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HYPERGEOMETRIC FORMS OF SOME TYPICAL MATHEMATICAL FUNCTIONS AND THEIR SUCCESSIVE DIFFERENTIATIONS

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Abstract In this paper, we derive hypergeometric forms of following typical trigonometric and inverse trigonometric functions:

$$covers^{-1}(x), \quad (covers^{-1}(x))^2 \quad covers(x), \quad \frac{vers^{-1}(x)}{\sqrt{(2x)}}, \quad \left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2, \quad \frac{(vers^{-1}(x))^2}{2}, \quad vers(x)$$

and successive differential coefficients:

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right), \quad \frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2, \quad \frac{d^n}{dx^n}\left(\frac{(vers^{-1}(x))^2}{2}\right), \quad \frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right),$$

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right)^2, \quad \frac{d^n}{dx^n}\left(covers\left(\frac{\pi}{2}\pm 2\sqrt{x}\right)\right), \quad \frac{d^n}{dx^n}\left(vers(\pm 2\sqrt{x})\right).$$

1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0, -1, -2, -3, \cdots\}$.
Also, the symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.
The well-known Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$[8, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [9, p.23, eq(22) and eq(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ 0 \leqq k \leqq n) \\ 0 & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ k > n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that$(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array} z\right] = {}_pF_q\left[\begin{array}{c}\alpha_1, \alpha_2, \ldots, \alpha_p;\\ \\ \beta_1, \beta_2, \ldots, \beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}, \quad (1.1)$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots \;;\; j = 1, 2, \ldots, q.$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$,

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$,

(iv) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) > 0$,

(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.2}$$

$\Re(\omega)$ being the real part of complex number ω.

Each of the following results will be needed in our present study.

- **Some Basic Results for** covers(x), covers$^{-1}(x)$, vers(x) and vers$^{-1}(x)$: [5, p.36 ,eqns (v) (vi), and p.40], see also [6, pp.44 and 46]

$$covers(x) = 1 - \sin(x) \tag{1.3}$$

Putting $x = 0$ in (1.3), we have

$$covers(0) = 1 \tag{1.4}$$

or

$$0 = covers^{-1}(1) \tag{1.5}$$

Putting $x = \frac{\pi}{2}$ in (1.3), we have

$$covers\left(\frac{\pi}{2}\right) = 0 \tag{1.6}$$

or

$$\frac{\pi}{2} = covers^{-1}(0) \tag{1.7}$$

Suppose

$$y = covers^{-1}(x) \tag{1.8}$$

$$covers(y) = 1 - \sin(y) = x \tag{1.9}$$

Now differentiating (1.9) w.r.t x, we get

$$\frac{dy}{dx} = \frac{-1}{\cos(y)} = \frac{-1}{\sqrt{1 - \sin^2(y)}}$$

$$= \frac{-1}{\sqrt{1-(1-x)^2}}$$

[5, p.40]:

$$\frac{d}{dx} covers^{-1}(x) = \frac{-1}{\sqrt{2x-x^2}} \tag{1.10}$$

[5, p.36, eqns (v) and (vi)] , see also [6, pp.44 and 46]

$$y = vers(x) = 1 - \cos(x) \tag{1.11}$$

Putting $x = 0$ in (1.11), we obtain

$$vers(0) = 0 \tag{1.12}$$

or

$$0 = vers^{-1}(0) \tag{1.13}$$

Putting $x = \frac{\pi}{2}$ in (1.11), we get

$$vers\left(\frac{\pi}{2}\right) = 1 \tag{1.14}$$

Suppose

$$y = vers^{-1}(x) \tag{1.15}$$

$$vers(y) = x = 1 - \cos(y) \tag{1.16}$$

or

$$y = vers^{-1}(x) = \cos^{-1}(1-x) \tag{1.17}$$

Now differentiating (1.16) w.r.t y, we get

$$\frac{dx}{dy} = \sin(y)$$

$$\frac{dy}{dx} = \frac{1}{\sin(y)} = \frac{1}{\sqrt{1-\cos^2(y)}} = \frac{1}{\sqrt{1-(1-x)^2}}$$

Therefore[5, p.40]:

$$\frac{d}{dx}(vers^{-1}(x)) = \frac{1}{\sqrt{2x-x^2}} \tag{1.18}$$

- **Some Basic Formulae For Pochhammer's Symbol:**

$$5.7.9.11\ldots(2n+3) = 2^n\left(\frac{5}{2}\right)_n \tag{1.19}$$

$$3.5.7.9\ldots(2n+1) = 2^n\left(\frac{3}{2}\right)_n \tag{1.20}$$

$$1.3.5.7.9\ldots(2n-1) = 2^n\left(\frac{1}{2}\right)_n \tag{1.21}$$

$$\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) = \prod_{i=1}^{n}(\alpha+i-1) = (\alpha)_n \tag{1.22}$$

- **Decomposition Formula For Infinite Series:**

$$\sum_{n=0}^{\infty}\Phi(n)=\sum_{n=0}^{\infty}\Phi(2n)+\sum_{n=0}^{\infty}\Phi(2n+1), \tag{1.23}$$

provided that series in both sides, are absolutely convergent .

