\cdot\left(\frac{u^{\tau r+\text{Re}(1+\gamma)}}{\tau r+\text{Re}(1+\gamma)}\right)_0^{b-a}|f(t)|\,dt.$$

Alternatively,

$$\left\|J^{\xi\,;\,\alpha,\beta}_{a+;\tau,\gamma}\,f\right\|_1$$
$$=(b-a)^{\text{Re}(1+\gamma)}\left[\left|\frac{(\alpha+1)_n}{n!}\right|\sum_{r=0}^{n}\frac{\left|(-n)_r\right|\left|(n+\alpha+\beta+1)_{\tau r}\right|}{\left|(1+\gamma)_{\tau r}\right|\left[\tau r+\text{Re}\,(1+\gamma)\right]}\frac{\left|\xi(b-a)^{\tau}\right|^{r}}{r!}\right]$$
$$\cdot\int_a^b|f\,(t)|\,dt$$
$$=B\,\|f\|_{\,1};$$

where,

$$B=(b-a)^{\text{Re}(1+\gamma)}\left[\left|\frac{(\alpha+1)_n}{n!}\right|\sum_{r=0}^{n}\frac{\left|(n)_r\right|\left|(n+\alpha+\beta+1)_{\tau r}\right|}{\left|(1+\gamma)_{\tau r}\right|\left[\tau r+\text{Re}\,(1+\gamma)\right]}\frac{\left|\xi(b-a)^{\tau}\right|^{r}}{r!}\right].$$

This completes the proof of Theorem 3.2. □

**Theorem 3.3.** *Let* $x, \alpha, \beta, \gamma, \mu, \xi \in \mathbb{C}$ *and* $\tau > 0$; $n \in \mathbb{N}$, $\mathrm{Re}(\alpha)$ , $\mathrm{Re}(\gamma) > (-1)$ *and* $\mathrm{Re}(\beta) > 0$ ; *and* $b > a$ *then*

$$\begin{aligned}&\left(I_{a+}^{\mu}\left[J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} f\right]\right)(x)\\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)}\left(J_{a+;\tau,\gamma+\mu}^{\xi\,;\,\alpha,\beta} f\right)(x) = \left(J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta}\left[I_{a+}^{\mu} f\right]\right)(x)\,;\end{aligned} \tag{3.4}$$

*is valid for any summable function* $f \in L(a,b)$.

*Proof.* By combining (1.6) and (1.3), we get

$$\begin{aligned}&\left(I_{a+}^{\mu}\left[J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} f\right]\right)(x)\\
&= \frac{1}{\Gamma(\mu)}\int_a^x \frac{\left[\left(J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} f\right)(t)\right]}{(x-t)^{1-\mu}}\,dt\\
&= \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\left(\int_a^t (t-u)^{\gamma} P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1-2\left(\xi(t-u)^{\tau}\right)\right) f(u)\,du\right)dt\\
&= \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\\
&\left(\int_a^t (t-u)^{\gamma}\frac{(\alpha+1)_n}{n!}\,{}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; \xi(t-u)^{\tau}\right) f(u)\,du.\right)dt\end{aligned}$$

$$\begin{aligned}&\therefore \left(I_{a+}^{\mu}\left[J_{a+;\tau,\gamma}^{\xi;\alpha,\beta} f(x)\right]\right)\\
&= \int_a^x \frac{1}{\Gamma(\mu)}\frac{(\alpha+1)_n}{n!}\int_u^x (x-t)^{\mu-1}(t-u)^{\gamma}\\
&\quad\cdot {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; \xi(t-u)^{\tau}\right) dt\, f(u)\,du.\end{aligned}$$

By replacing $t-u$ with $\zeta$, we obtain:

$$\begin{aligned}&\left(I_{a+}^{\mu}\left[J_{a+;\tau,\gamma}^{\xi;\alpha,\beta} f(x)\right]\right)\\
&= \int_a^x \frac{1}{\Gamma(\mu)}\frac{(\alpha+1)_n}{n!}\int_0^{x-u} (x-u-\zeta)^{\mu-1}\zeta^{\gamma}\\
&\quad\cdot {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; \xi\zeta^{\tau}\right) d\zeta\, f(u)\,du\\
&= \int_a^x \frac{1}{\Gamma(\mu)}\frac{(\alpha+1)_n}{n!}\\
&\quad\cdot \int_0^{x-u} \frac{(\zeta)^{\gamma}{}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; \xi(\zeta)^{\tau}\right)}{((x-u)-\zeta)^{1-\mu}} d\zeta f(u)\,du\end{aligned} \tag{3.5}$$