- **Maclaurin's Expansion:**

$$F(x)=\sum_{n=0}^{\infty}\left(\frac{d^nF(x)}{dx^n}\right)_{x=0}\frac{x^n}{n!}=F(0)+xF^{'}(0)++\frac{x^2}{2!}F^{''}(0)+\frac{x^3}{3!}F^{'''}(0)+\ldots. \tag{1.24}$$

- **Taylor's Expansion:**

$$F(x)=\sum_{n=0}^{\infty}\left(\frac{d^nF(x)}{dx^n}\right)_{x=a}\frac{(x-a)^n}{n!}=$$

$$=F(a)+(x-a)F^{'}(a)+\frac{(x-a)^2}{2!}F^{''}(a)+\frac{(x-a)^3}{3!}F^{'''}(a)+\ldots. \tag{1.25}$$

- **Leibnitz's Theorem:**

$$D^n[U(x)T(x)]=({}^nc_0)(D^nU)(D^0T)+({}^nc_1)(D^{n-1}U)(D^1T)+({}^nc_2)(D^{n-2}U)(D^2T)+\ldots$$
$$+({}^nc_{n-1})(D^1U)(D^{n-1}T)+({}^nc_n)(D^0U)(D^nT). \tag{1.26}$$

- n^{th} **Derivative of Binomial Function:**

$$\frac{d^n}{dx^n}\left({}_1F_0\left[\begin{matrix}a; & \\ & \frac{x}{2}\\ -; & \end{matrix}\right]\right)=\frac{d^n}{dx^n}\left(\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\left(\frac{x}{2}\right)^r\right) \tag{1.27}$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{r!\,2^r}\frac{d^n}{dx^n}(x^r)$$

$$=\sum_{r=n}^{\infty}\frac{(a)_r}{2^r}\frac{x^{r-n}}{(r-n)!}$$

Replacing r by $r+n$, we get

$$\frac{d^n}{dx^n}\left(1-\frac{x}{2}\right)^{-a}=\frac{d^n}{dx^x}\left({}_1F_0\left[\begin{matrix}a; & \\ & \frac{x}{2}\\ -; & \end{matrix}\right]\right)=\sum_{r=0}^{\infty}\frac{(a)_{r+n}}{2^{r+n}}\frac{x^r}{r!}$$

$$=\frac{(a)_n}{2^n}\sum_{r=0}^{\infty}\frac{(a+n)_r}{r!}\left(\frac{x}{2}\right)^r$$

$$=\frac{(a)_n}{2^n}{}_1F_0\left[\begin{matrix}a+n; & \\ & \frac{x}{2}\\ -; & \end{matrix}\right] \tag{1.28}$$

$$=\frac{(a)_n}{2^n}\left(1-\frac{x}{2}\right)^{-a-n} \tag{1.29}$$

- **Clausen's Formula**[8, p.275,Q.N.9] see also [3, p.377] and [2, p.86,eq. 4]:

$$\left({}_2F_1\left[\begin{array}{ccc} a, b; & & \\ & & x \\ a+b+\frac{1}{2}; & & \end{array}\right]\right)^2 = {}_3F_2\left[\begin{array}{ccc} 2a, 2b, a+b; & & \\ & & x \\ 2a+2b, a+b+\frac{1}{2} & & \end{array}\right] \tag{1.30}$$

where $a+b+\frac{1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|x| < 1$.

- If $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

then

$$\sin(n\pi) = 0, \ \ \cos(n\pi) = (-1)^n, \ \ \sin(2n+1)\frac{\pi}{2} = (-1)^n, \ \ \cos(2n+1)\frac{\pi}{2} = 0 \tag{1.31}$$

- **Fractional Derivative Formula**[9, p.285, eq.(5)]:

$$\frac{d^\mu}{dz^\mu}\, z^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, z^{\lambda-\mu}, \tag{1.32}$$

where μ and λ are arbitrary complex numbers such that $\Gamma(\lambda+1)$ and $\Gamma(\lambda-\mu+1)$ are well defined and meaningful.

This paper is organized as follows:
Motivated by the hypergeometric forms of some mathematical functions recorded in the literature [1],[2],[7],[8] and [9], in **section 2** we mention, hypergeometric forms of some typical trigonometric functions $covers^{-1}(x)$, $(covers^{-1}(x))^2$ $covers(x)$, $\frac{vers^{-1}(x)}{\sqrt{(2x)}}$, $\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2$, $\frac{(vers^{-1}(x))^2}{2}$ *and* $vers(x)$ with suitable convergence conditions . Also we have given their detailed derivations by using Taylor's expansion , Maclaurin's expansion, Decomposition formula, Leibnitz Theorem and Clausen's formula. In **section 3** we have given nth differential coefficients of $\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)$, $\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2$, $\frac{(vers^{-1}(x))^2}{2}$, $covers^{-1}(1-\sqrt{x})$, $\left(covers^{-1}(1-\sqrt{x})\right)^2$, $covers(\frac{\pi}{2} \pm 2\sqrt{x})$ and $(vers(\pm 2\sqrt{x}))$ using series rearrangement techniques.

2 Hypergeometric Forms of Some Trigonometric Functions and Inverse Trigonometric Functions

Under associated convergence conditions, the following results hold true:

$$covers^{-1}(x) = (1-x)\ {}_2F_1\left[\begin{array}{ccc} \frac{1}{2}, \frac{1}{2}; & & \\ & & (x-1)^2 \\ \frac{3}{2}; & & \end{array}\right] ; \ |x-1| \leq 1 \tag{2.1}$$

$$\left(covers^{-1}(x)\right)^2 = (1-x)^2\ {}_3F_2\left[\begin{array}{ccc} 1, 1, 1; & & \\ & & (x-1)^2 \\ 2, \frac{3}{2}; & & \end{array}\right] ; \ |x-1| \leq 1 \tag{2.2}$$

$$covers(x) = \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 {}_1F_2\left[\begin{array}{ccc} 1; & & \\ & & -\frac{1}{4}\left(x-\frac{\pi}{2}\right)^2 \\ 2, \frac{3}{2}; & & \end{array}\right] ; \forall\ x \tag{2.3}$$

$$\frac{vers^{-1}(x)}{\sqrt{(2x)}} = {}_2F_1\left[\begin{array}{ccc} \frac{1}{2}, \frac{1}{2}; & & \\ & & \frac{x}{2}; \\ \frac{3}{2}; & & \end{array}\right] ; \ |x| \leq 2 \tag{2.4}$$

$$\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2 = {}_3F_2\left[\begin{matrix} 1,1,1; \\ \\ 2,\frac{3}{2}; \end{matrix} \quad \frac{x}{2}\right] \; ; \; |x| \le 2 \tag{2.5}$$

$$\frac{(vers^{-1}(x))^2}{2} = x + \frac{x^2}{6}\; {}_3F_2\left[\begin{matrix} 1,2,2; \\ \\ 3,\frac{5}{2}; \end{matrix} \quad \frac{x}{2}\right] \; ; \; |x| \le 2 \tag{2.6}$$

$$vers(x) = \frac{x^2}{2}\; {}_1F_2\left[\begin{matrix} 1; \\ \\ 2,\frac{3}{2}; \end{matrix} \quad \frac{-x^2}{4}\right] \; ; \; \forall \; x \tag{2.7}$$

Proof of the result (2.1):

Let

$$y = covers^{-1}(x), \quad (y)_1 = 0 \tag{2.8}$$

$$covers(y) = 1 - \sin(y) = x \tag{2.9}$$

Differentiating (2.9) w.r.t x, we have

$$-\cos(y)\frac{dy}{dx} = 1$$

$$y_1 = \frac{-1}{\cos(y)} = \frac{-1}{\sqrt{1-\sin^2(y)}} = \frac{-1}{\sqrt{1-(1-x)^2}}$$

Therefore

$$y_1 = \frac{-1}{\sqrt{2x - x^2}} \tag{2.10}$$

Putting $x = 1$ in (2.10), we get

$$(y_1)_1 = -1 \tag{2.11}$$

Again equation (2.10) can be written as:

$$\sqrt{(2x - x^2)}\;\; y_1 = -1 \tag{2.12}$$

Again differentiate with respect to x, we get

$$\sqrt{(2x - x^2)}\;\; y_2 + \frac{(1-x)\, y_1}{\sqrt{2x - x^2}} = 0$$

or

$$(2x - x^2)\;\; y_2 + (1-x)\;\; y_1 = 0 \tag{2.13}$$

Putting $x = 1$ in (2.13), we get

$$(y_2)_1 = 0 \tag{2.14}$$

Now differentiate (2.13) with respect to x, we have

$$(2x - x^2)\;\; y_3 + y_2\,(2 - 2x) + (1-x)\, y_2 + y_1(-1) = 0 \tag{2.15}$$

Putting $x = 1$ in (2.15) gives:

$$(y_3)_1 = (y_1)_1 = -1 \tag{2.16}$$

Now using Leibnitz's theorem (1.26) in (2.13), we get

$$(2x - x^2)\;\; y_{n+2} + n(2-2x)\;\; y_{n+1} + \frac{n(n-1)}{2}(-2)\, y_n + (1-x)\, y_{n+1} + n(-1)\, y_n = 0 \tag{2.17}$$

Putting $x = 1$ in (2.17), we get

$$(y_{n+2})_1 + 0 - n(n-1)\ (y_n)_1 - n(y_n)_1 = 0 \tag{2.18}$$

or

$$(y_{n+2})_1 = n^2 (y_n)_1 \tag{2.19}$$

Again putting $n = 2, 3, 4, 5, 6, 7, 8, 9\dots$ in (2.19) respectively, we get

$$\begin{aligned}
(y_4)_1 &= 2^2\ (y_2)_1 = 0, \quad ;\ (y_2)_1 = 0\\
(y_5)_1 &= 3^2 (y_3)_1 = (-1)\ 3^2\ 1^2, \quad ;\ (y_3)_1 = -1\\
(y_6)_1 &= 4^2\ (y_4)_1 = 0,\ ;\ (y_4)_1 = 0\\
(y_7)_1 &= 5^2\ (y_5)_1 = -5^2\ 3^2\ 1^2,\ ;\ (y_5)_1 = -3^2.1^2\\
(y_8)_1 &= 6^2\ (y_6)_1 = 0,\ ; (y_6)_1 = 0\\
(y_9)_1 &= 7^2\ (y_7)_1 = -7^2\ 5^2\ 3^2\ 1^2,\ ;\ (y_7)_1 = -5^2.3^2.1^2
\end{aligned} \tag{2.20}$$

Replacing n by $(n-2)$ in (2.19), we get

$$(y_n)_1 = (n-2)^2 (y_{n-2})_1 \tag{2.21}$$

Again replacing n by $(2n+1)$ in (2.21), we have

$$\begin{aligned}
(y_{2n+1})_1 &= (2n-1)^2\ (y_{2n-1})_1 \qquad (2.22)\\
&= (2n-1)^2 (2n-3)^2 (y_{2n-3})_1\\
&= (2n-1)^2 (2n-3)^2 (2n-5)^2 \dots 5^2\ 3^2\ 1^2\ (-1)^{2n+1}\\
&= -\left(1\ 3\ 5\dots(2n-1)\right)^2
\end{aligned}$$

Therefore,

$$(y_{2n+1})_1 = -2^{2n} \left[\left(\frac{1}{2}\right)_n\right]^2 \tag{2.23}$$

Using Taylor's expansion (1.25) around the point $x = 1$, we have

$$y(x) = \sum_{n=0}^{\infty} (y_n)_1\ \frac{(x-1)^n}{n!} \tag{2.24}$$

Using decomposition formula for infinite series in (2.24), we get

$$y(x) = \sum_{n=0}^{\infty} (y_{2n})_1\ \frac{(x-1)^{2n}}{2n!} + \sum_{n=0}^{\infty} (y_{2n+1})_1 \frac{(x-1)^{2n+1}}{(2n+1)!} \tag{2.25}$$

we know that all even order differential coefficients at $x = 1$ are zero. Therefore in view of result (2.23) above equation reduces to:

$$y(x) = -\sum_{n=0}^{\infty} \frac{2^{2n}\left[\left(\frac{1}{2}\right)_n\right]^2 (x-1)^{2n+1}}{(2n+1)!},$$

which on simplification gives:

$$y(x) = -(x-1)\ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n (x-1)^{2n}}{n!\ \left(\frac{3}{2}\right)_n}\ .$$