use of (1.6) and applying (2.1) yield

$$\begin{aligned}&\left(I_{\alpha+}^{\mu}\left[J_{a+;\tau,\gamma}^{\xi\,;\,\alpha,\beta} f\right]\right)(x)\\
&= \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)}\left[\frac{(\alpha+1)_n}{n!}\right]\\
&\quad\cdot \int_a^x \left[(x-u)^{\mu+\gamma}{}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma+\mu; \tau; \xi(x-u)^{\tau}\right)\right] f(u)\,du.\end{aligned}$$

As a result,

$$\left(I^{\mu}_{\alpha+}\left[J^{\xi\,;\,\alpha,\beta}_{a+;\tau,\gamma}\,f\right]\right)(x)=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)}J^{\xi\,;\,\alpha,\beta}_{a+;\tau,\gamma+\mu}\,f(x)\,.$$

To prove the second part of Theorem (3.4), we start by examining the right-hand side of (3.4). By applying (1.3), we obtain

$$\begin{aligned}
&\left(J^{\xi\,;\,\alpha,\beta}_{a+;\tau,\gamma}\left[I^{\mu}_{a+}f\right]\right)(x)\\
&=\int_a^x (x-t)^{\gamma}P^{(\alpha,\,\gamma,\,\beta)}_{n,\,\tau}\left(1-2\left(\xi(x-t)^{\tau}\right)\right)\left(I^{\mu}_{a+}f\right)(t)\,dt\\
&=\int_a^x (x-t)^{\gamma}\frac{(\alpha+1)_n}{n!}\\
&\cdot\,{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(x-t)^{\tau}\right)\left(I^{\mu}_{a+}f\right)(t)\,dt\\
&=\frac{(\alpha+1)_n}{n!}\int_a^x (x-t)^{\gamma}{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(x-t)^{\tau}\right)\\
&\cdot\left(\frac{1}{\Gamma(\mu)}\int_a^t\frac{f(u)}{(t-u)^{1-\mu}}du\right)dt.
\end{aligned}$$

$$\begin{aligned}
&\therefore\left(J^{\xi\,;\,\alpha,\beta}_{a+;\tau,\gamma}\left[I^{\mu}_{a+}f\right]\right)(x)\\
&=\frac{(\alpha+1)_n}{n!}\int_{u=a}^{x}\frac{1}{\Gamma(\mu)}\int_{t=u}^{x}(x-t)^{\gamma}(t-u)^{\mu-1}\\
&\cdot\,{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(x-t)^{\tau}\right)dt f(u)\,du.
\end{aligned}$$

Substituting $(x-t)=\zeta$ yields

$$\begin{aligned}
&\left(J^{\xi;\alpha,\beta}_{a+;\tau,\gamma}\left[I^{\mu}_{a+}f\right]\right)(x)\\
&=\frac{(\alpha+1)_n}{n!}\int_{u=a}^{x}\frac{1}{\Gamma(\mu)}\int_{\zeta=x-u}^{0}(\zeta)^{\gamma}(x-\zeta-u)^{\mu-1}\\
&\cdot\,{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(\zeta)^{\tau}\right)(-d\zeta)\,f(u)\,du\\
&=\frac{(\alpha+1)_n}{n!}\int_{u=a}^{x}\frac{1}{\Gamma(\mu)}\int_{\zeta=0}^{x-u}(\zeta)^{\gamma}(x-\zeta-u)^{\mu-1}\\
&\cdot\,{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;\xi(\zeta)^{\tau}\right)d\zeta f(u)\,du.
\end{aligned}$$

We have established the proof for (3.5). By applying the same method, we also derive the second identity of (3.4). □

## 4 Product formula of $P_{n,\tau}^{(\alpha,\gamma,\beta)}(1-2x)$

**Theorem 4.1.** *When $\tau = m$ where $m \in \mathbb{N}$, the generalized Jacobi polynomial $P_{n,\tau}^{(\alpha,\gamma,\beta)}(1-2x)$ takes the form of:*

$$\begin{aligned} P_{n,\tau=m}^{(\alpha,\gamma,\beta)}(1-2x) &= \left[\frac{(\alpha+1)_n}{n!}\right] \frac{(2\pi)^{(m-1)/2}}{m^{\gamma+\frac{1}{2}}} \frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)} \\ &\cdot \prod_{t=0}^{m-1} \left( \frac{1}{\Gamma\left(\frac{1+\gamma+t}{n}\right)} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+mr)}{\left(\frac{1+\gamma+t}{n}\right)_r} \frac{x^r}{r! m^{mr}} \right); \end{aligned} \tag{4.1}$$