. Hence

$$covers^{-1}(x) = (1-x)\ {}_2F_1\left[\begin{array}{c} \frac{1}{2},\frac{1}{2}; \\ \\ \frac{3}{2}; \end{array} (x-1)^2\right]. \tag{2.26}$$

Proof of the result (2.2):

Using equation (2.26), we have

$$.\left(covers^{-1}(x)\right)^2 = (1-x)^2\left({}_2F_1\left[\begin{array}{c} \frac{1}{2},\frac{1}{2}; \\ \\ \frac{3}{2}; \end{array} (x-1)^2\right]\right)^2 \tag{2.27}$$

$$= (1-x)^2\ {}_2F_1\left[\begin{array}{c} \frac{1}{2},\frac{1}{2}; \\ \\ \frac{3}{2}; \end{array} (x-1)^2\right]\ {}_2F_1\left[\begin{array}{c} \frac{1}{2},\frac{1}{2}; \\ \\ \frac{3}{2}; \end{array} (x-1)^2\right] \tag{2.28}$$

Using Clausen's formula (1.30) in (2.28), we get

$$\left(covers^{-1}(x)\right)^2 = (1-x)^2\ {}_3F_2\left[\begin{array}{c} 1,1,1; \\ \\ 2,\frac{3}{2}; \end{array} (x-1)^2\right]. \tag{2.29}$$

Proof of the result (2.3):
We know that:

$$y(x) = covers(x) = 1-\sin(x), \qquad y\left(\frac{\pi}{2}\right) = 0 \tag{2.30}$$

On differentiating (2.30) n times, w.r.t x ,we have

$$y^{(n)}(x) = 0 - \frac{d^n}{dx^n}\sin(x) = -\sin\left(\left(\frac{n\pi}{2}\right)+x\right) \tag{2.31}$$

Putting $x=\frac{\pi}{2}$ in (2.31), we get

$$y^{(n)}\left(\frac{\pi}{2}\right) = -\sin\left[\frac{n\pi}{2}+\frac{\pi}{2}\right] = -\cos\left(\frac{n\pi}{2}\right) \tag{2.32}$$

Using Taylor's expansion (1.25) around the point $x=\frac{\pi}{2}$, we have

$$y(x) = 0 - \sum_{n=1}^{\infty}\frac{(x-\frac{\pi}{2})^n}{n!}\cos\left(\frac{n\pi}{2}\right) \tag{2.33}$$

$$= -\sum_{n=0}^{\infty}\frac{(x-\frac{\pi}{2})^{n+1}}{(n+1)!}\cos\left((n+1)\frac{\pi}{2}\right) \tag{2.34}$$

Now using decomposition identity (1.23) in (2.34), we have

$$y(x) = -\sum_{n=0}^{\infty}\frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}\cos\left((2n+1)\frac{\pi}{2}\right) - \sum_{n=0}^{\infty}\frac{(x-\frac{\pi}{2})^{2n+2}}{(2n+2)!}(-1)^{n+1} \tag{2.35}$$

$$= 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}\sum_{n=0}^{\infty}\frac{(-1)^n\left(x-\frac{\pi}{2}\right)^{2n}}{2^{2n}\left(\frac{3}{2}\right)_n(2)_n}\frac{(1)_n}{n!}$$

$$covers(x) = \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2\ {}_1F_2\left[\begin{array}{c} 1; \\ \\ 2,\frac{3}{2}; \end{array} -\frac{1}{4}\left(x-\frac{\pi}{2}\right)^2\right]. \tag{2.36}$$

Proof of the result (2.4):

Suppose

$$y = \frac{vers^{-1}(x)}{\sqrt{(2x)}} \tag{2.37}$$

On using (1.17) in (2.37), we get

$$y = \frac{vers^{-1}(x)}{\sqrt{(2x)}} = \frac{\cos^{-1}(1-x)}{\sqrt{(2x)}} \tag{2.38}$$

or

$$(\sqrt{2x})\, y = \cos^{-1}(1-x) \tag{2.39}$$

Differentiate with respect to x, we have

$$\sqrt{2x}\, y_1 + y\,\frac{1}{\sqrt{2x}} = \frac{1}{\sqrt{(2x-x^2)}} \tag{2.40}$$

$$2x\, y_1 + y = \frac{1}{\sqrt{1-\frac{x}{2}}} = \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} \tag{2.41}$$

$$= {}_1F_0\left[\begin{matrix} \frac{1}{2}; \\ -; \end{matrix}\ \frac{x}{2}\right]. \tag{2.42}$$

Now using Leibnitz's theorem (1.26)and equation (1.29) in (2.41) , we get on simplification:

$$({}^nc_0)(y_{n+1})(2x) + ({}^nc_1)(y_n)(2) + y_n = \frac{\left(\frac{1}{2}\right)_n}{2^n}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}-n}. \tag{2.43}$$

Putting $x = 0$ in (2.43) , we get

$$2n\,(y_n)_0 + (y_n)_0 = \frac{\left(\frac{1}{2}\right)_n}{2^n} \tag{2.44}$$

or

$$(y_n)_0 = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{2^n\left(\frac{3}{2}\right)_n} \tag{2.45}$$

Now using Maclaurin's expansion, we have

$$y = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,(y_n)_0 \tag{2.46}$$

$$= \sum_{n-0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n 2^n}\frac{x^n}{n!} \tag{2.47}$$

Hence

$$\frac{vers^{-1}(x)}{\sqrt{(2x)}} = {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ \frac{x}{2}\right]. \tag{2.48}$$

Proof of the results (2.5) and (2.6):

Using the result (2.4), we have

$$\left(\frac{(vers^{-1}(x))}{\sqrt{(2x)}}\right)^2 = \frac{(vers^{-1}(x))^2}{2x} \tag{2.49}$$

$$= {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};\\ \frac{3}{2};\end{matrix}\ \frac{x}{2}\right]{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};\\ \frac{3}{2};\end{matrix}\ \frac{x}{2}\right]. \tag{2.50}$$

Further using Clausen's reduction formula (1.30) in (2.50), we get

$$\frac{(vers^{-1}(x))^2}{2x} = {}_3F_2\left[\begin{matrix}1,1,1;\\ 2,\frac{3}{2};\end{matrix}\ \frac{x}{2}\right], \tag{2.51}$$

which proves result (2.5).

Again,

$$\frac{(vers^{-1}(x))^2}{2} = x\ {}_3F_2\left[\begin{matrix}1,1,1;\\ 2,\frac{3}{2};\end{matrix}\ \frac{x}{2}\right] \tag{2.52}$$

$$= x\left[1+\sum_{r=1}^{\infty}\frac{(1)_r(1)_r(1)_r}{(2)_r\left(\frac{3}{2}\right)_r r!}\left(\frac{x}{2}\right)^r\right]$$

$$= x\left[1+\sum_{r=0}^{\infty}\frac{(1)_{r+1}(1)_{r+1}}{(2)_{r+1}\left(\frac{3}{2}\right)_{r+1}}\left(\frac{x}{2}\right)^{r+1}\right]$$

$$= x\left[1+\frac{(1)_1(1)_1\left(\frac{x}{2}\right)^1}{(2)_1\left(\frac{3}{2}\right)_1}\sum_{r=0}^{\infty}\frac{(2)_r(2)_r(1)_r}{(3)_r\left(\frac{5}{2}\right)_r r!}\left(\frac{x}{2}\right)^r\right]$$

$$= x\left([1+\frac{x}{6}\ {}_3F_2\left[\begin{matrix}1,2,2;\\ 3,\frac{5}{2};\end{matrix}\ \frac{x}{2}\right]\right)$$

$$\frac{(vers^{-1}(x))^2}{2} = x+\frac{x^2}{6}{}_3F_2\left[\begin{matrix}1,2,2;\\ 3,\frac{5}{2};\end{matrix}\ \frac{x}{2}\right]. \tag{2.53}$$

Proof of the result (2.7):
We know

$$y = vers(x) = 1-\cos(x),\ \ (y)_0 = 0 \tag{2.54}$$

Differentiating (2.54) w.r.t x, we get

$$\frac{dy}{dx} = y_1 = \sin(x) \tag{2.55}$$

Taking nth derivative of (2.55), we get

$$y_{n+1} = \sin\left(\frac{n\pi}{2}+x\right) \tag{2.56}$$

Putting $x=0$ in (2.56), we obtain

$$(y_{n+1})_0 = \sin\left(\frac{n\pi}{2}\right) \tag{2.57}$$

Now using Maclaurin's expansion (1.24) , we have

$$y = \sum_{n=0}^{\infty} (y_n)_0 \frac{x^n}{n!} \tag{2.58}$$

$$y = 0 + \sum_{n=1}^{\infty} \frac{x^n}{n!}(y_n)_0 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}(y_{n+1})_0$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \sin\left(\frac{n\pi}{2}\right) \tag{2.59}$$

Using decomposition formula (1.23) in (2.59), we have

$$vers(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \sin\left(\frac{2n\pi}{2}\right) + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{(2n+2)!} \sin\left((2n+1)\frac{\pi}{2}\right)$$

$$= 0 + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{(2n+2)!} \sin\left((2n+1)\frac{\pi}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{(2n+2)!}(-1)^n \tag{2.60}$$

which on simplification gives:

$$vers(x) = \frac{x^2}{2}\ {}_1F_2\left[\begin{array}{cc} 1; & \\ & -\frac{x^2}{4} \\ 2, \frac{3}{2}; & \end{array}\right] . \tag{2.61}$$

3 Successive Differentiations of Some Typical Functions

Under associated convergence conditions, the following results hold true:

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right) = \frac{\left((\frac{1}{2})_n\right)^2}{(\frac{3}{2})_n 2^n}\ {}_2F_1\left[\begin{array}{cc} n+\frac{1}{2}, n+\frac{1}{2}; & \\ & \frac{x}{2} \\ \frac{3}{2}+n; & \end{array}\right];\quad |x| < 2 \ and \ n \in \mathbb{N}_0 \tag{3.1}$$

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2 = \frac{\left((n!)\right)^3}{(\frac{3}{2})_n\ (2)_n\ 2^n}\ {}_3F_2\left[\begin{array}{cc} 1+n, 1+n, 1+n; & \\ & \frac{x}{2} \\ 2+n, \frac{3}{2}+n; & \end{array}\right];\quad |x| < 2 \ and \ n \in \mathbb{N}_0 \tag{3.2}$$

$$\frac{d^n}{dx^n}\left(\frac{(vers^{-1}(x))^2}{2}\right) = \frac{((n-1)!)^2}{(\frac{1}{2})_n\ 2^n}\ {}_2F_1\left[\begin{array}{cc} n, n; & \\ & \frac{x}{2} \\ n+\frac{1}{2}; & \end{array}\right];\quad |x| < 2 \ and \ n \in \mathbb{N} \tag{3.3}$$

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right) = (-1)^n \left(-\frac{1}{2}\right)_n x^{\frac{1}{2}-n}\ {}_2F_1\left[\begin{array}{cc} \frac{1}{2}, \frac{1}{2}; & \\ & x \\ \frac{3}{2}-n; & \end{array}\right],\quad |x| < 1 \ and \ n \in \mathbb{N}_0 \tag{3.4}$$