$x, \alpha, \beta, \gamma \in \mathbb{C}$ ; $n, m \in \mathbb{N}$, $\mathrm{Re}\,(\alpha)$ , $\mathrm{Re}\,(\gamma) > (-1)$ *and* $\mathrm{Re}\,(\beta) > 0$ .

*Proof.* Putting $x = r + \frac{(\gamma+1)}{m}$ in (1.12), we obtain

$$\begin{aligned} \frac{1}{\Gamma(mx)} = \frac{1}{\Gamma(\gamma+1+mr)} &= \frac{(2\pi)^{(m-1)/2}}{m^{m\,x-\frac{1}{2}}} \frac{1}{\prod_{t=1}^{m} \Gamma\left(x + \frac{t-1}{m}\right)} \\ &= \frac{(2\pi)^{(m-1)/2}}{m^{\gamma+1-\frac{1}{2}}} \frac{1}{m^{mr} \prod_{t=0}^{m-1} \Gamma\left(r + \frac{\gamma+1+t}{m}\right)} \\ &= \frac{(2\pi)^{(m-1)/2}}{m^{\gamma+\frac{1}{2}}} \frac{1}{m^{mr} \prod_{t=0}^{m-1} \Gamma\left(r + \frac{\gamma+1+t}{m}\right)}. \end{aligned}$$

Thus,

$$\frac{1}{\Gamma(\gamma+1+mr)} = \frac{(2\pi)^{(m-1)/2}}{m^{\gamma+\frac{1}{2}}} \frac{1}{m^{mr} \prod_{t=0}^{m-1} \Gamma\left(r + \frac{\gamma+1+t}{m}\right)}. \tag{4.2}$$

From (4.2) and (1.2) subsequently, for $\tau = m \in \mathbb{N}$, we obtain

$$\begin{aligned} &P_{n,\tau}^{(\alpha,\gamma,\beta)}(1-2x) \\ &= \frac{(\alpha+1)_n}{n!} {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; m; x\right) \\ &= \frac{(\alpha+1)_n}{n!} \frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)} \\ &\cdot \sum_{r=0}^{n} \left( \frac{(-n)_r \Gamma(n+\alpha+\beta+1+mr)}{m^{mr} \prod_{t=0}^{m-1} \Gamma\left(r + \frac{1+\gamma+t}{m}\right)} \frac{(2\pi)^{(m-1)/2}}{m^{\gamma+\frac{1}{2}}} \frac{x^r}{r!} \right) \\ &= \frac{(\alpha+1)_n}{n!} \frac{(2\pi)^{(m-1)/2}}{m^{\gamma+\frac{1}{2}}} \frac{\Gamma(1+\gamma)}{\Gamma(n+\alpha+\beta+1)} \\ &\cdot \prod_{t=0}^{m-1} \left( \frac{1}{\Gamma\left(\frac{1+\gamma+t}{m}\right)} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(n+\alpha+\beta+1+mr)}{\left(\frac{1+\gamma+t}{m}\right)_r} \frac{x^r}{r! m^{mr}} \right). \end{aligned}$$

This finalizes the proof of Theorem 4.1. □

**Remark 4.2.** By setting $m = 1$ in equation (4.1), it reduces to $P_n^{(\alpha,\beta)}(1-2x)$.

***Conclusion :***
*The present research focuses on the theoretical investigations and development of certain results involving generalized Jacobi polynomial* $P_{n,\tau}^{(\alpha,\gamma,\beta)}(x)$ *and the operator* $J_{a+;\tau,\gamma}^{\xi;\alpha,\beta}f(x)$. *This study aims to derive and analyze results that enhance our understanding about behaviour of the polynomial. The discussion revolves around advanced mathematical concepts involving fractional calculus. These topics collectively represent advanced mathematical tools that expand traditional calculus and integration theories, offering powerful frameworks for analyzing and understanding complex mathematical phenomena across various disciplines.*

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