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right)^2 = \frac{((n-1)!)^2}{2\left(\frac{1}{2}\right)_n}{}_2F_1\left[\begin{array}{cc} n, n; & \\ & x \\ n+\frac{1}{2}; & \end{array}\right],\quad |x| < 1 \ and \ n \in \mathbb{N} \tag{3.5}$$

$$\frac{d^n}{dx^n} covers\left(\frac{\pi}{2} \pm 2\sqrt{x}\right) = \frac{(-1)^{n-1}}{\left(\frac{1}{2}\right)_n} {}_0F_1\left[\begin{matrix} -; \\ n+\frac{1}{2}; \end{matrix} \; -x\right], \quad \forall\, x \; and \; n \in \mathbb{N}_0 \tag{3.6}$$

$$\frac{d^n}{dx^n} vers\left(\pm 2\sqrt{x}\right) = \frac{(-1)^{n-1}}{\left(\frac{1}{2}\right)_n} {}_0F_1\left[\begin{matrix} -; \\ n+\frac{1}{2}; \end{matrix} \; -x\right], \quad \forall\, x \; and \; n \in \mathbb{N}_0 \tag{3.7}$$

Proof of the result (3.1)**:**

Using the result (2.4), we have

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right) = \frac{d^n}{dx^n}\left({}_2F_1\left[\begin{matrix} \frac{1}{2},\frac{1}{2}; \\ \frac{3}{2}; \end{matrix} \; \frac{x}{2}\right]\right) \tag{3.8}$$

$$= \frac{d^n}{dx^n}\left(\sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r(\frac{1}{2})_r}{(\frac{3}{2})_r\, 2^r} \; \frac{x^r}{r!}\right) \tag{3.9}$$

$$= \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r\,(\frac{1}{2})_r}{(\frac{3}{2})_r\, 2^r} \; \frac{d^n}{dx^n}\frac{x^r}{r!} \tag{3.10}$$

$$= \sum_{r=n}^{\infty} \frac{(\frac{1}{2})_r\,(\frac{1}{2})_r\; r!}{(\frac{3}{2})_r\, 2^r\,(r-n)!} \; \frac{x^{r-n}}{r!}\,. \tag{3.11}$$

Replacing r by $r+n$ in equation (3.11), we get

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_{r+n}\,(\frac{1}{2})_{r+n}}{(\frac{3}{2})_{r+n}\, 2^{r+n}\,(r+n-n)!} \; x^{r+n-n} \tag{3.12}$$

$$= \frac{(\frac{1}{2})_n\,(\frac{1}{2})_n}{(\frac{3}{2})_n\, 2^n} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+n)_r\,(\frac{1}{2}+n)_r}{(\frac{3}{2}+n)_r\, 2^r} \; \frac{x^r}{r!} \tag{3.13}$$

$$= \frac{\left((\frac{1}{2})_n\right)^2}{(\frac{3}{2})_n 2^n} \; {}_2F_1\left[\begin{matrix} n+\frac{1}{2}, n+\frac{1}{2}; \\ \frac{3}{2}+n; \end{matrix} \; \frac{x}{2}\right]. \tag{3.14}$$

Proof of the result (3.2)**:**

Using the result (2.5), we have

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2 = \frac{d^n}{dx^n} \; {}_3F_2\left[\begin{matrix} 1,1,1; \\ 2,\frac{3}{2}; \end{matrix} \; \frac{x}{2}\right]. \tag{3.15}$$

$$= \frac{d^n}{dx^n}\left(\sum_{r=0}^{\infty} \frac{(1)_r(1)_r(1)_r}{(\frac{3}{2})_r\, 2^r\,(2)_r} \; \frac{x^r}{r!}\right) \tag{3.16}$$

$$= \sum_{r=0}^{\infty} \frac{(1)_r(1)_r(1)_r}{(\frac{3}{2})_r\, 2^r\,(2)_r} \; \frac{d^n}{dx^n}\frac{x^r}{r!} \tag{3.17}$$

$$= \sum_{r=n}^{\infty} \frac{(1)_r(1)_r(1)_r\; r!}{(\frac{3}{2})_r\, 2^r\,(2)_r\,(r-n)!} \; \frac{x^{r-n}}{r!}\,. \tag{3.18}$$

Replacing r by $r+n$ in equation (3.18), we get

$$\frac{d^n}{dx^n}\left(\frac{vers^{-1}(x)}{\sqrt{(2x)}}\right)^2 = \sum_{r=0}^{\infty} \frac{(1)_{r+n}(1)_{r+n}(1)_{r+n}}{(\frac{3}{2})_{r+n}\, 2^{r+n}\, (2)_{r+n}\, (r+n-n)!}\; x^{r+n-n} \tag{3.19}$$

$$= \frac{(1)_n\,(1)_n\,(1)_n}{(2)_n\, 2^n\, (\frac{3}{2})_n} \sum_{r=0}^{\infty} \frac{(1+n)_r(1+n)_r(1+n)_r}{(\frac{3}{2}+n)_r\, 2^r\, (2+n)_r\, (r)!}\; x^r \tag{3.20}$$

$$= \frac{((n!))^3}{(\frac{3}{2})_n\,(2)_n\, 2^n}\; {}_3F_2\left[\begin{array}{c} 1+n, 1+n, 1+n; \\ \\ 2+n, \frac{3}{2}+n; \end{array} \frac{x}{2}\right]. \tag{3.21}$$

Proof of the result (3.3):
From (2.52), we know that

$$\frac{d^n}{dx^n}\left(\frac{(vers^{-1}(x))^2}{2}\right) = \frac{d^n}{dx^n}\left(x\;{}_3F_2\left[\begin{array}{c} 1,1,1; \\ \\ 2, \frac{3}{2}; \end{array} \frac{x}{2}\right]\right) \tag{3.22}$$

$$= \frac{d^n}{dx^n}\left(x \sum_{r=0}^{\infty} \frac{(1)_r(1)_r(1)_r}{(\frac{3}{2})_r\, 2^r\, (2)_r}\; \frac{x^r}{r!}\right) \tag{3.23}$$

$$= \sum_{r=0}^{\infty} \frac{(1)_r(1)_r(1)_r}{(\frac{3}{2})_r\, 2^r\, (2)_r}\; \frac{d^n}{dx^n}\frac{x^{r+1}}{r!} \tag{3.24}$$

$$= \sum_{r=n-1}^{\infty} \frac{(1)_r(1)_r(1)_r\,(r+1)!}{(\frac{3}{2})_r\, 2^r\, (2)_r\, (r+1-n)!}\; \frac{x^{r+1-n}}{r!}. \tag{3.25}$$

Replacing r by $r+n-1$ in equation (3.25), we get

$$\frac{d^n}{dx^n}\left(\frac{(vers^{-1}(x))^2}{2}\right) = \sum_{r=0}^{\infty} \frac{(1)_{r+n-1}(1)_{r+n-1}(1)_{r+n-1}\,(r+n-1+1)!}{(\frac{3}{2})_{r+n-1}\, 2^{r+n-1}\, (2)_{r+n-1}\, (r+n-1+1-n)!}\; \frac{x^{r+n-1+1-n}}{(r+n-1)!} \tag{3.26}$$

$$= \sum_{r=0}^{\infty} \frac{(1)_{r+n-1}(1)_{r+n-1}\,(r+n)!}{(\frac{3}{2})_{r+n-1}\, 2^{r+n-1}\, (2)_{r+n-1}}\; \frac{x^r}{(r)!} \tag{3.27}$$

$$= \frac{(1)_{n-1}\,(1)_{n-1}}{(2)_{n-1}\,(\frac{3}{2})_{n-1}\, 2^{(n-1)}} \sum_{r=0}^{\infty} \frac{(1+n-1)_r(1+n-1)_r\,(r+n)!}{(2+n-1)_r(\frac{3}{2}+n-1)_r\, 2^r}\; \frac{x^r}{(r)!}. \tag{3.28}$$

On further simplification, we get

$$\frac{d^n}{dx^n}\left(\frac{(vers^{-1}(x))^2}{2}\right) = \frac{((n-1)!)^2\, \frac{1}{2}\, \Gamma(\frac{1}{2})\, 2}{\Gamma(\frac{1}{2}+n)\, 2^n} \sum_{r=0}^{\infty} \frac{(n)_r(n)_r}{(\frac{1}{2}+n)_r\, 2^r}\; \frac{x^r}{(r)!} \tag{3.29}$$

$$= \frac{((n-1)!)^2}{(\frac{1}{2})_n\, 2^n}\; {}_2F_1\left[\begin{array}{c} n, n; \\ \\ n+\frac{1}{2}; \end{array} \frac{x}{2}\right]. \tag{3.30}$$

Proof of the result (3.4):
In equation (2.1) replace x by $1-\sqrt{x}$, we get

$$covers^{-1}(1-\sqrt{x}) = \sqrt{x}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};\\ \frac{3}{2};\end{matrix}\ x\right]. \tag{3.31}$$

Now

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right) = \frac{d^n}{dx^n}\left\{\sqrt{x}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};\\ \frac{3}{2};\end{matrix}\ x\right]\right\} \tag{3.32}$$

$$= \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r}{\left(\frac{3}{2}\right)_r\ r!}\frac{d^n}{dx^n}\ x^{r+\frac{1}{2}}. \tag{3.33}$$

Now using fractional derivative formula (1.32), we get

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right) = \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r\Gamma(r+\frac{1}{2}+1)}{\left(\frac{3}{2}\right)_r\Gamma(r+\frac{1}{2}-n+1)\ r!}\ x^{r+\frac{1}{2}-n} \tag{3.34}$$

$$= \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r\Gamma\left(\frac{3}{2}\right)}{\Gamma(r+\frac{3}{2}-n)\ r!}\ x^{r+\frac{1}{2}-n} \tag{3.35}$$

$$= \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r\Gamma\left(\frac{3}{2}\right)}{\left(\frac{3}{2}-n\right)_r\Gamma(\frac{3}{2}-n)\ r!}\ x^{r+\frac{1}{2}-n}, \tag{3.36}$$

which on simplification gives:

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-n\right)}\ x^{\frac{1}{2}-n}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};\\ \frac{3}{2}-n;\end{matrix}\ x\right] \tag{3.37}$$

$$= (-1)^n\left(-\frac{1}{2}\right)_n\ x^{\frac{1}{2}-n}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};\\ \frac{3}{2}-n;\end{matrix}\ x\right] \tag{3.38}$$

Proof of the result (3.5):
In equation (2.2) replace x by $1-\sqrt{x}$, we get

$$\left(covers^{-1}(1-\sqrt{x})\right)^2 = x\ {}_3F_2\left[\begin{matrix}1,1,1;\\ 2,\frac{3}{2};\end{matrix}\ x\right]. \tag{3.39}$$

Now

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right)^2 = \frac{d^n}{dx^n}\left\{x\ {}_3F_2\left[\begin{matrix}1,1,1;\\ 2,\frac{3}{2};\end{matrix}\ x\right]\right\} \tag{3.40}$$

$$= \sum_{r=0}^{\infty}\frac{(1)_r(1)_r(1)_r}{(2)_r\left(\frac{3}{2}\right)_r\ r!}\frac{d^n}{dx^n}\ x^{r+1} \tag{3.41}$$

$$= \sum_{r=n-1}^{\infty}\frac{(1)_r(1)_r(r+1)!}{(2)_r\left(\frac{3}{2}\right)_r\ (r+1-n)!}\ x^{r+1-n}. \tag{3.42}$$

Replacing r by $r+n-1$ in equation (3.42), we get

$$\frac{d^n}{dx^n}\left(covers^{-1}(1-\sqrt{x})\right)^2 = \sum_{r=0}^{\infty}\frac{(1)_{r+n-1}(1)_{r+n-1}(r+n)!}{(2)_{r+n-1}\left(\frac{3}{2}\right)_{r+n-1}(r)!}\, x^r \tag{3.43}$$

$$= \frac{(1)_{n-1}(1)_{n-1}}{(2)_{n-1}\left(\frac{3}{2}\right)_{n-1}}\sum_{r=0}^{\infty}\frac{(n)_r(n)_r\Gamma(r+n+1)}{(1+n)_r\left(\frac{1}{2}+n\right)_r(r)!}\, x^r \tag{3.44}$$

$$= \frac{((n-1)!)^2\,\Gamma(2)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma(1+n)\Gamma\left(\frac{1}{2}+n\right)}\sum_{r=0}^{\infty}\frac{(n)_r(n)_r\Gamma(n+1)}{\left(\frac{1}{2}+n\right)_r(r)!}\, x^r \tag{3.45}$$

$$= \frac{((n-1)!)^2}{2\left(\frac{1}{2}\right)_n}\,{}_2F_1\left[\begin{matrix} n, n; \\ & x \\ n+\frac{1}{2}; \end{matrix}\right]. \tag{3.46}$$

Proof of the result (3.6)

In equation (2.3) replacing x by $\left(\frac{\pi}{2}\pm 2\sqrt{x}\right)$, we get

$$covers\left(\frac{\pi}{2}\pm 2\sqrt{x}\right) = 2x\;{}_1F_2\left[\begin{matrix} 1; \\ & -x \\ 2, \frac{3}{2}; \end{matrix}\right] \tag{3.47}$$

Now,

$$\frac{d^n}{dx^n}covers\left(\frac{\pi}{2}\pm 2\sqrt{x}\right) = \frac{d^n}{dx^n}\left\{2x\;{}_1F_2\left[\begin{matrix} 1; \\ & -x \\ 2, \frac{3}{2}; \end{matrix}\right]\right\} \tag{3.48}$$

$$= 2\sum_{r=0}^{\infty}\frac{(1)_r(-1)^r}{(2)_r\left(\frac{3}{2}\right)_r r!}\frac{d^n}{dx^n}\, x^{r+1} \tag{3.49}$$

$$= 2\sum_{r=n-1}^{\infty}\frac{(1)_r(-1)^r(r+1)!}{(2)_r\left(\frac{3}{2}\right)_r r!(r+1-n)!}\, x^{r+1-n}\, . \tag{3.50}$$

Replacing r by $r+n-1$ in equation (3.50), we get

$$\frac{d^n}{dx^n}covers\left(\frac{\pi}{2}\pm 2\sqrt{x}\right) = 2\sum_{r=0}^{\infty}\frac{(-1)^{r+n-1}(r+n)!}{(2)_{r+n-1}\left(\frac{3}{2}\right)_{r+n-1}(r)!}\, x^r \tag{3.51}$$

$$= \frac{2(-1)^{n-1}}{(2)_{n-1}\left(\frac{3}{2}\right)_{n-1}}\sum_{r=0}^{\infty}\frac{(-1)^r(r+n)!}{(1+n)_r\left(\frac{1}{2}+n\right)_r(r)!}\, x^r \tag{3.52}$$

On simplification equation (3.52) gives:

$$\frac{d^n}{dx^n}covers\left(\frac{\pi}{2}\pm 2\sqrt{x}\right) = \frac{2(-1)^{n-1}\,\Gamma(2)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma(1+n)\Gamma\left(\frac{1}{2}+n\right)}\sum_{r=0}^{\infty}\frac{(-1)^r(n+1)_r\Gamma(n+1)}{(1+n)_r\left(\frac{1}{2}+n\right)_r(r)!}\, x^r \tag{3.53}$$

$$\frac{d^n}{dx^n}covers\left(\frac{\pi}{2}\pm 2\sqrt{x}\right) = \frac{(-1)^{n-1}}{\left(\frac{1}{2}\right)_n}\,{}_0F_1\left[\begin{matrix} -; \\ & -x \\ n+\frac{1}{2}; \end{matrix}\right], \tag{3.54}$$

which proves the result (3.6),
In equation (2.7) replacing x by $\pm 2\sqrt{x}$, we get

$$vers(\pm 2\sqrt{x}) = (2x)\;{}_1F_2\left[\begin{matrix} 1; \\ & -x \\ 2, \frac{3}{2}; \end{matrix}\right]. \tag{3.55}$$

Similarly by adopting the same procedure (3.55) as used in the derivation of (3.6), we can prove the result (3.7).

4 Conclusion

In this paper we obtained hypergeometric forms and successive differential coefficients of some typical mathematical functions e.g. $covers(x)$, $covers^{-1}(x)$, $vers(x)$, $vers^{-1}(x)$ (given in the monographs of differential calculus by J. Edwards) and other related functions using the expansions of Taylor, Maclaurin, series rearrangement technique and Clausen's formula. From the above observations, we can obtain hypergeometric forms and successive differential coefficients of other new mathematical functions (may be found in the literature of Mathematics) in an analogous manner, which may be useful in the fields of Mathematical Physics, Statistics and other branches of Science and Engineering.

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$z = 1 + \frac{\alpha^{'}}{1} \cdot \frac{\beta^{'}}{\gamma^{'}} \cdot \frac{\delta^{'}}{\epsilon^{'}} x + \frac{\alpha^{'}.(\alpha^{'}+1)}{1.2} \cdot \frac{\beta^{'}.(\beta^{'}+1)}{\gamma^{'}.(\gamma^{'}+1)} \cdot \frac{\delta^{'}.(\delta^{'}+1)}{\epsilon^{'}.(\epsilon^{'}+1)} x^2 + \ldots$
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